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THE ZENGA EQUALITY CURVE: A NEW APPROACH TO MEASURING TAX REDISTRIBUTION AND PROGRESSIVITY

by Francesca Greselin

Università degli Studi di Milano-Bicocca

Simone Pellegrino*

Università degli Studi di Torino

and

Achille Vernizzi

Università degli Studi di Milano

We adopt and extend the new Zenga inequality curve to study the degree of progressivity as well as the redistributive and re-ranking effects of a personal income tax system. Moreover, we also establish the social welfare implications of these new inequality measures and compare them with the classical approach based on the Lorenz curve and the Gini coefficient. The Zenga methodology is based on comparing the mean income of the poorest income earners with the mean income of the remaining richest part of the population. To the best of our knowledge, this approach has never been applied to study the effects produced by a personal income tax. To fill this gap in the literature, we prove that the elasticity of the Zenga uniformity curve with respect to the Lorenz curve is always greater than 1, thus recasting—within the new paradigm—the most important curves and the corresponding tax indices, such as the Reynolds–Smolensky, the Kakwani, and the Atkinson–Plotnick–Kakwani indices. We then derive three important inequalities for the newly developed measures, inspired by the well-known properties of the classical approach. Finally, we show how some information, which could remain unnoticed by the cumulative approach inherent to the Lorenz curve, is instead highlighted by the new methodology. The advantages of complementing the classic indices with the new ones are discussed through an application to the Italian tax system.

JEL Codes: H23, H24

Keywords: bottom-to-top ratios, Gini index, personal income tax, Zenga index

1. Introduction

Over more than a century of economic literature, several approaches have been proposed to study the inequality of quantitative variables, primarily income distributions.

Within this research stream, many synthetic indices appeared to summarize and compare the inequality of distributions using a single scalar. Among them, the

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*Correspondence to: Simone Pellegrino, Department of Economics and Statistics – ESOMAS, Università degli Studi di Torino, Torino, Italy (simone.pellegrino@unito.it).

most famous inequality index is, undoubtedly, the Gini (1914) coefficient, which has also a graphical explanation through the Lorenz (1905) curve.

In addition, a Gini-based methodology has been proposed in the tax literature to measure the degree of tax progressivity, yielding the (Gini-based) Kakwani (1977b) index, considering that relative income differentials get compressed in the transition from the pre-tax to the post-tax distribution. Similarly, the redistributive effect produced by the tax is measured by the (Gini-based) Reynolds and Smolensky (1977) index.

Although the Lorenz curve is a fundamental tool for welfare comparisons (Atkinson, 1970; Shorrocks, 1983; Atkinson and Bourguignon, 1987), we have found only a few applications of it in the tax literature, because it is not easy to draw specific conclusions by examining and comparing the Lorenz and concentration curves of different distributions (i.e. pre-tax distribution, post-tax distribution, and tax distribution). Indeed, because of their inherent cumulative nature, the different Lorenz curves are hardly distinguishable. Therefore, in most of the existing empirical research, the overall effect of taxation and transfer policy is primarily derived from the Gini and concentration coefficients (for a recent application, see Guillaud et al., 2019).

A few years ago, Zenga (2007) proposed a new methodology to plot and measure inequality, in which the new curve and index are based on comparisons between the mean income of the poorest income earners and the mean income of the remaining richest part of the population. Several recent studies have pointed out the different features of the Zenga approach with respect to the standard viewpoint, based on the Lorenz curve. Therefore, we move further in this literature stream to explore the effectiveness of the former method in studying the effects of a personal income tax and the social welfare interpretations and implications of the tax progressivity (Son, 2013; Kakwani and Son, 2020) according to this new procedure. In particular, we mainly show that the graphical representation and some analytical tools based on the Zenga inequality curve provide an accurate instrument for understanding which part of a pre-tax distribution is mostly affected by the tax system or by a tax reform.

The remainder of this paper is organized as follows. Section 2 briefly reviews the Lorenz and Gini approaches and then recalls their implications about social welfare. Similarly, Section 3 outlines the new Zenga curve and index, reminds their properties, and presents the social welfare implications within the Zenga approach. In view of our main purpose, that is, to analyze the effects of taxation on income distributions, Section 4 extends the Zenga curve and index, defining three new tools for measuring the degree of progressivity of a personal income tax, as well as its re-ranking and redistributive effects. Important relationships among the new indices and curves have been derived, and the social welfare implications of taxation are discussed. Section 5 shows an application to a tax case; in particular, Section 5.1 introduces the microsimulation model used for the empirical estimations, whereas Section 5.2 develops a compared analysis with the new and standard approach, to show the interest and potential of the proposal; finally, Section 5.3 presents the social welfare implications. Section 6 offers some concluding remarks. We gather in Appendix A some technical proofs and in Appendix B formulas for the derivations of empirical curves and indices using survey data.

2. The Lorenz Curve and the Gini Index

2.1. *Basic Notation*

Given a random variable $Z \geq 0$ with nonnegatively supported *cdf* $F(Z)$ for $Z \geq 0$, representing gross or net incomes as well as taxes, we denote the corresponding population quantile function by $F^{-1}(p) = z_p = inf\{z: F(Z) \geq p\}$, for $0 < p < 1$.

The Lorenz (1905) curve plots the cumulative share of Z, denoted by $L_F(p)$, versus the cumulative share of the population p. In the ideal case of perfect equality (i.e. a society in which all people have the same income), the share of incomes equals the share of the population, so that $L_F(p) = p$, for all $0 < p < 1$. In this case, the Lorenz curve is the diagonal line from (0, 0) to (1, 1). Conversely, the lower the share of income $L_F(p)$ held by the share of income earners p, the higher the inequality. In the ideal case of perfect inequality (i.e. a society in which all people but one have a zero income), the share of incomes equals zero for $0 \leq p < 1$, so that $L_F(p) = 0$, and only for $p = 1$ we have $L_F(1) = 1$. It is given by $(p, L_F(p))$, where

$$(1) \qquad L_F(p) = \frac{\int_0^p F^{-1}(s)ds}{\int_0^1 F^{-1}(s)ds} = \frac{1}{\mu_F}\int_0^p F^{-1}(s)ds,$$

and $\mu_F = E(Z)$ denotes the mean value or the expectation of the random variable Z.

It seems very natural to express the degree of inequality through the deviation of the actual Lorenz curve from the line we get in the case of perfect equality, namely the diagonal line. The Gini coefficient is precisely given by twice the area between the equality line and the Lorenz curve.[1]

$$(2) \qquad G_F = 2\int_0^1 \left(p - L_F(p)\right) dp.$$

Now, considering the mean income of the poorer p percent of the population

$$(3) \qquad \mu_F^-(p) = \frac{1}{p}\int_0^p F^{-1}(s)\, ds,$$

the Gini index can be rewritten in terms of the relative deviation of the mean income of the poorer p percent of the population from the overall mean μ_F:

$$(4) \qquad G_F = \int_0^1 \frac{\mu_F - \mu_F^-(p)}{\mu_F} 2p\, dp.$$

[1]In the original works (Gini, 1912, 1914; Pietra, 1915), the coefficient is given in terms of discrete distributions.

2.2. *Social Welfare Implications*

Following the seminal paper by Atkinson (1970) and related literature (see Son,2013; Kakwani and Son, 2020) for a review of the literature and extensions), government policies should be judged based on their impact on social welfare, which is an aggregate measure of society's welfare derived from each individual's welfare levels. Within this framework, the area under twice the (generalized) Lorenz curve can be interpreted as follows:

$$W_G = 2\int_0^1 \mu_F L(p)dp = \mu_F(1-G) = 2\int_0^{+\infty} z_p(1-p)dp. \tag{5}$$

Equation (5) is the social welfare function implied by the Gini index that was proposed by Sen (1974). Note from the third term on the right-hand side of equation (5) that the Gini social welfare function is the weighted average of individual welfare levels z_p, with weight given by $w(p) = 2(1-p)$. It can easily be shown that the total weight adds up to 1 for the entire population. Furthermore, the weight is proportional to the welfare ranking of individuals: the poorest individual receives the maximum weight, and the richest individual gets the minimum weight. If this pro-poor weighting of welfare is acceptable to policymakers, then it is a useful tool to analyze government policies (see Kakwani and Son, 2020).

3. The New Zenga Curve and Index

3.1. *Basic Notation*

The increasing gap observed between the less fortunate and the more fortunate individuals (see, among many others, Piketty, 2013), motivated a fresh rethinking with respect to inequality and gave rise to many proposals in the literature (see Zenga, 2007; Gastwirth, 2014, 2016; Davydov and Greselin, 2018, and references wherein). There is a consensus that no measure can be considered superior to the others (Osberg, 2017); therefore, the choice of an inequality measure must rest on its appropriateness for specific substantive problems (Jasso, 1982). In particular, a few years ago, to capture the recent changes in the extreme parts of the income distribution, Zenga (2007) proposed a new inequality curve $I_F(p)$ based on contrasting the average income of the poorer p percent bottom earners $\mu_F^-(p)$ defined in equation (3) with the amount that is held, on average, by the richest top earners, namely the remaining $(1-p)$ percent of the population:

$$\mu_F^+(p) = \frac{1}{1-p}\int_p^1 F^{-1}(s)ds$$

Therefore, Zenga (2007) defined the curve $\left(p, I_F(p)\right)$, where

$$I_F(p) = \frac{\mu_F^+(p) - \mu_F^-(p)}{\mu_F^+(p)}$$

for $0 < p < 1$. When the random variable Z is equal to a constant, the corresponding quantile $F^{-1}(p)$ is also equal to the constant, as well as both the lower and the upper means $\mu_F^-(p)$ and $\mu_F^+(p)$; thus, $I_F(p) = 0 \forall p \in (0, 1)$, indicating perfect equality or an egalitarian society. The other extreme scenario is when, loosely speaking, there is only one member in the society who gets the entire income of the population; thus, $I_F(p) = 1 \forall p \in (0, 1)$. As illustrated by Greselin et al. (2010), this approach *considers that the notions of poor and rich are relative to each other* and summarizes, in a single measure, the amount of inequality in the population by the following index:[2]

$$I_F = \int_0^1 \frac{\mu_F^+(p) - \mu_F^-(p)}{\mu_F^+(p)} \, dp. \tag{6}$$

3.2. *Properties of the Zenga Index*

The Zenga index obeys a number of properties that can be regarded as being intrinsic to the concept of inequality. In what follows, we review such properties using the notation I_Z or I_F for the index, whenever this helps in simplifying the presentation.

- *Scale Invariance*. The Zenga inequality index is a relative inequality measure, as proportional changes in all incomes, where changing Z in cZ with $c > 0$, do not alter the level of inequality:

$$I_Z = I_{cZ}. \tag{7}$$

Technically, we say that it is homogeneous of degree zero in incomes. Evidently, an index satisfying equation (7) handles the money illusion; namely, if incomes are measured in pounds instead of dollars, then inequality does not change.

- *Sensitivity to translation*. By adding any constant $c > 0$ to the income Z, transforming Z into $Z + c$, the relative measure of inequality decreases:

$$I_{Z+c} \leq I_Z.$$

[2]At the time of paper submission, we found a similar approach in Jasso (2018), based on the ratio of average income among the top 1 percent to the average income among the bottom 99 percent, called "TopBot." The author states that "*TopBot is not constrained to the 99-1% split, but can be used with any percentage split of a population into two sub-populations, such as 90-10, 50-50, 10-90, 1-99, and so on, or whatever may be most appropriate for the substantive context*." To complete the comparison with Jasso (2018), the Zenga (2007) index integrates the standard form of TopBot as: "one minus the bottom-to-top ratio" $1 - \mu^-(p) / \mu^+(1 - p)$ over all percentiles p for the Bottom group, and the complementary group of $1 - p$ for the Top group.

In other words, a relative measure of inequality has to consider that the relative distances within incomes are reduced by adding a constant positive amount.

- *Lorenz ordering*. Following Aaberge (2001), the Lorenz ordering $Z \leq_L Y$ indicates the bound $L_Z(p) \geq L_Y(p)$ for all $p \in [0, 1]$. If the random variables Z and Y follow the Lorenz ordering, then

$$I_Z \leq I_Y.$$

- *Pigou–Dalton transfer principle*. Pigou–Dalton transfer principle states that progressive (i.e. from rich to poor) rank-order and mean-preserving transfers should decrease the value of inequality measures, changing Z into Y, yielding

$$I_Z \geq I_Y.$$

Comparing the Lorenz and the Zenga curves,[3] we observe that although the Zenga $I_F(p)$ contrasts the mean incomes of two disjoint and exhaustive subpopulations (i.e. the poor and the rich), $L_F(p)$ compares the economic condition of the poorer group to that of the entire population.

Finally, the most important consideration is related to the weight function $2p$ adopted to define the Gini index in equation (4), for normalization purposes. It places a less emphasis on the most crucial comparisons referring to the poorest portions of populations, and a greater emphasis on the comparisons between almost-coinciding subpopulations, which are likely to be less informative.[4] However, the Zenga index considers, with the same weight, any relative deviation from equality, measured by $I_F(p)$, in any part of the distribution.

3.3. *Comparison Between the Zenga and the Lorenz Curves*

In the analysis of real data and the effects of different taxation systems, we prefer to switch from the Zenga inequality curve to its complement to 1, that is, the Zenga uniformity curve. In the following sections, the latter plays a central role. It is given by $\left(p, U_F(p)\right) = \left(p, 1 - I_F(p)\right)$, where

$$U_F(p) = \frac{\mu_F^-(p)}{\mu_F^+(p)}.$$

[3]Even though the literature on the Zenga index and curve is obviously not as extensive as the one on the Gini index, we find research on various features of the index and curve (Polisicchio, 2008; Polisicchio and Porro, 2009; Jedrzejczak and Trzcinska, 2019; Maffenini and Polisicchio, 2014; Greselin et al., 2009; Arcagni and Porro, 2014), inferential results and their applications (Greselin and Pasquazzi, 2009; Greselin et al., 2010, 2013, 2014), subgroup decompositions of the index (Radaelli, 2008, 2010), longitudinal decomposition (Mussini and Zenga, 2013) decompositions by income sources (Zenga et al., 2012; Pasquazzi and Zenga, 2018), and many applications on real data (Arcagni and Zenga, 2013). The interested reader can find accurate discussions on the advantages of the Zenga index over the Gini index within both the descriptive and inferential frameworks. Langel and Tillé (2012) analyzed the sampling distribution of the empirical Zenga index, and Antal et al. (2011) extended inferential results to complex sampling designs.

[4]Because of this consideration, many generalization of the Gini index arose in the literature, emphasizing or de-emphasizing, depending on the problem under consideration, the difference $p - L(p)$ in some regions of the unit interval [0, 1]; among them, we cite Donaldson and Weymark (1980) and Yitzhaki (1983).

We introduced $I(p)$ as an inequality curve, being equivalent to the claim that $U(p)$ is a curve measuring the extent of equality across the opposite groups of poorer and richer individuals in the population.[5] Any departure from the perfect equality is represented by the deviation of the uniformity curve from 1. Moreover, $U(p)$ has a more direct interpretation, as we will see in the application to real economic data in Section 5. The Lorenz curve itself can be expressed in terms of means and percentiles as

$$L_F(p) = \frac{p\mu_F^-(p)}{\mu_F},$$

which also yields

$$L_F(p) = U_F(p)\,\frac{p\mu_F^+(p)}{\mu_F}. \tag{8}$$

To analyze the sensitivity of the uniformity curve with respect to a variation in the Lorenz curve, we start from another equivalent formulation of their analytical relationship (Zenga, 2007):

$$U_F(p) = \frac{(1-p)\,L_F(p)}{p\left(1-L_F(p)\right)},$$

and we obtain

$$\frac{\partial U_F(p)}{\partial L_F(p)} = \frac{(1-p)}{p(1-L_F(p))^2}. \tag{9}$$

The elasticity measures the proportional change in an economic variable in response to a change in another; therefore, the elasticity of the uniformity curve with respect to the Lorenz curve is given by

$$\frac{\partial U_F(p)\,/\partial L_F(p)}{U_F(p)\,/L_F(p)} = \frac{1}{1-L_F(p)} > 1. \tag{10}$$

This means that an increase of 1 percent in the Lorenz ordinate causes a change in the ordinate of the uniformity curve greater than 1 percent, for all $p \in (0,1)$. Moreover, we see that the elasticity in equation (10) increases as $L_F(p)$ approaches 1, that is, with the increase in percentile p. An increase of 1 percent in the income of the poorest p percent can be hardly noticed as a difference between

[5]An anonymous referee suggested us to use the word "equality" to designate the curve $U(p)$, being "equality" the opposite of "inequality." We opted for the name given to it by Zenga, in the original paper: "$U(p)$ measures the *uniformity* between the lower and the upper group" of poorer and richer, respectively.

the two Lorenz curves. However, because of the greater elasticity of $U_F(p)$ with respect to $L_F(p)$, the same 1 percent increase in the income of the poorest p percent may produce a visible shift in $U_F(p)$. We will empirically observe the effects of this higher sensitivity of the Zenga uniformity curve in Section 5, where the latter reacts with more evidence to a new tax system, generating a modification in the underlying distribution F.

The Lorenz curve captures, in a sense, the essence of inequality, by displaying the deviation of each person's welfare from perfect equality. The nearer the Lorenz curve is to the egalitarian line, the more equal the distribution of welfare. We end this section by showing the equivalence between Lorenz dominance and the ordering based on the $U_F(p)$ curve. Let F_X and F_Y be the distribution functions of the continuous nonnegative random variables X and Y, with both finite and positive expected values. We need to introduce here some notions of dominance:

Definition 1. *We say that X dominates Y under the Lorenz ordering, denoting it by $F_X \geq_L F_Y$ if and only if $L_{F_X}(p) \leq L_{F_Y}(p)\ \forall p \in (0,1)$.*

Definition 2. *We say that X dominates Y under the uniformity ordering, denoting it by $F_X \geq_Z F_Y$ if and only if $U_{F_X}(p) \leq U_{F_Y}(p)\ \forall p \in (0,1)$.*

Proposition 3. *The Lorenz ordering and the ordering based on the uniformity curve are equivalent.*[6]

Proof. We may rewrite equation (10) as follows:

$$L_F(p) = 1 - \frac{1-p}{1-p+pU_F(p)}.$$

Therefore, $L_{F_X}(p) \leq L_{F_Y}(p) \Leftrightarrow L_{F_X}(p) = 1 - \frac{1-p}{1-p+pU_{F_X}(p)} \leq 1 - \frac{1-p}{1-p+pU_{F_Y}(p)}, \Leftrightarrow U_{F_X}(p) \leq U_{F_Y}(p) \forall p \in (0,1)$.

Consequently, the uniformity curve can be used as a criterion for ranking government policies or programs.

3.4. *Social Welfare Implications*

Along the lines of Son (2013) and Kakwani and Son (2020), we introduce here the social welfare evaluation based on the Zenga uniformity curve

[6]Polisicchio and Porro (2009) have shown that the ordering based on the Zenga inequality curve is equivalent to the Lorenz ordering.

$$W_Z = \mu_F \int_0^1 \frac{L_F(p)}{(1-L_F(p))} \frac{(1-p)}{p} dp = \mu_F(1-I_Z) \tag{11}$$
$$= \int_0^1 z_p \frac{(-\ln p + p - 1)}{(1-L_F(p))^2} dp,$$

where the weight function $w_Z(p)$ is obtained after integrating by parts W_Z, as follows:

$$w_Z(p) = \frac{(-\ln p + p - 1)}{(1-L_F(p))^2}.$$

We can verify that the function described by equation (11) is homogeneous of degree one, implying that if we change all incomes by the same proportion, this function also varies by the same proportion. We then decompose $w_Z(p)$ into two multiplicative terms

$$w_Z(p) = \frac{(-\ln p + p - 1)}{(1-p)^2} \frac{(1-p)^2}{(1-L_F(p))^2}$$
$$:= w_Z^*(p)\,\beta_Z(p).$$

The first term

$$w_Z^*(p) = \frac{(-\ln p + p - 1)}{(1-p)^2}$$

is a *nonnegative, concave upward*, and *strictly decreasing* function of the rank p, like the social welfare functions

$$w_G(p) = \left[2(1-p)\right], \quad w_{G_k}(p) = \left[k\left(1-p\right)^k\right], \quad \text{and} \quad w_B(p) = \left[-\ln p\right].$$

that are implicit in the Gini index G, the generalized Gini G_k, and the Bonferroni index B, respectively (see Kakwani and Son, 2020). It can easily be shown that the total weight $w_Z^*(p)$ adds up to 1. Moreover, if everyone receives the same income, then the social welfare function W_Z must be equal to μ. To satisfy this requirement, the total weight implied by $w_Z(p)$ must add up to one. This is verified because, when all incomes are equal, we have that $L_F(p) = p$, so that

$$w(p) = \frac{-\ln p + p - 1}{(1-L_F(p))^2} = \frac{-\ln p + p - 1}{(1-p)^2} = w^*(p). \tag{12}$$

The function $w_Z^*(p)$ incorporates a society's distributional judgment, where the poorest individual receives the maximum weight and the richest individual gets the minimum weight (see Greselin et al.,2020). The second term

[Correction added on 12 March 2021 after first online publication: Equation 11 has been corrected in this version.]

$$\beta_Z(p) = \frac{(1-p)^2}{(1-\mathrm{L}_F(p))^2} = \left(\frac{\mu}{\mu_F^+(p)}\right)^2$$

depends, instead, on both p and $L(p)$. $\beta_Z(p)$ is a decreasing function of p, with $\beta_Z(0)=+1$ and $lim_{p\to 1}\beta_Z(p)=0$. The greater the ratio $\frac{\mu_F^+ p}{\mu_F}$, the greater the penalization given to the income z_p by $\beta_Z(p)$. Comparing to what we have recalled about the Gini and Bonferroni indexes, in the social welfare evaluation based on the Zenga uniformity curve, beyond the weight function based on the ranks p, we also have $\beta_Z(p)$. This difference arises from the fact that the denominator for the Gini and the Bonferroni indexes is a constant value, whereas in the present approach the denominator is a function of p.

4. The Lorenz and Zenga Approaches: Considering Tax Effects

4.1. *General Overview*

In the following, we jointly analyze the pre-tax and post-tax income, as well as the tax distribution. First, we briefly review the standard approach based on the Lorenz curves and the Gini coefficients, and then we construct analogous curves and indices by extending the Zenga approach.

We assume that the pre-tax incomes $\tilde{x}=\{x_1,x_2,\dots,x_n\}$ are arranged in increasing order (i.e. $x_i<x_{i+1}$ for $i=1,\dots,n-1$). Let $T(x)$ be the tax paid by an individual of income x. The post-tax or disposable income of the individual will then be $y(x)=x-T(x)$. Let us consider the after-tax incomes $\tilde{y}=\{y_1,y_2,\dots,y_n\}$ and taxes $\tilde{t}=\{t_1,t_2,\dots,t_n\}$, so that each triplet (x_i,y_i,t_i) refers to the i th individual of the sample. Because, for each pair of individuals i,j such that $x_i<x_j$, it is not granted that $y_i<y_j$ and $t_i<t_j$, we denote the post-tax incomes by Y when they are ordered in a nondecreasing order, and the same incomes by Y_X when units are ranked according to the pre-tax order. Similarly, we will denote the observations related to tax amounts by T and T_X.

Based on the seminal work by Musgrave and Thin (1948), several indices have been constructed to evaluate the redistributive and re-ranking effects and the degree of progressivity of a tax system. They are mainly functions of the Gini coefficients $G(X)$, $G(Y)$, and $G(T)$ and of the corresponding concentration (or pseudo-Gini) coefficients $C(Y_X)$ and $C(T_X)$.

Therefore, it may be useful to recall the definition of the concentration curve and coefficient (Kakwani, 1977a). In Section 2, the Lorenz curve has been defined as the relationship between the proportion p of the population having income less than or equal to x, and the corresponding proportion $L_F(p)$ of owned income, $(p, L_F(p))$. Here, we need to recast the Lorenz curve in terms of the income x, by exploiting the equality $p=F(x)$. Thus, $L_F(p)=L_F(F^{-1}(x))$ is a function of x with values in [0, 1], as a genuine *cdf*, which we may denote by $F_1(x)$ for short, and therefore, the Lorenz curve can be equivalently expressed by $(F(x), F_1(x))$.

Let us now consider the variable W, observed on the same sample (or population), and let $F_1[W(x)]$ be the share of W owned by the statistical units having a value of X less than or equal to x (i.e. $F_1[W(x)]$ cumulates the values of W along the ordering given by X). The concentration curve of W is the relationship between $F(x)$ and $F_1[W(x)]$, and the concentration index $C(W)$ is equal to one minus twice the area under the concentration curve.

Now, we are ready to recall the most important indices for analyzing tax effects.

A global measure of tax progressivity assesses the deviation of a given tax system from proportionality; therefore, it is related to the local index of liability progression, that is, the elasticity of the tax liability with respect to the pre-tax income evaluated at each pre-tax income level (Jakobsson, 1976). If the tax elasticity is equal to 1 at all income levels x, the two cdf curves $F_1(x)$ and $F_1(T(x))$ coincide, as the greater the distance between them, the larger the difference of the tax elasticity from unity.

The overall degree of progressivity is generally evaluated by the Kakwani (1977b) index

$$K = C\left(T_X\right) - G(X),$$

that is twice the area between the Lorenz curve of X and the concentration curve of T_X. Therefore, K measures the departure from proportionality of the actual tax system.

Similarly, we introduce an analogous tool for measuring tax progressivity, in accordance with the Zenga approach. To this end, we need to develop the concepts of the concentration curve and index within the Zenga approach. We evaluate the concentration of the taxes and denote the generic ordinate of the tax uniformity curve by $U_{T_X}(p)$ and the corresponding index by $U\left(T_X\right)$, considering the taxes amounts $\left\{t_i\right\}$, not in their natural sequence, but when the latter are sorted by the ordering induced by the sorted incomes $\left\{x_i\right\}$.[7] Now, we are ready to introduce the new curve $KI(p)$ and the synthetic measure KI, derived by the Zenga approach:

$$KI(p) = I_{T_X}(p) - I_X(p) = U_X(p) - U_{T_X}(p)$$
$$KI = I\left(T_X\right) - I(X) = U(X) - U\left(T_X\right).$$

We see that $KI(p)$ involves differences between the ordinates of the tax uniformity curve $U_{T_X}(p)$ and the pre-tax uniformity Zenga curve $U_X(p)$. If the concentration of taxes is greater than the concentration of pre-tax incomes, the post-tax income distribution is less concentrated than the pre-tax one, and the tax is progressive. The difference between the Gini coefficients of the pre-tax X and post-tax Y income distributions assesses the overall redistributive effect, measured by the *RE* index:

[7]It holds that $U_{T_X}(p) \geq U_T(p)$ and, equivalently, $I_{T_X}(p) \leq I_T(p)$ because $\mu^-_{T_X} \geq \mu^-_T$ and $\mu^+_{T_X} \leq \mu^+_T$.

$$RE = G(X) - G(Y) = \left(G(X) - C\left(Y_X\right)\right) - \left(G(Y) - C\left(Y_X\right)\right).$$

If we compare the Gini coefficient of the pre-tax distribution X, and post-tax concentration of Y_X, considering the units sorted according to the pre-tax incomes in both cases, we arrive at the Reynolds–Smolensky index RS:

$$RS = G(X) - C\left(Y_X\right). \tag{13}$$

Therefore, the overall redistributive effect RE is usually quantified as twice the area between the Lorenz curves for pre-tax and post-tax distributions, and the Reynolds–Smolensky index RS is given by twice the area between the Lorenz curve for the pre-tax distribution and the concentration curve for the post-tax distribution (Reynolds and Smolensky, 1977; Lambert, 2001).

If the tax determines re-ranking, then $G(Y) > C\left(Y_X\right)$ and $RS > RE$. The Atkinson–Plotnick–Kakwani index (Atkinson, 1980; Plotnick, 1981; Kakwani, 1984) is a measure of the overall re-ranking in the transition from pre-tax to post-tax income distribution, defined by:

$$R\left(Y_X\right) = G(Y) - C\left(Y_X\right). \tag{14}$$

Similarly, as we did before with taxes, beyond the uniformity curve and index $U_Y(p)$ and $U(Y)$ evaluated on the sorted values $\{y_i\}$, the concentration curve $U_{Y_X}(p)$ and index $U\left(Y_X\right)$ arise when we consider the post-tax amounts $\{y_i\}$, not in their natural sequence, but when the latter are sorted by the ordering induced by sorted incomes $\{x_i\}$.

Following the Zenga approach, we introduce three new curves and synthetic indices:

$$\begin{aligned} REI(p) &= I_X(p) - I_Y(p) = U_Y(p) - U_X(p) = RSI(p) - RI(p) \\ REI &= I(X) - I(Y) = U(Y) - U(X) = RSI - RI \\ RSI(p) &= I_X(p) - I_{Y_X}(p) = U_{Y_X}(p) - U_X(p) \\ RSI &= I(X) - I\left(Y_X\right) = U\left(Y_X\right) - U(X) \\ RI(p) &= I_Y(p) - I_{Y_X}(p) = U_{Y_X}(p) - U_Y(p) \\ RI &= I(Y) - I\left(Y_X\right) = U\left(Y_X\right) - U(Y). \end{aligned} \tag{15}$$

Three important and well-known results hold for the classical measures of the degree of progressivity, the redistributive and the re-ranking effects of a tax system. First, we recall them, and then derive their analogous counterparts in the new setting.

- The Kakwani progressivity index K is related to the Reynolds–Smolensky RS index by (see, e.g. Lambert (2001)):

$$RS = K\overline{T}/\overline{Y},$$

where we switch notation to improve readability and we denote, from now on, with $\overline{T}$ and $\overline{Y}$ the averages of T and Y, respectively.[8] This means that RS is a function of two variables (i.e. the Kakwani index K and the overall average tax rate); therefore, RS can increase even if the overall average tax rate decreases, and if K more than compensates for the effect of the tax rate. Thus, an analogous relation holds between RSI and KI, based on the corresponding curves (see the Appendix A).

$$(16) \qquad RSI(p) = I_X(p) - I_{Y_X}(p) = \lambda(p)\left[I_{T_X}(p) - I_X(p)\right] = \lambda(p)\, KI(p).$$

In equation (16), $KI(p) = I_{T_X}(p) - I_X(p)$ measures the tax progressivity at percentile p, whereas the factor $\lambda\, p = \mu^+_{T_X} / \mu^+_{Y_X}$ measures the tax incidence on incomes greater than x_p; note that the term $\lambda(p)$, which multiplies $\left(I_{T_X}(p) - I_X(p)\right)$, is generally different for each p: if the tax is progressive, $\lambda(p)$ is an increasing function of X, reflecting the tax system progressivity. In analogy with equation (13), the synthetic measure based on the curve in equation (16) can be expressed as $RSI = I(X) - I\left(\mathrm{Y}_X\right)$
.

- The Atkinson–Plotnick–Kakwani takes only nonnegative values. In analogy with this property, we will show that $RI \geq 0$. By recalling that $\mu^-_{Y_X}(p) \geq \mu^-_Y(p)$ and $\mu^+_{Y_X}(p) \leq \mu^+_Y(p)$, because Y is in nondecreasing order, then

$$RI(p) = \frac{\mu^+_Y(p) - \mu^-_Y(p)}{\mu^+_Y(p)} - \frac{\mu^+_{Y_X}(p) - \mu^-_{Y_X}(p)}{\mu^+_{Y_X}(p)}$$
$$= \frac{\mu^-_{Y_X}(p)}{\mu^+_{Y_X}(p)} - \frac{\mu^-_Y(p)}{\mu^+_Y(p)} \geq 0.$$

The last inequality strictly holds if and only if the tax system induces some re-ranking effect among the poorest p percent individuals in the population; otherwise, $RI(p) = 0$. This property gives RI the role of a measure of the re-ranking effect produced by the tax.

- In analogy with the well-known inequality $RS \geq RE$, we have that $RSI(p) \geq REI(p)$ for all p, and therefore:

$$(17) \qquad RSI \geq REI.$$

To prove our claim, by conveniently decomposing $REI(p)$ as follows:

$$REI(p) = I_X(p) - I_Y(p)$$
$$= \left[I_X(p) - I_{Y_X}(p)\right] - \left[I_Y(p) - I_{Y_X}(p)\right]$$
$$= RSI(p) - RI(p),$$

[8]As we will discuss later (see Section 4.2), $\overline{T}/\overline{X} = \alpha$ and $\overline{T}/\overline{Y} = \alpha/(1-\alpha)$.

we get the thesis using (ii). Therefore, the equality in equation (17) holds if and only if the tax does not determine the re-ranking in Y.

4.2. *Comparing the Lorenz- and Zenga-Based Approaches for Tax Analysis*

To understand the different behaviors of the curves $RS(p)$ and $RSI(p)$ as well as $K(p)$ and $KI(p)$, and to emphasize their relationships, we get a step further by decomposing the corresponding equations. Similar to $\mu_Z^-(p)$ and $\mu_Z^+(p)$, we define the lower and upper average tax rates

(18)
$$\alpha^-(p) = \frac{\mu_T^-(p)}{\mu_X^-(p)}$$

(19)
$$\alpha^+(p) = \frac{\mu_T^+(p)}{\mu_X^+(p)}.$$

In addition, we denote the overall pre-tax mean income by $\overline{X}$, and the overall average tax rate by $\alpha = \overline{T}/\overline{X}$.

Thus, we can reinterpret $RS(p)$, $RSI(p)$, $K(p)$, and $KI(p)$ as follows:

(20)
$$RS(p) = \frac{\mu_X^-(p)}{\overline{X}} \frac{\alpha - \alpha^-(p)}{1-\alpha} p = L_X(p) \frac{\alpha - \alpha^-(p)}{1-\alpha}$$

(21)
$$K(p) = \frac{\mu_X^-(p)}{\overline{X}} \frac{\alpha - \alpha^-(p)}{\alpha} p = L_X(p) \frac{\alpha - \alpha^-(p)}{\alpha}$$

(22)
$$RSI(p) = \frac{\mu_X^-(p)}{\mu_X^+(p)} \frac{\alpha^+(p) - \alpha^-(p)}{1-\alpha^+(p)} = U_X(p) \frac{\alpha^+(p) - \alpha^-(p)}{1-\alpha^+(p)}$$

(23)
$$KI(p) = \frac{\mu_X^-(p)}{\mu_X^+(p)} \frac{\alpha^+(p) - \alpha^-(p)}{\alpha^+(p)} = U_X(p) \frac{\alpha^+(p) - \alpha^-(p)}{\alpha^+(p)}.$$

In the Lorenz–Gini approach, the redistributive effect $RS(p)$ in equation (20) and the degree of progressivity $K(p)$ in equation (21) depend, for each p, on three elements: the ordinates of the Lorenz curve for the pre-tax income distribution $L_X p$, the overall average tax rate α, and the average tax rate obtained by considering only the bottom p percent income units $\alpha^-(p)$.

Moreover, we notice that the ratio between $RS(p)$ and $K(p)$ is constant, because $\frac{RS(p)}{K(p)} = \frac{\alpha}{1-\alpha}$. Therefore, the two curves convey the same information; however, the same cannot be observed for the ratio between $RSI(p)$ and $KI(p)$, because

(24)
$$\frac{RSI(p)}{KI(p)} = \frac{\alpha^{+}(p)}{1-\alpha^{+}(p)}.$$

The Zenga approach offers a different tool, because in equations (22) and (23) there is no constant term; thus, both $RSI(p)$ and $KI(p)$ contribute to explaining the impact of the tax, with each of them conveying distinct information. The redistributive effect $RSI(p)$ and the degree of progressivity $KI(p)$ depend, for each p, on three elements: the ordinates of the Zenga curve for the pre-tax income distribution $U_X(p)$, the average tax rate for the bottom p percent income units $\alpha^{-}(p)$, and the average tax rate for the top $(1-p)$ percent income units $\alpha^{+}(p)$.

We have observed that the Lorenz curve in equations (20) and (21) is replaced by the uniformity curve in equations (22) and (23). In addition, the uniformity curve lies over the Lorenz curve for the lowest percentiles, whereas the Lorenz curve lies over the uniformity curve for the highest percentiles. Evidently, there exist two values $p^{*}, p^{**} \in (0,1)$ such that $U_X(p) \geq L_X(p)$ for all $p < p^{*}$ and $U_X(p) \leq L_X(p)$ for $p > p^{**}$ (the proof is given in Appendix A).

Moreover, in the Gini–Lorenz approach in equations (20) and (21), $\alpha^{-}(p)$ is compared with the overall tax rate α, whereas in the Zenga-based approach in equations (22) and (23), the comparison is made with the average tax rate for the higher percentiles $\alpha^{+}(p)$.

Furthermore, $\frac{\alpha-\alpha^{-}(p)}{1-\alpha}$ and $\frac{\alpha-\alpha^{-}(p)}{\alpha}$ decrease with p, whereas the ratios $\frac{\alpha^{+}(p)-\alpha^{-}(p)}{1-\alpha^{+}(p)}$ and $\frac{\alpha^{+}(p)-\alpha^{-}(p)}{\alpha^{+}(p)}$ generally have a more complex behavior. In terms of their interpretation, $\frac{\alpha-\alpha^{-}(p)}{\alpha}$ represents the (relative) comparison between the overall average tax rate α and the average tax rate of the poorest p percent income earners. Conversely, the ratio $\frac{\alpha^{+}(p)-\alpha^{-}(p)}{\alpha^{+}(p)}$ represents the (relative) comparison between the average tax rate of the poorest p percent income earners and the average tax rate of the upper $(1-p)$ percent part of the distribution (i.e. the richest income earners over the percentile p).

We may add a second interpretation of the same two ratios, in terms of disposable incomes. Because $\alpha-\alpha^{-}(p)=(1-\alpha^{-}(p))-(1-\alpha)$ and $\alpha^{+}(p)-\alpha^{-}(p)=(1-\alpha^{-}(p))-\left(1-\alpha^{+}(p)\right)$, $\frac{\alpha-\alpha^{-}(p)}{1-\alpha}$ is the difference between the average disposable income after taxation, up to percentile p and the average disposable income of the entire population. On the contrary, in $\frac{\alpha^{+}(p)-\alpha^{-}(p)}{1-\alpha^{+}(p)}$, the average disposable income after taxation, up to percentile p, is compared to the average disposable income of the complementary $(1-p)$ percent richest part of the population. In other words, we propose to assess the redistribution and progressivity of a tax system by comparing their effects on the poorest and richest segments of the population.

4.3. *Social Welfare Implications of Taxation*

Following the detailed analysis shown in Kakwani and Son (2020), in this section we present how social welfare gains and losses because of taxation can be measured and quantified. Basically, the cited authors propose *a social welfare function framework to derive measures of tax progressivity and explore their normative* *properties*. These measures can be absolute and relative: absolute measures evaluate the deviation of an observed tax system to a situation in which all tax units pay the same amount of taxes; relative ones measure the deviation of an observed tax system to a situation in which all tax units are affected by the same average tax rate. Moreover, by considering increasing and concave social welfare functions, and functions characterized by homogeneity of degree one, evaluated social welfare levels have a money-level interpretation.

In particular, focusing on equations (5) and (11), we denote by W_G^X and W_Z^X the overall social welfare levels when pre-tax incomes are considered, where the labels G and Z refer to the Gini and the Zenga approaches, respectively. Similarly, we call W_G^Y and W_Z^Y the overall social welfare levels when post-tax incomes are considered according to the post-tax order and $W_G^{Y|X}$ and $W_Z^{Y|X}$ the overall social welfare levels when post-tax incomes are considered according to the pre-tax order.

Then Kakwani and Son (2020) derive specific equations for the social welfare gains and losses of taxation and their decompositions. In particular, $W_G^Y - W_G^X < 0$ and $W_Z^Y - W_Z^X < 0$ measure the absolute difference in social welfare levels before and after taxation, whereas the corresponding relative measures, $\frac{W_G^Y - W_G^X}{\overline{T}} < 0$ and $\frac{W_Z^Y - W_Z^X}{\overline{T}} < 0$, are simply obtained by dividing the previous values by the average tax amount $\overline{T}$.

Three relevant relations hold for the Gini approach: $W_G^X = \overline{X}(1 - G(X))$; $W_G^Y = \overline{Y}(1 - G(Y))$; and $W_G^{Y|X} = \overline{Y}\left(1 - C\left(Y_X\right)\right)$. A same set of equations can also be derived for the Zenga methodology: $W_Z^X = \overline{X}(1 - I(X))$; $W_Z^Y = \overline{Y}(1 - I(Y))$; and $W_Z^{Y|X} = \overline{Y}\left(1 - I\left(Y_X\right)\right)$.

Having defined the overall average tax rate by $e = \frac{\overline{T}}{\overline{X}}$, Kakwani and Son (2020) decompose the absolute loss of welfare because of taxation, $W_G^Y - W_G^X$, in three parts, as follows:

$$W_G^Y - W_G^X = H_G^A - eW_G^X + \left(W_G^{Y|X} - (1-e)\, W_G^X\right). \tag{25}$$

For the Zenga approach, we similarly obtain

$$W_Z^Y - W_Z^X = H_Z^A - eW_Z^X + \left(W_Z^{Y|X} - (1-e)\, W_Z^X\right). \tag{26}$$

In particular,

- $H_G^A = W_G^Y - W_G^{Y|X} = -\overline{Y}R\left(Y_X\right) < 0$ and $H_Z^A = W_Z^Y - W_Z^{Y|X} = -\overline{Y}RI < 0$ capture the social welfare loss because of re-ranking: therefore, they are measures of absolute horizontal inequity;
- $eW_G^X = \overline{T}(1 - G(X)) > 0$ and $eW_Z^X = \overline{T}(1 - I(X)) > 0$ can be thought as the social welfare loss when adopting a proportional tax that yields the same tax revenue of the actual progressive tax, and
- $W_G^{Y|X} - (1-e)\, W_G^X = \overline{Y}RS > 0$ and $W_Z^{Y|X} - (1-e)\, W_Z^X = \overline{Y}RSI > 0$ measure the progressivity of the tax system.

The corresponding relative decompositions and measures are simply obtained by dividing each term by the average tax amount $\overline{T}$ as mentioned earlier. As a consequence, decomposing $\frac{W_G^Y - W_G^X}{\overline{T}}$ and $\frac{W_Z^Y - W_Z^X}{\overline{T}}$, we obtain

$$H_G^R = \frac{W_G^Y - W_G^{Y|X}}{\overline{T}} = -\frac{\overline{Y}}{\overline{T}} R\left(Y_X\right) \quad \text{and} \quad H_Z^R = \frac{W_Z^Y - W_Z^{Y|X}}{\overline{T}} = -\frac{\overline{Y}}{\overline{T}} RI,$$

they basically measure the average loss of social welfare because of re-ranking for one euro of tax collected by the government. Similarly, we have

$$\frac{eW_G^X}{\overline{T}} = \frac{W_G^X}{\overline{X}} = 1 - G(X) \quad \text{and} \quad \frac{eW_Z^X}{\overline{T}} = \frac{W_Z^X}{\overline{X}} = 1 - I(X),$$

and we arrive, finally, at

$$\frac{W_G^{Y|X} - (1-e)\, W_G^X}{\overline{T}} = \frac{\overline{Y}}{\overline{T}} RS \text{ and } \frac{W_Z^{Y|X} - (1-e)\, W_Z^X}{\overline{T}} = \frac{\overline{Y}}{\overline{T}} RSI.$$

5. An Application to Real Tax Data

5.1. *The Data and the Micro-Simulation Model*

To compare the Gini- and Zenga-based approaches applied to a real-world tax system, we use a static micro-simulation model of the Italian personal income tax, updated to the 2014 fiscal year. The model considers the most important taxes and contributions in the Italian fiscal system. It has been developed by Pellegrino (2007) about 10 years ago using the statistical software Stata, and it is constantly updated to incorporate changes in the tax code.

Here we focus on the module of the microsimulation model concerning the personal income tax that is an updated version of the model described in Pellegrino et al. (2011). Technical details regarding the structure and main results of this version of the micro-simulation model can be found in Pellegrino et al. (2019). The model uses, as input, income data provided by the Bank of Italy (2015) Survey on Household Income and Wealth (hereafter SHIW). This survey collected information on individual and household post-tax income and wealth in 2014, covering 8156 households and 19,366 individuals. The sample is representative of the Italian population, which is composed of about 24.7 million households and 60.8 million individuals.

The raw data contained in the original survey must be first reworked appropriately to determine the post-tax income subject to the personal income tax. Then it is possible to apply the net-to-gross procedure, following the methodology proposed in Immervoll and O'Donoghue (2001), and to consequently determine the pre-tax distribution.

Considering individual taxpayers, results concerning the gross income distribution, the distribution of all tax variables, and the overall tax revenue are very close to the official statistics of the Department of Finance, Ministry of Economy and Finance (2016). Moreover, inequality indices, both for the taxpayers and for equivalent households, are also very close to the ones evaluated by the official micro-simulation model of the Italian Department of Finance (Di Nicola et al., 2015). Therefore, the selected instrument is suitable for our empirical analysis.

Finally, to conduct our study, individual nominal incomes must be transformed into equivalent household incomes using a proper equivalence scale. We choose to adopt the equivalence scale given by the square root of the number of the household components.

5.2. *Basic Results on the Italian Personal Income Tax*

We begin our discussion by evaluating the inequality indices, presented in Section 4, using the data obtained from the micro-simulation model applied on the SHIW. We get $G(X)=0.42058$, and by going back to the definition of the Gini coefficient in equation (4), this means that, before tax, on average, the mean income of the poorest groups is equal to 57.94 percent of the overall mean. By considering income data before tax, and interpreting the definition of the Zenga inequality index in equation (6), $I(X)=0.77869$ indicate that by splitting the population into two complementary groups at each percentile and averaging over all ratios, the mean income of the poorest groups is 22.13 percent of the mean income of the richest groups.

In addition, were the tax scheme proportional, the concentration coefficient for the tax liability $C\left(T_X\right)$ and the corresponding Gini coefficient $G(T)$ would be equal to the Gini coefficient for the pre-tax income $G(X)$, and, analogously, $I\left(T_X\right)$ and $I(T)$ would be equal to $I(X)$. Because the tax is progressive, we have instead $G(T)=0.64179$ and $C\left(T_X\right)=0.63541$, and thus $K=0.21483$. Similarly, $I(T)=0.92849$ and $I\left(T_X\right)=0.92465$, so that $KI=0.14596$. They measure the departure from proportionality of the tax. Moreover, were the tax scheme proportional, the concentration coefficient for post-tax incomes $C\left(Y_X\right)$ would be equal to the Gini coefficient for the pre-tax income. Because the tax is progressive, we have that $C\left(Y_X\right)<G(X)$, namely $C\left(Y_X\right)=0.37046$ (and $I\left(Y_X\right)=0.73453<I(X)$). Furthermore, we get $RS=0.05012$ to measure the reduction in the inequality because of progressive taxation. A similar interpretation applies to $RSI=0.04416$. With reference to the re-ranking (see equations (14) and (15)), we get $R\left(Y_X\right)=0.00058$ and $RI=0.00051$.

In the first part of the paper, we emphasized that the value added of the Zenga approach can be mainly appreciated by focusing on the uniformity curve. Therefore, Figure 1 plots the Lorenz and Zenga curves for the pre-tax distribution and the concentration and the Zenga curves for the post-tax and the tax liability distributions.

Looking at $U_X(p)$, we see that the bottom 25 percent of households earn a mean gross income that is equal only to 20.0 percent of the mean gross income of the top 75 percent; similarly, the bottom 50 percent earns a mean gross income equal to 28.2 percent of the one earned, on average, by the top half, with the corresponding percentages characterizing the bottom 75 percent and the bottom 99 percent being 29.6 percent and 11.6 percent, respectively.

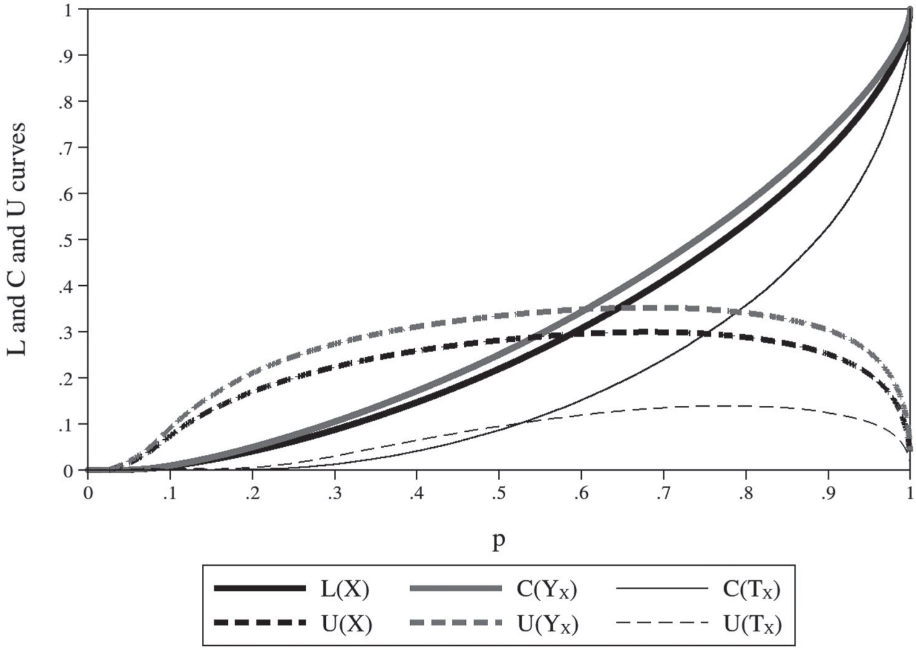

Figure 1. LorenzL,Zenga *U* and concentration Curves Cfor *X*, *Y*, and *T* Distributions

This informative set complements the one derived by the Lorenz curve for pre-tax incomes, stating that the bottom 25 percent of households gets only 6 percent of the overall pre-tax incomes, the bottom 50 percent gets 22 percent, and so on. A similar interpretation is given by considering $U_{T_X}(p)$ and $U_{Y_X}(p)$. The tax liability distribution is more concentrated than the pre-tax income distribution; thus, $U_{T_X}(p)$ lies below the curve $U_X(p)$: the mean value of the tax liability paid by the bottom 25 percent of households is 1.5 percent of the mean value paid by the remaining 75 percent of households; similarly, the mean value of the tax liability paid by the bottom 50 percent of households is only 9.5 percent of the mean value paid by the remaining 50 percent of households, with the corresponding percentages of the bottom 75 percent and 99 percent being 13.9 percent and 5.7 percent, respectively. Conversely, the curve $U_{Y_X}(p)$ lies above the curve $U_X(p)$, because the relative position of households gets compressed in the transition from pre-tax to post-tax incomes.

The usefulness of the Zenga approach is apparent when studying the effect of progressive taxation in greater detail. The graphical representation of the Lorenz and concentration curves for tax liability (here omitted) does not add information: the two curves are indistinguishable, as they lie approximately one above the other. In contrast, Figure 2 plots the corresponding curves $U_T(p)$ and $U_{T_X}(p)$ derived by the Zenga approach, and the difference between the two Zenga curves is apparent when the ordering differs.

The re-ranking of taxes starts from very low values of the percentile (in the abscissa, about 0.05). It increases until 0.20, becomes constant up to 0.80, and then

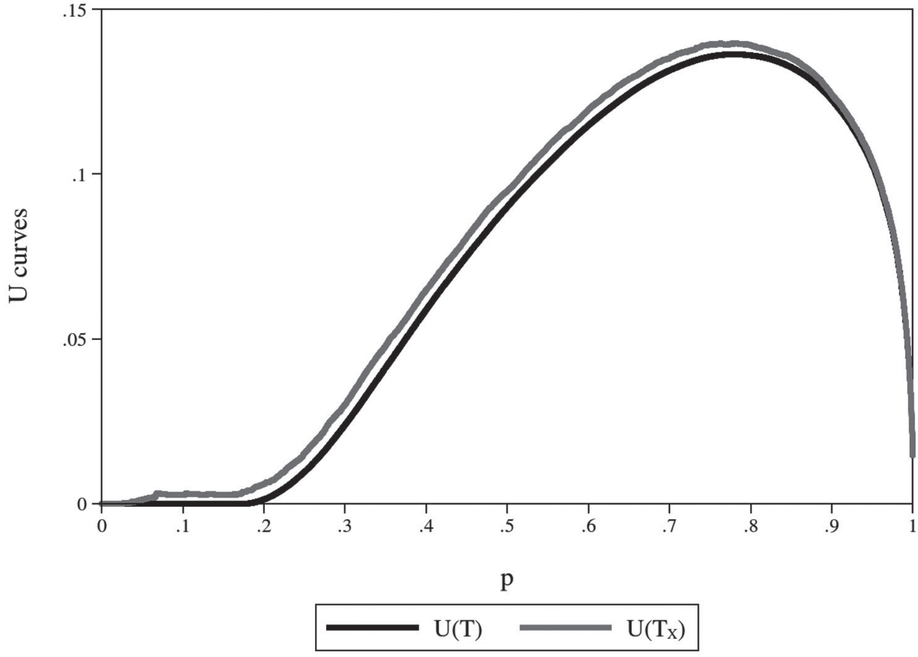

Figure 2. The Zenga uniformity Curves for$T$

starts decreasing. Therefore, we can appreciate the effectiveness of adopting the Zenga approach to study the effects of the progressive taxation.

By construction, the Zenga-based approach allows the observation of such tiny differences, because $U_T(p)$ and $U_{T_X}(p)$ span from 0 to 0.15, compared to the corresponding Gini-based approach, where the Lorenz and concentration curves always span from 0 to 1.

Finally, Figures 3 and 4 plot the *RS* and *K* effects, comparing the two approaches. The standard Lorenz-based analysis, in both cases, generates curves (see their definitions in equations (19–22)) that monotonically increase up to the 85th percentile, and then decrease. In other words, the share of post-tax income accruing to each poorest portion of population, up to the 8th decile, is greater than the corresponding share of pre-tax incomes, because of progressive taxation.

The Zenga approach provides us with different information. The $RSI(p)$ effect, shown in Figure 3, increases up to the 40th percentile, then more or less becomes constant up to the 90th percentile. In particular, the curve $RSI(p)$ increases more sharply than the curve $RS(p)$ up to the 30th percentile, showing that the poor benefit the most from progressive taxation; however, as the Italian tax system does not consider a negative income taxation, the absolute benefit depends on the available pre-tax income. Finally, starting from the 90th percentile, the $RSI(p)$ effect sharply decreases, because the top 10 percent of households pay a very high share of the overall tax revenue when compared to the tax revenue provided by the bottom 90 percent of the population, with this share increasing with income.

Moreover, $U_X(p)$ and $U_{Y_X}(p)$ increase up to the 68th percentile, with these increases being remarkably high up to the 40th percentile, underlining that, up

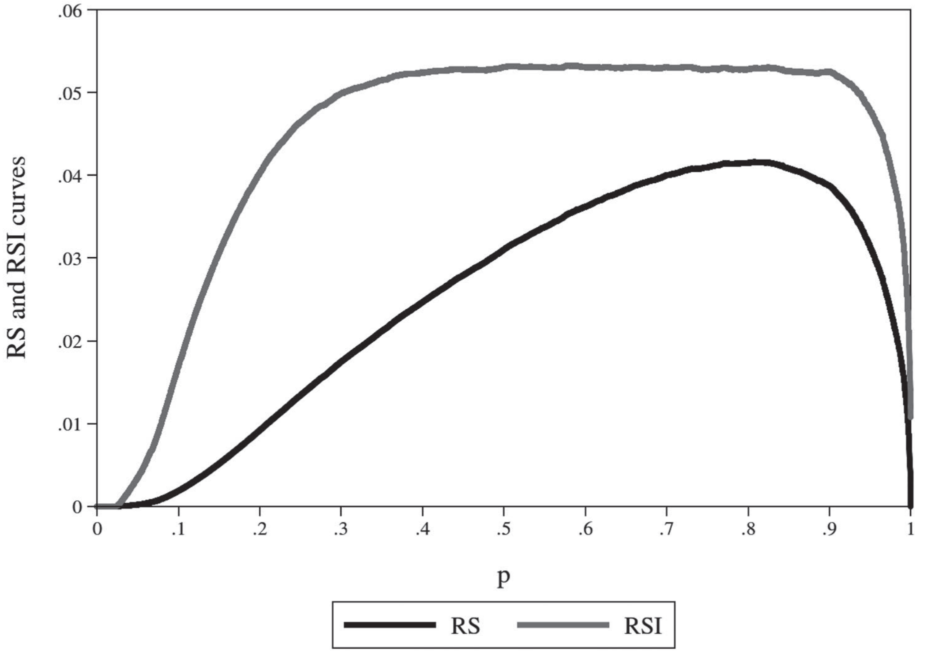

Figure 3. *RS* According to the Lorenz and Zenga Approaches

to the 4th decile, the mean pre-tax and post-tax income of the poorest part of the distribution increase faster than the mean pre-tax and post-tax income of the richest part of the distribution. From the 40th percentile to the 70th percentile, this relationship remains, but loses its power; then from the 70th percentile, it is reversed, particularly from the 90th percentile, as observed before. On the contrary, the distance between $U_X(p)$ and $U_{T_X}(p)$ decreases from the 30th percentile onward, before what is observed for the curve $RSI(p)$ (Figure 4). Evidently, by comparing the share of gross income and taxes accruing to each decile, it can be observed that their distance is increasing up to the 30th percentile, then it decreases, being even negative in the top part of the income distribution.

Given these observations, $U_X(p)$, $RSI(p)$, and $KI(p)$ place an emphasis on the part of the income distribution that benefits the most and the part that is harmed the most by progressive taxation (with respect to a proportional tax yielding the same revenue).

In addition, the shape of the curve $RSI(p)$ seems to show a pattern related to the shape of the average tax rate in each percentile, with the latter being zero up to the 18th percentile, and then it sharply increases up to the 30th percentile. Afterward, it continues increasing with less intensity up to the 90th percentile. Finally, in the top part of the income distribution, the average tax rate is very high.

To understand this relation, note that, according to 19, the curves $RS(p)$ and $K(p)$ are related by the fixed coefficient $\frac{\alpha}{1-\alpha}$. However, according to 23, the curves $RSI(p)$ and $KI(p)$ are related by the factor $\frac{\alpha^{+}(p)}{1-\alpha^{+}(p)}$, which depends on the percentile, and allows to determine RSI from KI.

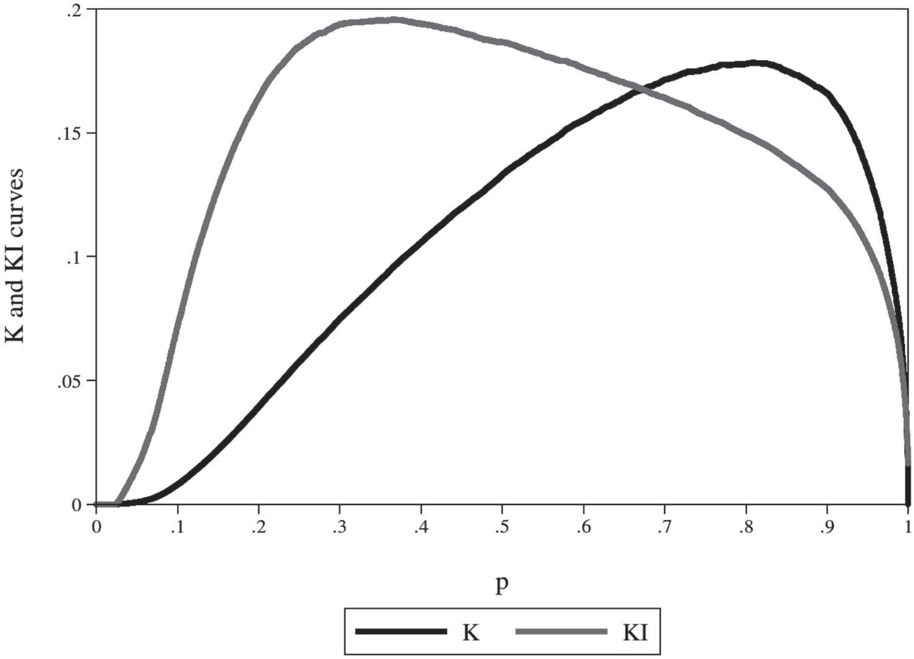

Figure 4. *K* According to the Lorenz and Zenga Approaches

Focusing on this remark, Figure 4 compares the two Kakwani effects. As mentioned before, by adopting the Lorenz approach, the progressivity effect increases up to the 90th percentile, whereas according to the Zenga curve, it increases only up to the 30th percentile.

The shape of the curve $KI(p)$ is very intuitive, as in the bottom part of the income distribution, $U_X(p)$ is strictly increasing, whereas about 20 percent of the households pay no taxes, so that the curve $U_{T_X}(p)$ is zero. Therefore, their difference, that is $KI(p)$, strictly increases once all households with zero tax liability have been considered. Starting from the 40th percentile onward, $U_{T_X}(p)$ increases at a higher rate than $U_X(p)$, so that the curve $KI(p)$ decreases.

Following the discussion presented in Section 4.2, we now observe the behavior of the most important components of $RS(p)$ and $RSI(p)$, in particular, in the range $0.4 < p < 0.9$.

Recalling equations (22) and (23), Figure 5 shows all the components that participate in the definition of the RSI and KI curves. The uniformity curve $U_X(p)$ (dashed black line) and $\frac{\alpha^+(p)-\alpha^-(p)}{1-\alpha^+(p)}$ (solid black line) contribute to define $RSI(p)$, whereas the uniformity curve $U_X(p)$ and $\frac{\alpha^+(p)-\alpha^-(p)}{\alpha^+(p)}$ (solid gray line) contribute to define $KI(p)$. Recalling equations (16) and (24), we get $\lambda_i = \frac{\alpha^+(p)}{1-\alpha^+(p)}$ (dashed gray line).

For $0.3 < p < 0.9$, the dashed and solid black lines of Figure 5 are almost symmetric, and the product of their ordinates is almost constant, explaining the constant gray line ($RSI(p)$) of Figure 3. Conversely, for $0 < p < 0.3$, the product of the

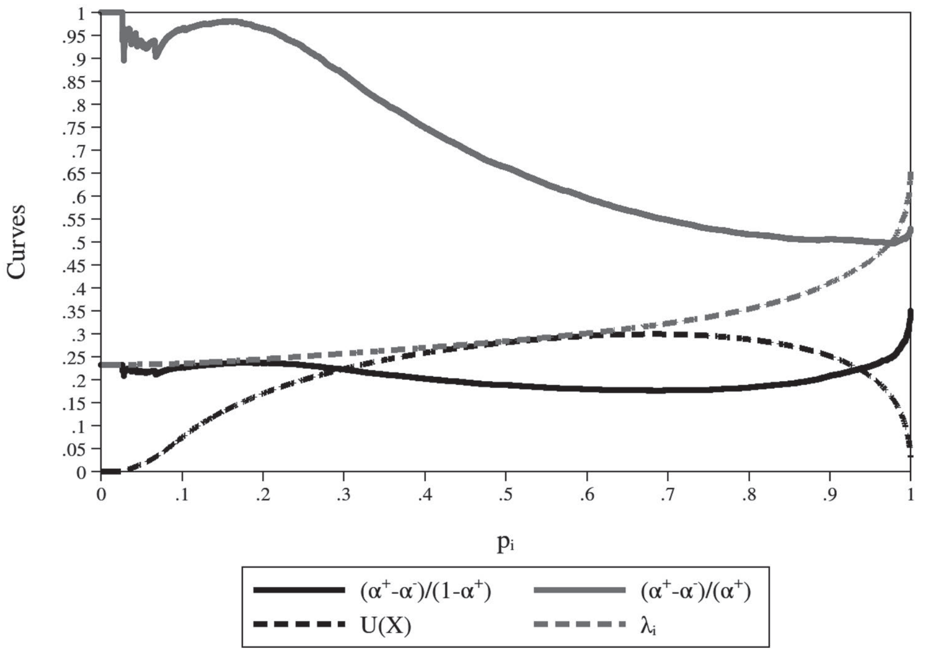

Figure 5. The Contributions to *RSI*

ordinates of the solid and dashed black lines is increasing, whereas the opposite happens for $p > 0.9$.

The shapes (here omitted) of the curves $\alpha^+(p) - \alpha^-(p)$ and $\frac{\alpha^+(p)-\alpha^-(p)}{1-\alpha^+(p)}$ are very close to each other, with $\alpha^-(p)$ growing approximately at a constant rate in the range $0.4 < p < 0.9$. In other words, the curve $RSI(p)$ shows that the progressivity of the tax is not expansionist for this portion of population. Detailed information on the tax progressivity can be observed analyzing the $KI(p)$ curve (gray line in Figure 4). Its ordinates are given by the product of the ordinates of the dashed black line and the solid gray line in Figure 5), which first increases (up to the 30th percentile) and then decreases.

5.3. *Social Welfare Implications of Taxation*

Having described results according to the usual statistical framework, we discuss here the social welfare implications of both Gini and Zenga approaches (see Section 4.3).

Focusing on equation (5), the overall social welfare when pre-tax incomes are considered is $W_G^X = 12,477.86$. The overall social welfare for post-tax incomes is lower, because here we focus our attention only on the (personal income) taxation side, without considering the welfare gains because of public expenditures. In particular, the overall social welfare when post-tax incomes are considered, according to the posttax order, is $W_G^Y = 10,982.50$; the overall social welfare when post-tax incomes are considered, according to the pre-tax order, is $W_G^{Y|X} = 10,992.63$. In the transition from the pre-tax to the post-tax incomes, a social loss of about 11.9 percent is registered: $W_G^Y - W_G^X = -1,495.36$.

Decomposing this overall value can give more details on the welfare loss because of taxation (equation (25)):

$$W_G^Y - W_G^X = -eW_G^X + \left(W_G^{Y|X} - (1-e)\, W_G^X \right) + H_G^A.$$

We get:[9]

$$-1,495.36 = -\,(2,360.38) + (875.16) + (-10.13)\,.$$

This means that, according to the Gini social welfare function, were all income units are asked to pay the same amount of taxes yielding the same tax revenue observed in the actual tax system, the per capita social welfare loss would be $-2,360 = -eW_G^X$ euro. However, the tax system is progressive, so that the tax progressivity of the Italian personal income tax, measured by $W_G^{Y|X} - (1-e)\, W_G^X$, determines a per capita increase in social welfare by about 875 euro; this increase only partially overcomes the negative effect because of $-eW_G^X$. The application of the tax also determines inefficiencies; in particular, in the transition from the pre-tax to the post-tax, income re-ranking occurs: this unpleasant outcome is measured by the welfare loss because of horizontal inequity H_G^A, which is very small, about 10 euro per capita.

A similar picture emerges whenever the Zenga approach is considered. Focusing on equation (11), the overall social welfare when pre-tax incomes are considered is $W_Z^X = 4,765.66$, clearly lower than W_G^X. The welfare level according to the Zenga approach is lower than the one evaluated according to the Gini methodology in our empirical exercise: in the latter the weights $2(1-p)$ are linearly decreasing from 2 to 0 when p increases, whereas in the former the values of the weights $w_Z(p) = \frac{-lnp + p - 1}{(1 - L_F(p))^2}$ are higher than $w_G(p) = 2(1-p)$ for $p < 0.059$, and they are always lower elsewhere.

Similarly, the overall social welfare for post-tax incomes, according to the post-tax order, is lower than both W_Z^X and W_G^X, and it is equal to 4, 626.34. Finally, the overall social welfare when post-tax incomes are considered, according to the pre-tax order, is instead $W_Z^{Y|X} = 4,635.18$.

In the transition from the pre-tax to the post-tax incomes, an overall social loss (see equation (26)) of about 2.9 percent is registered: $W_Z^Y - W_Z^X = -139.32 = H_Z^A - eW_Z^X + \left(W_Z^{Y|X} - (1-e)\, W_Z^X \right)$. Using the latter decomposition, $-eW_Z^X = -901.50$, $W_Z^{Y|X} - (1-e)\, W_Z^X = 771.02$, and $H_Z^A = -8.85$. According to the Zenga social welfare function, the tax progressivity determines a per capita increase in social welfare by about 771 euro; like in the Gini approach, this increase only partially overcomes the other negative effects. The loss of social welfare due to the application of a proportional tax yielding the same tax revenue is equal to 901 euro per capita. The welfare loss induced by the horizontal inequity is small also in this case, about 9 euro per capita.

[9]By letting $-1,495.36$ be equal to 100, the first term is about 0.68, the second one is -157.85, and the third one is -58.53.

We do not discuss the results according to the relative decomposition: as mentioned earlier, the measures of such decomposition are simply obtained by dividing each term of the absolute decomposition by the average tax amount $\overline{T}=4,073.68$.

6. Concluding Remarks

In this paper, we have applied a recently proposed approach to measure inequality, namely the Zenga index and curve, to study the degree of progressivity and the redistributive and re-ranking effects of taxation. The novel approach to inequality proposed by Zenga (2007) is based on contrasting the economic conditions of opposite and exhaustive parts of the population.

After analyzing how the Zenga uniformity curve reacts with respect to variations in the Lorenz curve, we have derived the first original result of our research: the elasticity of the Zenga uniformity curve exceeds the elasticity of the Lorenz curve and increases with the population percentile.

Therefore, following the existing tax literature, we have replicated the most important curves and the corresponding tax indices, namely the Reynolds–Smolensky, the Kakwani, and the Atkinson–Plotnick–Kakwani indices, adopting the new paradigm. Our proposal is motivated by the fact that valid information for policy makers is obtained by comparing the degree of progressivity and the redistributive and the re-ranking effects of a personal income tax system observed in the poorest part of the population, with the corresponding degree and effects occurring in the richest part. Furthermore, along the lines of the well-known properties of the three aforementioned indices, we have derived the corresponding properties that hold true for the newly introduced measures, and we have shown how social welfare gains and losses because of taxation can be assessed in the new approach.

We then discussed the strengths and weaknesses of our approach, by comparing the information conveyed by the classical indices and the new ones through applying it to the Italian tax system, based on a micro-simulation model of the personal income tax, updated to the 2014 fiscal year. We have used, as input data for the model, the Bank of Italy (2015) Survey on Household Income and Wealth. We have shown that the new curves provide an insight into information that could be hidden (or at least diminished) in the cumulative approach intrinsic to the Lorenz curve. In light of the obtained results, the analysis of the effects of a tax system has been enriched by the considerations derived from the new approach.

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Supporting Information

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APPENDIX A: TECHNICAL PROOFS

APPENDIX B: EMPIRICAL ESTIMATES OF THE GINI AND ZENGA INDICES