the review of income and wealth

Review of Income and Wealth
Series 65, Number. 4 December 2019
DOI: 10.1111/roiw.12370

AN AXIOMATIC FOUNDATION OF THE MULTIPLICATIVE HUMAN DEVELOPMENT INDEX

BY YOKO KAWADA and YUTA NAKAMURA*

Graduate School of Economics, Keio University

AND

SHUHEI OTANI

Department of Economics, University of Wisconsin–Madison

Abstract

The aggregation formula in the Human Development Index (HDI) was changed to a geometric mean in 2010. In this paper, we search for a theoretical justification for employing this new HDI formula. First, we find a maximal class of index functions, what we call *quasi-geometric means*, that satisfy *symmetry for the characteristics*, *normalization*, and *separability*. Second, we show that power means are the only quasi-geometric means satisfying *homogeneity*. Finally, the new HDI is the only power mean satisfying *minimal lower boundedness*, which is a local complementability axiom proposed by Herrero *et al*. (2010).

JEL Codes: D63, I32

Keywords: aggregation theory, geometric mean, Human Development Index, quasi-geometric mean, power mean

1. Introduction

Building on Sen's idea of capabilities (Sen, 1985), the Human Development Index (HDI) measures well-being in a society by aggregating the degrees of achievements in three characteristics: health, education, and income. However, over the two decades since its introduction, it has been pointed out that the aggregation formula has a serious drawback: the three characteristics are treated as completely substitutable (e.g., Desai, 1991; Sagar and Najam, 1998; Herrero *et al*., 2010). For example, no matter how bad the state of health is, it can be compensated by further education or additional income. Since achievements in each of the different characteristics contribute to different functionings, their measurements are not in fact completely substitutable. To limit the possibility of such substitutability, Herrero *et al*. (2010)

Note: The authors would thank Toyotaka Sakai for his supervision and Kazuhiko Hashimoto, Takumi Kongo, Toshifumi Kuroda, Maki Michinaka, Shuhei Morimoto, Noriaki Okamoto, Tadashi Sekiguchi, Shuhei Shiozawa, participants at 2016 Graduate Summer Workshop on Game Theory held at Seoul National University in August 2016, and two anonymous referees for their helpful comments. They would especially like to thank Tsuyoshi Adachi and Kohei Kamaga for their insightful comments on Theorem 1 in the previous version of the present paper. Y. Nakamura's research is financially supported by JSPS KAKENHI, Grant-in-Aid for JSPS Fellows (16J00342).

*Correspondence to: Yuta Nakamura, Graduate School of Economics, Keio University, Tokyo 108-8345, Japan (yuta.nakamura0223@gmail.com).

defined *minimal lower boundedness* and explored index functions that satisfy this property and other standard axioms: *symmetry for the characteristics*, *normalization*, and *separability*. In their main result, they claimed that a class of multiplicative index functions can be characterized by those axioms.

In 2010, the United Nations Development Programme revised the aggregation formula for HDI by replacing "arithmetic mean" with "geometric mean" in its definition. This new HDI belongs to the class of multiplicative index functions by Herrero, Martínez, and Villar (hereinafter, Herrero *et al.*). Therefore, their result seems to provide a rationale for the revision. In fact, Zambrano (2014) used their result as a key ingredient in providing a rationale for the revision.

Nevertheless, we show that Herrero *et al.*'s claim does not hold. We provide examples of non-multiplicative index functions satisfying all their axioms. This means, in particular, that the rationale provided by Zambrano (2014) for the new HDI needs to be fixed.

The purpose of this paper is to find index functions that treat characteristics as non-substitutable, following Herrero *et al.* However, we focus on the case where achievements have already been aggregated across individuals; so that, as in practice, an index function only aggregates them across characteristics. We thus abstract away from considerations regarding the distribution of human development among individuals (e.g., Foster *et al.*, 2005; Seth, 2013).

In this setting, we introduce a class of index functions, which we call *quasi-geometric means*. A quasi-geometric mean has a common function for all characteristics, based on which the index function takes the inverse of the geometric mean across characteristics. We first show that quasi-geometric means are the only index functions satisfying *symmetry for the characteristics*, *normalization*, and *separability*. Second, we prove that power means are the only quasi-geometric means satisfying *homogeneity*. Finally, it is shown that the new HDI (geometric mean) is the only power mean satisfying *minimal lower boundedness*, while the old HDI (arithmetic mean) is the only one satisfying *local substitutability*. Therefore, they can be interpreted as opposite extremes in the class of power means in terms of complementability and substitutability. This contrast provides a theoretical justification for the use of the new HDI.

The rest of the present study is organized as follows. Section 22 presents the model of Herrero *et al.* Section 33 introduces their main result and provides counterexamples. In Section 44, we provide characterizations, the proofs of which are relegated to Appendix A (in the Online Supporting Information). Section 55 concludes the discussion. Appendix B provides examples showing the tightness of the axioms in our theorems, and in Appendix C we present Zambrano's model and a rationale for the new HDI in his setting, which builds on our Theorem 1 below.

2. The Model

A *society* consists of a finite number of *individuals* $N \equiv \{1,2,\dots,n\}$ $(n \geq 1)$. Let $K \equiv \{1,2,\dots,k\}$ $(k \geq 1)$ be a finite set of *characteristics*.

In thecase of the new HDI, the characteristics are health, education, and income.[1]

For each $i \in N$ and each $j \in K$, a *measurement of i's achievement for j* is a value $y_{ij} \in [0,1]$. Note that the y_{ij}'s are normalized so that they are comparable independently of the units in which they are originally measured.[2] A *measurement vector for* $j \in K$ is a vector

$$\boldsymbol{y}_j \equiv \begin{pmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{nj} \end{pmatrix} \in [0,1]^n.$$

Then a *social state* is a matrix

$$Y \equiv (\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_k) = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1k} \\ \vdots & \vdots & \cdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nk} \end{bmatrix} \in \Omega \equiv [0,1]^{nk}.$$

An *index function* is a continuous function $I: \Omega \to \mathbb{R}$ that assigns each social state $Y \in \Omega$ to an *index* $I(Y) \in \mathbb{R}$. The higher the index, the better is the social state.

3. Counterexamples to the Theorem of Herrero *et al.* (2010)

Herrero *et al.* (2010) claim that a class of multiplicative index functions can be characterized by the following five axioms. *Monotonicity* requires that in any social state, if all the measurements increase, then its index also increases.

Monotonicity. For each $X, Y \in \Omega$, if $X \gg Y$, then $I(X) > I(Y)$.[3]

For each $Y \in \Omega$ and each permutation π on K, let $\pi(Y)$ be the social state that is obtained by arranging Y's columns according to π. *Symmetry for the characteristics* requires that an index function be independent of the labels of the characteristics.

Symmetry for the characteristics. For each $Y \in \Omega$ and each π, $I(\pi(Y)) = I(Y)$.

For convenience, define

$$\boldsymbol{1}_n \equiv \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{n \text{ rows}} \quad \text{and} \quad \boldsymbol{0}_n \equiv \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{n \text{ rows}}.$$

[1]In practice, achievements in health, education, and income are measured by life expectancy at birth, mean years and expected years of schooling, and gross national income per capita, respectively.

[2]It is worth noting that careful thought has to be given to normalizing data to values on the unit interval because the choice of normalization method affects the ordering over the values (Zambrano, 2014).

[3]For each $X, Y \in \Omega$, $X \gg Y$ means that $x_{ij} > y_{ij}$ for all $i \in N$ and all $j \in K$.

In addition, define

$$\boldsymbol{1} \equiv [\underbrace{\boldsymbol{1}_n, \boldsymbol{1}_n, \ldots, \boldsymbol{1}_n}_{k \text{ columns}}] \quad \text{and} \quad \boldsymbol{0} \equiv [\underbrace{\boldsymbol{0}_n, \boldsymbol{0}_n, \ldots, \boldsymbol{0}_n}_{k \text{ columns}}].$$

Normalization requires that in any social state, if all the measurements take the same value, then its index also takes the value.

Normalization. For each $\alpha \in [0,1]$, $I(\alpha \cdot \boldsymbol{1}) = \alpha$.

For each $Y \in \Omega$ and each $j \in K$, let

$$Y_{-j} \equiv (\boldsymbol{y}_i)_{i \in K \setminus \{j\}} \in [0,1]^{n(k-1)}.$$

Minimal lower boundedness requires that in any social state, if there exists a characteristic for which the measurement vector is at the lowest level, then its index is not more than that of any other social state.

Minimal lower boundedness. For each $X, Y \in \Omega$ and each $j \in K$, $I(X) \geq I(Y_{-j}, \boldsymbol{0}_n(j))$.

Consider two arbitrary social states with a common measurement vector for some characteristic. *Separability* requires that if this common measurement vector is replaced with another one, then an index function preserves the order between the two social states.

Separability. For each $X, Y \in \Omega$ with $X, Y \gg \boldsymbol{0}$ and each $j \in K$,

$$I(X_{-j}, \boldsymbol{x}_j) \geq I(Y_{-j}, \boldsymbol{x}_j) \Rightarrow I(X_{-j}, \boldsymbol{y}_j) \geq I(Y_{-j}, \boldsymbol{y}_j).$$

Consider any index function $I: \Omega \rightarrow \mathbb{R}$ satisfying the aforementioned five axioms. Herrero *et al.* (2010) define the *egalitarian equivalent value function* $\xi_j: \Omega \rightarrow \mathbb{R}$ *for each* $j \in K$ implicitly by such an I. That is, for each $Y \in \Omega$,

$$I(Y) = I\left(Y_{-j}, \xi_j(Y_{-j}, \boldsymbol{y}_j) \cdot \boldsymbol{1}_n\right).$$

For each $j \in K$, if $\xi_j: \Omega \rightarrow \mathbb{R}$ is independent of Y_{-j}, that is,

$$\forall \boldsymbol{y}_j \in [0,1]^n, \forall X_{-j}, Y_{-j} \in [0,1]^{n(k-1)}, \xi_j(X_{-j}, \boldsymbol{y}_j) = \xi_j(Y_{-j}, \boldsymbol{y}_j),$$

then $\xi_j(Y_{-j}, \boldsymbol{y}_j)$ is simply denoted by $\xi_j(\boldsymbol{y}_j)$. In addition, if all the egalitarian equivalent value functions are independent of their characteristic, that is,

$$\forall j, \ell \in K, \xi_j = \xi_\ell,$$

then the *common egalitarian equivalent value function* ξ_{j} is simply denoted by $\xi: [0,1]^n \rightarrow \mathbb{R}$.

Herrero *et al.* (2010, Theorem) claim that an index function satisfies the set of the five axioms if and only if it takes the multiplicative form of a common egalitarian equivalent value function, which contains the new HDI.

*Claim 1. (*Herrero et al., 2010, Theorem). For each index function $I:\Omega \to \mathbb{R}$, the following statements (1) and (2) are equivalent:

1. $I:\Omega \to \mathbb{R}$ satisfies *monotonicity*, *symmetry for the characteristics*, *normalization*, *minimal lower boundedness*, and *separability*;
2. there exists $\xi:[0,1]^n \to \mathbb{R}$ such that, for each $Y \in \Omega$,

$$I(Y) = \prod_{j \in K} \xi(\boldsymbol{y}_j)^{\frac{1}{k}}.$$

We provide two counterexamples to this claim which show that equation 1 does not imply equation 2 for any $k \geq 2$. We assume n = 1 for simplicity, but the ideas of these counterexamples work for any $n \geq 2$.

Example 1. ($k = 2$). Let $\hat{I}:[0,1]^2 \to \mathbb{R}$ be an index function such that, for each $(y_1, y_2) \in [0,1]^2$,

$$\hat{I}(y_1, y_2) \equiv \frac{1}{2} y_1^{\frac{2}{3}} y_2^{\frac{1}{3}} + \frac{1}{2} y_1^{\frac{1}{3}} y_2^{\frac{2}{3}}.$$

Then $\hat{I}$ satisfies equation (1) but violates equation (2).

Proof **Step 1***:* **$\hat{I}$ satisfies (1).** One can easily show that $\hat{I}$ satisfies monotonicity, symmetry for the characteristics, normalization, and minimal lower boundedness. To show separability, take any $(x_1, x_2), (y_1, y_2) \gg \boldsymbol{0}$. We need to prove that

$$\hat{I}(x_1, x_2) \geq \hat{I}(y_1, x_2) \Rightarrow \hat{I}(x_1, y_2) \geq \hat{I}(y_1, y_2); \text{ and}$$

$$\hat{I}(x_1, x_2) \geq \hat{I}(x_1, y_2) \Rightarrow \hat{I}(y_1, x_2) \geq \hat{I}(y_1, y_2).$$

We only offer a proof for the first equation, since the second can be proven in a similar way. Suppose that $\hat{I}(x_1, x_2) \geq \hat{I}(y_1, x_2)$. Then

$$0 \leq \hat{I}(x_1, x_2) - \hat{I}(y_1, x_2)$$

$$= \left[\frac{1}{2} x_1^{\frac{2}{3}} x_2^{\frac{1}{3}} + \frac{1}{2} x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}\right] - \left[\frac{1}{2} y_1^{\frac{2}{3}} x_2^{\frac{1}{3}} + \frac{1}{2} y_1^{\frac{1}{3}} x_2^{\frac{2}{3}}\right]$$

$$=\frac{1}{2}x_2^{\frac{1}{3}}\left[\left(x_1^{\frac{2}{3}}-y_1^{\frac{2}{3}}\right)+\left(x_1^{\frac{1}{3}}-y_1^{\frac{1}{3}}\right)x_2^{\frac{1}{3}}\right]$$

$$=\frac{1}{2}x_2^{\frac{1}{3}}\left[\left(x_1^{\frac{1}{3}}+y_1^{\frac{1}{3}}\right)\left(x_1^{\frac{1}{3}}-y_1^{\frac{1}{3}}\right)+\left(x_1^{\frac{1}{3}}-y_1^{\frac{1}{3}}\right)x_2^{\frac{1}{3}}\right]$$

$$=\frac{1}{2}x_2^{\frac{1}{3}}\left(x_1^{\frac{1}{3}}+y_1^{\frac{1}{3}}+x_2^{\frac{1}{3}}\right)\left(x_1^{\frac{1}{3}}-y_1^{\frac{1}{3}}\right).$$

Hence, by $x_1,x_2,y_1>0$, $x_1^{\frac{1}{3}}\geq y_1^{\frac{1}{3}}$. Therefore,

$$\hat{I}(x_1,y_2)-\hat{I}(y_1,y_2)=\frac{1}{2}y_2^{\frac{1}{3}}\left(x_1^{\frac{1}{3}}+y_1^{\frac{1}{3}}+y_2^{\frac{1}{3}}\right)\left(x_1^{\frac{1}{3}}-y_1^{\frac{1}{3}}\right)\geq 0,$$

which means that $\hat{I}(x_1,y_2)\geq\hat{I}(y_1,y_2)$.

Step 2: ***Î*** **violates (2).** Suppose, by contradiction, that there exists $\xi:[0,1]\rightarrow\mathbb{R}$ such that, for each $(y_1,y_2)\in[0,1]^2$,

$$\hat{I}(y_1,y_2)=\xi(y_1)^{\frac{1}{2}}\xi(y_2)^{\frac{1}{2}}. \tag{1}$$

Then, for each $\alpha\in[0,1]$,

$$\alpha=\hat{I}(\alpha,\alpha)=\xi(\alpha)^{\frac{1}{2}}\xi(\alpha)^{\frac{1}{2}}=\xi(\alpha). \tag{2}$$

By computation,

$$\hat{I}\left(1,\frac{1}{64}\right)=\frac{1}{2}\cdot 1^{\frac{2}{3}}\cdot\left(\frac{1}{64}\right)^{\frac{1}{3}}+\frac{1}{2}\cdot 1^{\frac{1}{3}}\cdot\left(\frac{1}{64}\right)^{\frac{2}{3}}$$

$$=\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{16}$$

$$=\frac{5}{32},$$

and thus, by equation 1,

$$\xi(1)^{\frac{1}{2}}\xi\left(\frac{1}{64}\right)^{\frac{1}{2}}=\frac{5}{32}.$$

However, by equation 2,

$$\xi(1)^{\frac{1}{2}}\xi\left(\frac{1}{64}\right)^{\frac{1}{2}}=1^{\frac{1}{2}}\cdot\left(\frac{1}{64}\right)^{\frac{1}{2}}=\frac{1}{8},$$

a contradiction.

A natural extension of $\hat{I}$ to the case $k=3$ is that, for each $(y_1,y_2,y_3)\in[0,1]^3$,

$$\hat{I}(y_1,y_2,y_3)\equiv\frac{1}{3}y_1^{\frac{2}{4}}y_2^{\frac{1}{4}}y_3^{\frac{1}{4}}+\frac{1}{3}y_1^{\frac{1}{4}}y_2^{\frac{2}{4}}y_3^{\frac{1}{4}}+\frac{1}{3}y_1^{\frac{1}{4}}y_2^{\frac{1}{4}}y_3^{\frac{2}{4}}.$$

However, this violates *separability*, since

$$\hat{I}(1,0.082,0.01)\fallingdotseq 0.1044>0.1037\fallingdotseq\hat{I}(0.3,0.3,0.01),$$

$$\hat{I}(1,0.082,1)\fallingdotseq 0.4522<0.4528\fallingdotseq\hat{I}(0.3,0.3,1),$$

which means that the counterexample works only for k = 2. Therefore, we provide another counterexample for k = 3, which can be modified to be a counterexample to Theorem 1 of Zambrano (2014). This counterexample works for any $k\geq 2$ with straightforward generalization.

Example 2. ($k=3$). Let $\tilde{I}:[0,1]^3\rightarrow[0,1]$ be an index function such that, for each $(y_1,y_2,y_3)\in[0,1]^3$,

$$\tilde{I}(y_1,y_2,y_3)\equiv\log\left[(e^{y_1}-1)^{\frac{1}{3}}(e^{y_2}-1)^{\frac{1}{3}}(e^{y_3}-1)^{\frac{1}{3}}+1\right].$$

Then $\tilde{I}$ satisfies equation 1 but violates equation 2.

Proof Step 1: $\tilde{I}$ satisfies (1). *One can easily show that $\tilde{I}$ satisfies monotonicity, symmetry for the characteristics, normalization, and minimal lower boundedness. To show separability, take any* $(x_1,x_2,x_3),(y_1,y_2,y_3)\gg\boldsymbol{0}$. We need to prove that

$$\hat{I}(x_1,x_2,x_3)\geq\hat{I}(y_1,y_2,x_3)\Rightarrow\hat{I}(x_1,x_2,y_3)\geq\hat{I}(y_1,y_2,y_3);$$

$$\hat{I}(x_1,x_2,x_3)\geq\hat{I}(y_1,x_2,y_3)\Rightarrow\hat{I}(x_1,y_2,x_3)\geq\hat{I}(y_1,y_2,y_3);\text{ and}$$

$$\hat{I}(x_1,x_2,x_3)\geq\hat{I}(x_1,y_2,y_3)\Rightarrow\hat{I}(y_1,x_2,x_3)\geq\hat{I}(y_1,y_2,y_3).$$

We only offer a proof for the first equation, since the second and third can be proven in a similar way. Suppose that $\tilde{I}(x_1,x_2,x_3)\geq\tilde{I}(y_1,y_2,x_3)$. Then

$$\log\left[(e^{x_1}-1)^{\frac{1}{3}}(e^{x_2}-1)^{\frac{1}{3}}(e^{x_3}-1)^{\frac{1}{3}}+1\right]\geq\log\left[(e^{y_1}-1)^{\frac{1}{3}}(e^{y_2}-1)^{\frac{1}{3}}(e^{x_3}-1)^{\frac{1}{3}}+1\right].$$

Since the base e is not less than 1,

$$(e^{x_1}-1)^{\frac{1}{3}}(e^{x_2}-1)^{\frac{1}{3}}(e^{x_3}-1)^{\frac{1}{3}}\geq(e^{y_1}-1)^{\frac{1}{3}}(e^{y_2}-1)^{\frac{1}{3}}(e^{x_3}-1)^{\frac{1}{3}}.$$

Since $e^{x_3}-1>0$ by $x_3>0$,

$$(e^{x_1}-1)(e^{x_2}-1) \geq (e^{y_1}-1)(e^{y_2}-1).$$

Hence,

$$\tilde{I}(x_1,x_2,y_3) - \tilde{I}(y_1,y_2,y_3)$$

$$= \log\left[(e^{x_1}-1)^{\frac{1}{3}}(e^{x_2}-1)^{\frac{1}{3}}(e^{y_3}-1)^{\frac{1}{3}}+1\right] - \log\left[(e^{y_1}-1)^{\frac{1}{3}}(e^{y_2}-1)^{\frac{1}{3}}(e^{y_3}-1)^{\frac{1}{3}}+1\right]$$

$$\geq \log\left[(e^{y_1}-1)^{\frac{1}{3}}(e^{y_2}-1)^{\frac{1}{3}}(e^{y_3}-1)^{\frac{1}{3}}+1\right] - \log\left[(e^{y_1}-1)^{\frac{1}{3}}(e^{y_2}-1)^{\frac{1}{3}}(e^{y_3}-1)^{\frac{1}{3}}+1\right]$$

$$= 0.$$

Therefore, $\tilde{I}(x_1,x_2,y_3) \geq \tilde{I}(y_1,y_2,y_3)$.

Step 2: ***I* violates (2).** Suppose, by contradiction, that there exists $\xi:[0,1] \to \mathbb{R}$ such that, for each $(y_1,y_2,y_3) \in [0,1]^3$,

(3) $$\tilde{I}(y_1,y_2,y_3) = \xi(y_1)^{\frac{1}{3}}\xi(y_2)^{\frac{1}{3}}\xi(y_3)^{\frac{1}{3}}.$$

Then, for each $\alpha \in [0,1]$,

(4) $$\alpha = \tilde{I}(\alpha,\alpha,\alpha) = \xi(\alpha)^{\frac{1}{3}}\xi(\alpha)^{\frac{1}{3}}\xi(\alpha)^{\frac{1}{3}} = \xi(\alpha).$$

By equation (3) and computation,

$$\xi(0.1)^{\frac{1}{3}}\xi(0.5)^{\frac{1}{3}}\xi(0.9)^{\frac{1}{3}} = \tilde{I}(0.1,0.5,0.9) \fallingdotseq 0.3808.$$

However, by equation (4),

$$\xi(0.1)^{\frac{1}{3}}\xi(0.5)^{\frac{1}{3}}\xi(0.9)^{\frac{1}{3}} = (0.1)^{\frac{1}{3}} \cdot (0.5)^{\frac{1}{3}} \cdot (0.9)^{\frac{1}{3}} \fallingdotseq 0.3557,$$

a contradiction.

4. Characterizations

In this section, we search for an axiomatic foundation of the new HDI. To that end, we focus on the real-use situation by assuming $n=1$ and $k \geq 3$. Assumption $n=1$ simplifies a social state to a vector $\mathbf{y} \equiv (y_1, \dots, y_k) \in [0,1]^k$, which consists of the *aggregated measurements* in each characteristic as in practice.[5]

In aggregation theory, a number of studies have investigated the class of quasi-arithmetic means, which was introduced by Aczél (1948) in a fixed characteristic model.

[5]Even when $n \geq 2$, we can aggregate all individuals' measurements for each characteristic by using *egalitarian equivalent value functions* defined in Section 33, and then we obtain the situation $n=1$. Therefore, the assumption $n=1$ is not restrictive in the real-use situation.

Definition 1. An index function $I:[0,1]^k \to [0,1]$ is a quasi-arithmetic mean if there exists a continuous and strictly increasing function $\eta:[0,1] \to \mathbb{R}$ such that, for each $\boldsymbol{y} \in [0,1]^k$,

$$I(\boldsymbol{y}) \equiv \eta^{-1}\Big(\frac{1}{k}\sum_{j\in K}\eta(y_j)\Big).$$

On the one hand, it is worth noting that the quasi-arithmetic mean with $\eta(y_j)=y_j$ turns out to be an arithmetic mean. On the other hand, as long as the domain is restricted to $(0,1]^k$, the quasi-arithmetic mean with $\eta(y_j)=\log y_j$ turns out to be a geometric mean. However, since the domain includes 0 in our model, a geometric mean does not belong to the class of quasi-arithmetic means.
So, as a counterpart in our model, we define quasi-geometric means. It is then shown that they are the only index functions satisfying *symmetry for the characteristics*, *normalization*, and *separability*.[6]

Definition 2. An index function $I:[0,1]^k \to [0,1]$ is a **quasi-geometric mean** if there exists a continuous and strictly increasing function $\eta:[0,1] \to \mathbb{R}$ such that, for each $\boldsymbol{y} \in [0,1]^k$,

$$I(\boldsymbol{y}) \equiv \eta^{-1}\Big(\prod_{j\in K}\eta(y_j)^{\frac{1}{k}}\Big).$$

Proposition 1. Suppose that $n=1$ and $k \geq 3$. Then quasi-geometric means are the only index functions satisfying symmetry for the characteristics, normalization, and separability.

Proof See Appendix A.1.

Let us sketch the proof for the uniqueness in our first proposition. Consider any index function I satisfying the trio of axioms. First, we generate a continuous and complete preordering from I. It inherits *symmetry for the characteristics* and *separability* of I, which enables us to apply Debreu's representation theorem (Debreu, 1959, Theorem 3): the preordering can be represented by an additively separable function. By transforming this function monotonically, we obtain a quasi-geometric mean, which is ordinally equivalent to I. Finally, by continuity and *normalization*, it can be shown that this quasi-geometric mean is in fact cardinally equivalent to I.

The index function $\tilde{I}$ in Example 2 is a quasi-geometric mean, where $\eta(y_j) \equiv e^{y_j} - 1$ for each $y_j \in [0,1]$. Thus, it satisfies all the three axioms in Proposition 1.

[6]Characterizations of quasi-arithmetic means are often applied to constructing social indices, such as poverty measures or inequality measures (Shorrocks, 1980; Foster and Shorrocks, 1991). In this context, population is usually treated as *variable* so that *subgroup consistency* or *subgroup decomposability* can be imposed. However, in our model, these axioms require characteristics to be variable, which seems quite strange. So, we impose *separability* instead of the axioms, but it plays a similar role.

However, it violates *homogeneity*.[7] This axiom requires that if all achievements grow by the same fraction of magnitude, then an index function also varies by the fraction of magnitude.

Homogeneity. For each $\boldsymbol{y} \in [0,1]^k$ and each $\lambda > 0$ with $\lambda \cdot \boldsymbol{y} \in [0,1]^k$,

$$I(\lambda \cdot \boldsymbol{y}) = \lambda \cdot I(\boldsymbol{y}).$$

By adding *homogeneity* to the set of axioms in Proposition 1, the class of power means can be characterized.[8]

Definition 3. An index function $I:[0,1]^k \to \mathbb{R}$ is a **power mean with exponent** $p \in \mathbb{R}$ if, for each $\boldsymbol{y} \in [0,1]^k$,

$$I(\boldsymbol{y}) = \begin{cases} \prod_{j \in K} y_j^{\frac{1}{k}} & \text{if } p = 0, \\ \left(\frac{1}{k} \sum_{j \in K} y_j^p \right)^{\frac{1}{p}} & \text{if } p \neq 0. \end{cases}$$

Proposition 2. Suppose that n=1 and $k \geq 3$. Then power means are the only index functions satisfying symmetry for the characteristics, normalization, separability, and homogeneity.

Proof See Appendix A.2.

Both the geometric mean ($p=0$) and the arithmetic mean ($p=1$) belong to the class of power means, but they are opposite extremes in the class in terms of complementability and substitutability. First, recall that *minimal lower boundedness* requires that poor achievement in a characteristic should not be compensated by good achievement in other characteristics. This is a weak axiom for local complementability, but a geometric mean (new HDI) is the only power mean that satisfies it.

On the other hand, we consider *local substitutability* as defined below. It requires that a state exist in which achievements in some two characteristics are fully substitutable.

Local substitutability. There exist $\boldsymbol{y} \in (0,1]^k$, $i,j \in K$ and $t > 0$ such that $y_i \neq y_j + t \in [0,1]$ and $y_j \neq y_i + t \in [0,1]$ for which

$$I(y_i - t, y_j + t, \boldsymbol{y}_{-i,j}) = I(\boldsymbol{y}).$$

Note that *local substitutability* is weaker than the following notion of *full substitutability*: for each $\boldsymbol{y} \in [0,1]^k$, each $i,j \in K$, and each $t > 0$, if $y_i - t, y_j + t \in [0,1]$, then

[7]The authors would like to thank an anonymous referee for suggesting this axiom.

[8]Power means are also known as generalized means or Hölder means.

$$I(y_i - t, y_j + t, \boldsymbol{y}_{-i,j}) = I(\boldsymbol{y}).$$

The arithmetic mean (old HDI) satisfies both local and full substitutabilities. However, *local substitutability* is sufficient to pin it down within the class of power means.

To summarize these observations, the next theorem contrasts the new and old HDIs within the class of power means: the new HDI is the only power mean satisfying *minimal lower boundedness*, while the old HDI is the only one satisfying *local substitutability*. Therefore, these aggregation formulas are opposite extremes in the class of power means in terms of complementability and substitutability. This contrast provides a theoretical justification for the use of the new HDI.

Theorem 1. Suppose that $n=1$ and $k \geq 3$.

(i) The new HDI (geometric mean) is the only index function satisfying *symmetry for the characteristics*, *normalization*, *separability*, *homogeneity*, and *minimal lower boundedness*.
(ii) The old HDI (arithmetic mean) is the only index function satisfying *symmetry for the characteristics*, *normalization*, *separability*, *homogeneity*, and *local substitutability*.

Proof (i) Take any index function $I:[0,1]^k \to \mathbb{R}$ satisfying *symmetry for the characteristics, normalization, separability, homogeneity, and minimal lower boundedness.* Then, by Proposition 2, I is a power mean with some exponent $p \in \mathbb{R}$. Suppose, by contradiction, that $p \neq 0$. Then,

$$I(0, \underbrace{1, 1, \ldots, 1}_{k-1}) = \left(\frac{k-1}{k}\right)^{\frac{1}{p}} > 0,$$

which contradicts *minimal lower boundedness.* Therefore, $p=0$, and hence I must be the geometric mean.

(ii) Take any index function $I:[0,1]^k \to \mathbb{R}$ satisfying *symmetry for the characteristics*, *normalization*, *separability*, *homogeneity*, and *local substitutability*. Then, by Proposition 2, I is a power mean with some exponent $p \in \mathbb{R}$.

Step 1. There exist $\boldsymbol{y} \in (0,1]^k, i,j \in K$ and $t \in \mathbb{R} \backslash \{0\}$ such that $y_i \geq y_j$ and $1 \geq y_i - t \geq y_j + t \geq 0$ for which

$$I(y_i - t, y_j + t, \boldsymbol{y}_{-i,j}) = I(\boldsymbol{y}).$$

By *local substitutability*, there exist $\boldsymbol{y} \in (0,1]^k, i,j \in K$ and $t > 0$ with $y_i \neq y_j + t \in [0,1]$ and $y_j \neq y_i + t \in [0,1]$ such that

$$I(y_i - t, y_j + t, \boldsymbol{y}_{-i,j}) = I(\boldsymbol{y}).$$

Consider the case with $y_i \geq y_j$. If $y_i - t \geq y_j + t$, then the claim holds. Thus, suppose that $y_i - t < y_j + t$. Let

$$t' = y_i - t - y_j.$$

Since $y_i \neq y_j + t$, $t' \in \mathbb{R} \backslash \{0\}$. Moreover,

$$y_j + t' = y_j + (y_i - t - y_j) = y_i - t,$$

$$y_i + t' = y_i - (y_i - t - y_j) = y_j + t.$$

Therefore, by *symmetry for characteristics*,

$$I(y_i - t', y_j + t', \mathbf{y}_{-i,j}) = I(y_j + t', y_i - t', \mathbf{y}_{-i,j}) = I(y_i - t, y_j + t, \mathbf{y}_{-i,j}) = I(\mathbf{y}).$$

By a similar argument, we can show the case with $y_j \geq y_i$.

Step 2 (show that $p \neq 0$) Suppose, by contradiction, that $p = 0$. Then, by Step1,

(5) $$(y_i - t)(y_j + t) = y_i y_j.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be such that, for each $s \in \mathbb{R}$,

$$f(s) = (y_i - s)(y_j + s).$$

By differentiating this function for s, we have

$$f'(s) = (y_i - s) - (y_j + s).$$

Then, for any $s \in \mathbb{R}$ with $y_i - s > y_j + s$, $f'(s) > 0$, that is, f is strictly increasing on $(-\infty, \frac{y_i - y_j}{2}]$. Therefore, since $t \in (-\infty, \frac{y_i - y_j}{2}]$ and $t \neq 0$,

$$(y_i - t)(y_j + t) \neq y_i y_j,$$

which contradicts equation 58.

Step 3 ($p = 1$). Suppose, by contradiction, that $p \neq 1$. Then, by Steps 1 and 2,

(6) $$(y_i - t)^p + (y_j + t)^p = (y_i)^p + (y_j)^p.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be such that, for each $s \in \mathbb{R}$,

$$g(s) = (y_i - s)^p + (y_j + s)^p.$$

By differentiating this function for s, we have

$$g'(s) = -p(y_i - s)^{p-1} + p(y_j + s)^{p-1}.$$

Hence, for any $s \in \mathbb{R}$ with $y_i - s > y_j + s$, if $p > 1$, then $g'(s) < 0$ so g is strictly decreasing on $(-\infty, \frac{y_i - y_j}{2}]$; on the other hand, if $p < 1$, then $g'(s) > 0$ so g is strictly increasing on $(-\infty, \frac{y_i - y_j}{2}]$. In either case, since $t \in (-\infty, \frac{y_i - y_j}{2}]$ and $t \neq 0$, equation 6 never holds, a contradiction.

5. Discussion

We have considered the problem of designing an aggregation formula used in the HDI. First, we have shown that quasi-geometric means are the only index functions satisfying *symmetry for the characteristics*, *normalization*, and *separability*. Second, we have shown that power means are the only index functions satisfying *homogeneity* as well as the trio of axioms. Although both the new HDI (geometric mean) and the old HDI (arithmetic mean) belong to the class of power means, they can be interpreted as opposite extremes in terms of complementability and substitutability: that is, the new HDI is the only power mean satisfying *minimal lower boundedness*, while the old HDI is the only one satisfying *local substitutability*. This contrast provides a theoretical justification for the use of the new HDI.

We have also proved that the axiomatization of (Herrero et al., 2010, Theorem) does not hold. Consequently, one might question which index functions satisfy their set of axioms: monotonicity, symmetry for the characteristics, normalization, separability, and minimal lower boundedness. Let us conclude the present paper by answering this question in our setting $n=1$ and $k \geq 3$. Note that, as seen in the proof of Proposition 1 (Appendix A.1), under $n=1$, monotonicity is implied by the trio of symmetry for the characteristics, normalization, and separability.[9] *Thus, monotonicity can be dropped. It then follows from Proposition 1 that by adding minimal lower boundedness to the trio of axioms therein, quasi-geometric means with $\eta(0)=0$ can be characterized. This observation clarifies the importance of homogeneity in Theorem 1(i).1* Corollary Suppose that $n=1$ and $k \geq 3$. Then quasi-geometric means with $\eta(0)=0$ are the only index functions satisfying *symmetry for the characteristics*, *normalization*, *separability*, and *minimal lower boundedness*.Proof Let us only show the uniqueness. Consider any index function $I:[0,1]^k \to \mathbb{R}$ satisfying *symmetry for the characteristics*, *normalization*, *separability*, and *minimal lower boundedness*. Then, by Proposition 1, there exists a continuous and strictly increasing function $\eta:[0,1] \to \mathbb{R}_+$ such that, for each $\boldsymbol{y} \in [0,1]^k$,

$$I(\boldsymbol{y}) = \eta^{-1}\Big(\prod_{j \in K} \eta(y_j)^{\frac{1}{k}}\Big). \tag{7}$$

Take any $x \in (0,1]$. By equation 65 and by using *minimal lower boundedness* twice and *normalization*,

$$\eta^{-1}\big(\eta(0)^{\frac{1}{k}} \eta(x)^{\frac{k-1}{k}}\big) = I(0,x,\ldots,x) = I(\boldsymbol{0}) = 0.$$

Hence,

$$\eta(0)^{\frac{1}{k}} \eta(x)^{\frac{k-1}{k}} = \eta(0),$$

[9]Any index function satisfying the trio of axioms is a quasi-geometric mean with a strictly increasing function $\eta:[0,1] \to \mathbb{R}$. Hence, the inverse function η^{-1} is also strictly increasing, and in turn the quasi-geometric mean satisfies *monotonicity*.

which means that

$$\eta(0)^{\frac{1}{k}}\left(\eta(x)^{\frac{k-1}{k}}-\eta(0)^{\frac{k-1}{k}}\right)=0.$$

Note that since η is strictly increasing, $x>0$ implies $\eta(x)^{\frac{k-1}{k}}>\eta(0)^{\frac{k-1}{k}}$. Therefore,

$$\eta(0)=0.$$

References

Aczél, J., "On Mean Values," *Bulletin of the American Mathematical Society*, 54, 392–400, 1948.

Debreu, G., "Topological Methods in Cardinal Utility Theory," in K. J. Arrow, S. Karlin, and P. Suppes (eds), *Mathematical Methods in the Social Sciences: Proceedings of the First Stanford Symposium*, Stanford University Press, Stanford, CA, 16–26, 1959.

Desai, M., "Human Development: Concepts and Measurement," *European Economic Review*, 35, 350–7, 1991.

Foster, J. E. and A. F. Shorrocks, "Subgroup Consistent Poverty Indices," *Econometrica*, 59, 687–709, 1991.

, L. F. López-Calva, and M. Székely, "Measuring the Distribution of Human Development: Methodology and Application to Mexico," *Journal of Human Development*, 6, 5–25, 2005.

Herrero, C., R. Martínez, and A. Villar, "Multidimensional Social Evaluation: An Application to the Measurement of Human Development," *Review of Income and Wealth*, 56, 483–97, 2010.

Sagar, A. D. and A. Najam, "The Human Development Index: A Critical Review," *Ecological Economics*, 25, 249–64, 1998.

Sen, A. K., *Commodities and Capabilities*, North-Holland, Amsterdam, 1985.

Seth, S., "A Class of Distribution and Association Sensitive Multidimensional Welfare Indices," *Journal of Economic Inequality*, 11, 133–62, 2013.

Shorrocks, A. F., "The Class of Additively Decomposable Inequality Measures," *Econometrica*, 48, 613–25, 1980.

Zambrano, E., "An Axiomatization of the Human Development Index," *Social Choice and Welfare*, 42, 853–72, 2014.

Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Additional supporting information may be found in the online version of this article at the publisher's web site:

Appendix A Omitted Proofs in Section 44

A.1 Proof of Proposition 1

A.2 Proof of Proposition 2

Appendix B Tightness of the Axioms

Appendix C Amendment of Zambrano's (2014) Theorem 1

References