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GETTING RENTAL PRICES RIGHT FOR COMPUTERS: RECONCILING DIFFERENT PERSPECTIVES ON DEPRECIATION

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National statistical agencies frequently assume high geometric depreciation rates for computers. However, typically the service flow that a computer generates over its useful life is roughly constant, which contradicts the geometric model of depreciation. A one hoss shay model of depreciation seems to be more appropriate for computers. The paper uses Australian data on computer investment over the past 25 years to construct one hoss shay estimates of computer capital stocks and flows and considers how best to approximate these more realistic models of depreciation with a geometric model. The paper shows that under certain assumptions, a geometric model can provide an exact approximation to an underlying one hoss shay model. This exactness result is extended to a more general model of depreciation, the *Constant Efficiency Profile* model. Finally, the paper shows how well the geometric approximation fits a one hoss shay model when the assumptions are not satisfied.

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1. INTRODUCTION

Many statistical agencies use the geometric model of depreciation. But the geometric model of depreciation does not seem to be very realistic for computers: the service flow of a new computer rarely lasts longer than 4 years due to obsolescence. Moreover, since constant quality computer prices decline somewhere around 15 percent per year, this high negative rate of asset price change becomes a positive addition to a typically high geometric depreciation rate in the geometric user cost formula, leading to the possibility that the resulting user cost is too high which in turn could lead to a value of computer capital services which is also too high.[1]

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[1]Diewert (2012, p. 63) suggested this possibility but also suggested that more research was required.

Basically, the geometric model of depreciation is not plausible for computers. A one hoss shay model of depreciation with a short length of life (equal to 3 or 4 years) seems to be much more plausible.[2] In the present paper, we attempt to determine how well the geometric model of depreciation can approximate one hoss shay models under somewhat idealized conditions (steady growth in asset investments, steady rates of growth in constant quality prices and constant nominal costs of capital). Somewhat surprisingly, we find that under these idealized conditions, the geometric model of depreciation can provide a very good approximation to a one hoss shay model, provided that the "right" geometric depreciation rate is chosen.[3]

The geometric and one hoss shay models of depreciation are described in Sections 2 and 3 respectively. In Section 4, we assume that a one hoss shay model of depreciation is the "truth" and we show how a geometric depreciation rate can be picked so that either geometric capital services will be exactly equal to one hoss shay capital services or so that geometric capital stocks will be exactly equal to the corresponding one hoss shay capital stocks. However, it turns out that under our simplifying assumptions, these two equalizing depreciation rates are identical. We explain why this puzzling result holds in Sections 5 and 6. It turns out that we can show that this result will hold in a much more general class of models. Thus in Section 5, we introduce a generalization of the one hoss shay model of depreciation, the *Constant Efficiency Profile* (CEP) model of depreciation. In this model, a new asset delivers services for only a finite lifetime but it is assumed that the service flow of an older asset relative to a new asset remains constant.[4] In Section 6, we again show how a geometric model of depreciation can approximate this model exactly under our simplifying assumptions and the puzzle encountered in Section 4 is explained in the context of this more general model. Section 7 concludes. An Appendix uses economy wide data on computer investments in Australia for the past 25 years and shows how good the geometric approximations are to one hoss shay models of depreciation when our simplifying assumptions are not satisfied.

2. The Geometric Model of Depreciation

In this section, we develop the algebra that describes the geometric model of depreciation under some simplifying assumptions, which are as follows:

- The rate of growth of investment g in the asset is constant over time;
- The rate of growth in the price of a constant quality unit of the asset i is constant over time;
- The cost of financial capital for firms or the nominal rate of interest r is constant over time;

[2]The "one hoss shay" or "light bulb" model of depreciation assumes that the service flow of the asset is constant over the lifetime L of the asset and then it is retired at the end of L periods of use.

[3]In the Appendix, we show that under more realistic conditions, our geometric model approximation is adequate to approximate one hoss shay capital services but not so good at approximating one hoss shay capital stocks.

[4]For a one hoss shay asset, an older asset (that is less than L periods old) delivers the same service flow as a new asset. The CEP model allows for an arbitrary pattern of service flows as the asset ages.

- When the geometric model of depreciation is used, it is assumed that the depreciation rate δ is constant over time.

In what follows, we will consider levels of prices, quantities and values for two periods, 0 and 1, and rates of change going from period 0 to 1. We assume that the amount of investment in the asset under consideration in period 0, q_I^0, is 1 and that the corresponding period 0 investment price, p_I^0, is also 1.[5] Thus the corresponding value of investment in period 0, v_I^0, is also equal to 1. Under the above assumptions, the period 1 quantity, price and value of investment in the asset are given by $p_I^1 \equiv (1+i)p_I^0 = 1+i$; $q_I^1(1+g)q_I^0 = 1+g$ and $v_I^1 \equiv p_I^1 q_I^1 = (1+i)(1+g)$.[6] The price, quantity and value of investment data for the two periods under consideration are summarized in equations (1) below.

$$
\begin{aligned}
&p_I^0 = 1;\, p_I^1 = (1+i); && p_I^1/p_I^0 = (1+i);\\
&q_I^0 = 1;\, q_I^1 = (1+g); && q_I^1/q_I^0 = (1+g);\\
&v_I^0 = 1;\, v_I^1 = (1+i)(1+g); && v_I^1/v_I^0 = (1+i)(1+g).
\end{aligned} \tag{1}
$$

We assume that the investment in the prior period becomes an addition to the capital stock of the current period. Under our constant rate of growth of investment assumption and using the constant geometric depreciation rate assumption, the capital stock at the beginning of period 0, q_K^0, is given by the following expression:[7]

$$
\begin{aligned}
q_K^0 &\equiv [1/(1+g)] + [(1-\delta)/(1+g)^2] + [(1-\delta)^2/(1+g)^3] + \cdots\\
&= (1+g)^{-1}[1+\alpha+\alpha^2+\alpha^3+\cdots]\\
&= 1/(g+\delta)
\end{aligned} \tag{2}
$$

where we assume that $\alpha \equiv (1-\delta)/(1+g)$ is between 0 and 1. Under our constant rate of growth of investment assumption and using the constant geometric depreciation rate assumption, the capital stock at the beginning of period 1, q_K^1, is given by the following expression:

$$
\begin{aligned}
q_K^1 &\equiv 1 + [(1-\delta)/(1+g)] + [(1-\delta)^2/(1+g)^2] + \cdots\\
&= [1+\alpha+\alpha^2+\alpha^3+\cdots]\\
&= (1+g)/(g+\delta)\\
&= (1+g)q_K^0.
\end{aligned} \tag{3}
$$

[5]We can choose units of measurement for the investment good that justify these assumptions.

[6]We assume that $1+i>0$ and $1+g>0$ so that prices, quantities and values are positive for the two periods.

[7]This method for obtaining a starting value for the geometric capital stock is due to Griliches (1980, p. 427) and Kohli (1982); see also Fox and Kohli (1998).

Thus the starting capital stock for period 1 will be equal to q_K^0 times $(1+g)$. The period 0 and 1 prices of the starting capital stocks, p_K^0 and p_K^1, are equal to the corresponding investment prices, p_I^0 and p_I^1. The period t values for geometric capital stocks, v_K^t, are defined as $v_K^t \equiv p_K^t q_K^t$ for $t=0,1$. Using these definitions and (1)-(3), the capital stock price, quantity and value information for the geometric depreciation model is summarized in equations (4) below:

$$
(4)\qquad \begin{array}{lll}
p_K^0=1; & p_K^1=(1+i); & p_K^1/p_K^0=(1+i); \\
q_K^0=1/(g+\delta); & q_K^1=(1+g)q_K^0; & q_K^1/q_K^0=(1+g); \\
v_K^0=1/(g+\delta); & v_K^1=(1+i)(1+g)v_K^0; & v_K^1/v_K^0=(1+i)(1+g).
\end{array}
$$

Note that the rates of growth for investment prices, quantities and values for the geometric depreciation model are equal to the corresponding rates of growth for capital stock prices, quantities and values.

We turn now to user costs or rental prices for the geometric depreciation model. The beginning of the period user cost of using the *services* of a unit of capital for period 0, u_s^0, is defined as the cost of purchasing one unit of capital at the beginning of period 0, using the services of the asset during period 0 and then subtracting the discounted market value of the used asset at the end of the period. Thus the period 0 user cost of capital is equal to the following expression:[8]

$$
(5)\qquad \begin{aligned} u_S^0 &\equiv p_K^0-(1-\delta)p_K^1/(1+r) \\ &=1-(1-\delta)(1+i)/(1+r) \end{aligned}
$$

where the second equation in (5) follows using equations (4). Rather than discounting end of period prices to the beginning of the period, it is more convenient to revalue beginning of the period prices to their end of period equivalents.[9] Thus the period 0 *end of period user cost*, p_s^0, is defined as $(1+r)u_s^0$. Thus using (5), p_s^0 is equal to the following (familiar) expression for the geometric model of depreciation:[10]

$$
(6)\qquad p_S^0 \equiv (1+r)-(1-\delta)(1+i)=r-i+(1+i)\delta.
$$

The corresponding quantity of capital services for period 0, q_s^0, is equal to the period 0 starting capital stock, $q_K^0=1/(g+\delta)$ and thus the period 0 value of capital services is $v_S^0 \equiv p_S^0 q_S^0=1/(g+\delta)$. The period 1 user cost of capital, u_S^1, is defined as $p_K^1-(1-\delta)p_K^2/(1+r)=(1+i)p_K^0-(1-\delta)(1+i)p_K^1/(1+r)$ using our assumption

[8]The concept of the user cost of capital dates back to Walras (1954, p. 267–269) but the modern development of the user cost concept is due to Jorgenson (1963,1989). This discrete time method for deriving the user cost (5) is due to Diewert (1974, p. 504, 1980, p. 471).

[9]See Diewert (2005, p. 485–486) for a more detailed discussion on the merits of discounting to either the beginning or end of an accounting period. End of period user costs are more consistent with accounting conventions; see Peasnell (1981, p. 56).

[10]Jorgenson and his coworkers derived this user cost formula in a continuous time framework; see Jorgenson (1963, 1989), Jorgenson and Griliches (1967, p. 256) and Christensen and Jorgenson (1969).

that investment (and asset) prices grow at the constant inflation rate i. Thus $u_S^1=(1+i)u_S^0$. Since the quantity of capital services for period 1, q_S^1, is equal to the period 1 starting capital stock, q_K^1, using (4), we have $q_S^1=q_K^1=(1+g)q_K^0=(1+g)q_S^0$. The period t value of capital services, v_s^t, is defined as $p_s^t q_s^t$ for $t = 0,1$. The price, quantity and value of capital services data for the two periods under consideration are summarized in equations (7) below.

$$(7)\quad \begin{array}{lll} p_S^0=r-i+(1+i)\delta; & p_S^1=(1+i)p_S^0; & p_S^1/p_S^0=(1+i); \\ q_S^0=1/(g+\delta); & q_S^1=(1+g)q_S^0; & q_S^1/q_S^0=(1+g); \\ v_S^0=[r-i+(1+i)\delta]/(g+\delta); & v_S^1=(1+i)(1+g)v_S^0; & v_S^1/v_S^0=(1+i)(1+g). \end{array}$$

Note that the rates of growth for capital services prices, quantities and values for the geometric depreciation model are equal to the corresponding rates of growth for investment and capital stock prices, quantities and values. The geometric model of depreciation is easy to implement and has the advantage that it is not necessary to compute separate prices, quantities and values for each vintage of the assets in use. We turn now to the one hoss shay or light bulb model of depreciation, where it is necessary to keep track of vintages of the asset.

3. The one Hoss Shay Model of Depreciation

We will illustrate the computations for the one hoss shay model of depreciation.[11] As there are various computers and peripherals, we have to base our model on a representative computer, which can be regarded as an aggregate over individual computers and peripherals used in the production process. We assume a length of life for a new representative asset of 3 or 4 years. This is the likely length of time that a computer lasts before it is retired. We will start off with the 4 year length of life. Of course, we can extend our analysis to other years of service lives.

We make the same long run basic assumptions about the price and quantity of investments that were made in the previous section. Thus equations (1) in the previous section can still be used in order to describe the price, quantity and value of investments in the asset for periods 0 and 1. However, the algebra that describes the evolution of capital stocks and service flows for the one hoss shay model is different (and more complicated).

The basic idea behind the one hoss shay model of depreciation and capital services is that a new unit of the asset provides a constant flow of services for L periods and then is retired. In the present section, we will assume that the asset class is computers and we assume initially that the length of life is 4 years so that the period length is one year.

We will start our analysis by assuming that the length of life of an asset is 4 periods.[12] In general, the value of an asset should equal the discounted flow of

[11]The one hoss shay model of depreciation is due to Böhm-Bawerk (1891, p. 342). For a more detailed analysis of this model, see Hulten (1990, 1996); Diewert and Lawrence (2000) and Diewert (2005).

[12]The special assumptions that define the one hoss shay model will be made below in equations (12).

the service flows that it yields over its useful life. Denote the expected value of the services provided by a unit of the asset purchased at the beginning of period 0 over periods 0, 1, 2 and 3 by v^t for t = 0,1,2,3. Let the price of the asset purchased at the beginning of period 0 be p_K^0. Then p_K^0 should equal the discounted value of its future service flows so that the following relationship between the v^t and p_K^0 should hold:

$$(8) \qquad p_K^0 = v^0 + (1+r)^{-1}v^1 + (1+r)^{-2}v^2 + (1+r)^{-3}v^3$$

where $r > 0$ is the one period nominal discount rate or cost of capital which is assumed to remain constant over time. The above equation assumes that the rental payments v^0, v^1, v^2 and v^3 are received at the beginning of each period. It is more convenient to assume that the rental payments are made at the end of each period. Denote these end of period expected rental prices by u^0, u^{1*}, u^{2*} and u^{3*}. Using these end of period rental prices, we replace equation (8) by equation (9) below:

$$(9) \qquad p_K^0(1+r) = u^0 + (1+r)^{-1}u^{1*} + (1+r)^{-2}u^{2*} + (1+r)^{-3}u^{3*}.$$

The period 0 rental price for a new unit of this fixed life asset, u^0, will be a counterpart to the end of period 0 geometric model period 0 rental price p_s^0 defined in the previous section by (6).

We assume that the period 0 rental prices for units of the asset that are 1, 2 and 3 years old at the beginning of period 0 are u_1^0, u_2^0 and u_3^0. The *relative efficiency* or utility, e_i, of an older asset of age i relative to a new asset in period 0 is defined as the ratio of the period 0 older asset rental price u_i^0 to the period 0 rental price of a new asset u^0:[13]

$$(10) \qquad e_i \equiv u_i^0/u^0; \qquad i = 1, 2, 3.$$

We assume that this pattern of relative efficiencies will persist through all future periods. Using the asset price growth assumptions made in Section 2, the price of a new asset at the beginning of period 0 was unity and this price was expected to grow at the inflation rate i so that the price of a new asset over the next 3 periods would be $(1+i)$, $(1+i)^2$ and $(1+i)^3$. With these assumptions, $u^{1*} = u^0(1+i)e_1$, $u^{2*} = u^0(1+i)^2e_2$, and $u^{3*} = u^0(1+i)^3e_3$. Substituting these equations into equation (9) leads to the following equation, which relates the period 0 new asset price p_K^0 to the corresponding period 0 rental price for a unit of the new asset u^0:

$$(11) \qquad p_K^0(1+r) = u^0[1 + (1+r)^{-1}(1+i)e_1 + (1+r)^{-2}(1+i)^2e_2 + (1+r)^{-3}(1+i)^3e_3].$$

For the *one hoss shay model of depreciation* with length of life L = 4, the relative efficiencies of the assets of age 0, 1, 2 and 3 are all equal to one:

[13]The sequence of (cross sectional) vintage rental prices u^0, u_1^0, u_2^0, u_3^0 is called the *age-efficiency profile* of the asset; see Schreyer (2001, 2009).

$$e_1 = e_2 = e_3. \tag{12}$$

Substitute equations (12) into (11) and using the fact that $p_K^0 \equiv 1$, we obtain the following expression for the (end of period) one hoss shay user cost or rental price for a new (and old) unit of capital:

$$u^0 = (1+r)/(1+\beta+\beta^2+\beta^3) \tag{13}$$

where β is defined as follows:

$$\beta \equiv (1+i)/(1+r) > 0. \tag{14}$$

The price of a new unit of capital at the beginning of period 0, p_K^0, is equal to the investment price for a new unit of the asset, p_I^0, and both of these prices are set equal to 1. The one hoss shay asset prices at the beginning of period 0 for assets that are 1, 2 and 3 years old are defined to be p_{K1}^0, p_{K2}^0 and p_{K3}^0. These vintage asset prices are also set equal to their discounted stream of future expected rentals and so the older is the asset, the fewer terms will be in the stream of discounted rentals. It turns out that the period 0 vintage asset prices can be defined as follows:

$$\begin{aligned} p_K^0 &\equiv (1+r)^{-1}u^0(1+\beta+\beta^2+\beta^3) &&= 1; \\ p_{K1}^0 &\equiv (1+r)^{-1}u^0(1+\beta+\beta^2) &&= (1+\beta+\beta^2)/(1+\beta+\beta^2+\beta^3) \equiv f_1; \\ p_{K2}^0 &\equiv (1+r)^{-1}u^0(1+\beta) &&= (1+\beta)/(1+\beta+\beta^2+\beta^3) \equiv f_2; \\ p_{K3}^0 &\equiv (1+r)^{-1}u^0 &&= 1/(1+\beta+\beta^2+\beta^3) \equiv f_3 \end{aligned} \tag{15}$$

where β is defined by (14). Note that the price for a one period old asset is the fraction f_1 of the new asset price, which is $p_K^0 = 1$, and the asset prices for 2 and 3 year old assets at the beginning of period 0 are the progressively smaller fractions f_2 and f_3.

The beginning of period 0 total value, v_K^0, of the one hoss shay capital stock can now be calculated using the vintage asset prices defined in (15). The quantity of new assets at the start of period 0 is equal to the previous period's quantity of investment, $1/(1+g)$, and the quantity of 1, 2 and 3 period old assets at the start of period 0 is $1/(1+g)^2$, $1/(1+g)^3$ and $1/(1+g)^4$ under our assumptions. Thus v_K^0 is equal to the following expression:

$$\begin{aligned} v_K^0 &= p_K^0(1+g)^{-1} + p_{K1}^0(1+g)^{-2} + p_{K2}^0(1+g)^{-3} + p_{K3}^0(1+g)^{-4} \\ &= (1+g)^{-1}[1 + f_1(1+g)^{-1} + f_2(1+g)^{-2} + f_3(1+g)^{-3}] \\ &\equiv q_K^0 \end{aligned} \tag{16}$$

where we have defined the period 0 price of the one hoss shay capital stock to be $p_K^0 = 1$ and hence the value of the period 0 capital stock v_K^0 is equal to the period 0 quantity, q_K^0. Hence we have defined the vintage quantity components of the period 0 capital stock in terms of equivalent units of new capital stock components using the f_i as relative weights.[14]

Repeating the above analysis for period 1 shows that $p_K^1=(1+i)p_K^0=1+i$, $q_K^1=(1+g)q_K^0$ and $v_K^1=(1+i)(1+g)v_K^0$. Using the above analysis, the one hoss shay capital stock price, quantity and value data are summarized in equations (17) below.

$$\begin{aligned}
&p_K^0=1; && p_K^1=(1+i);\\
(17)\quad &q_K^0=(1+g)^{-1}[1+f_1(1+g)^{-1}+f_2(1+g)^{-2}+f_3(1+g)^{-3}]; && q_K^1=(1+g)q_K^0;\\
&v_K^0=q_K^0; v_K^0=(1+i)(1+g)v_K^0; && v_K^1/v_K^0=(1+i)(1+g)
\end{aligned}$$

where the f_i were defined in equations (15) and β was defined by (14). Note that the rates of growth for one hoss shay capital stock prices, quantities and values are equal to the corresponding rates of growth for investment prices, quantities and values and these rates of growth are also equal to the corresponding rates of growth for investments and stocks for the geometric model of depreciation. However, *it is not the case that the period 0 capital stock quantities and values necessarily coincide for the geometric and one hoss shay models.* Later, we will look for conditions that make the models consistent with each other.

Finally, we need to compute the prices, quantities and values for the one hoss shay service flows for periods 0 and 1. This is relatively straightforward. The user cost u^0 defined by (13) is the one hoss shay price of capital services, p_s^0, for period 0. The quantity of capital services that corresponds to this user cost is q_s^0 and it is equal to the sum of the lagged investments for 4 periods, $(1+g)^{-1}+(1+g)^{-2}+(1+g)^{-3}+(1+g)^{-4}$. The period 1 one hoss shay price of capital services under our assumptions turns out to be $p_S^1=u^0(1+i)$ and the corresponding quantity of capital services, q_S^1, is equal to $(1+g)q_s^0$. The one hoss shay capital services price, quantity and value data are summarized in equations (18) below.

$$\begin{aligned}
&p_S^0=(1+r)/(1+\beta+\beta^2+\beta^3); && p_S^1=(1+i)p_S^0;\\
(18)\quad &q_S^0=(1+g)^{-1}+(1+g)^{-2}+(1+g)^{-3}+(1+g)^{-4}; && q_S^1=(1+g)q_S^0;\\
&v_S^0=p_S^0q_S^0; && v_S^1=(1+i)(1+g)v_S^0
\end{aligned}$$

where $\beta\equiv(1+i)/(1+r)$. As usual, the rates of growth for capital services prices, quantities and values for the one hoss shay depreciation model are equal to the corresponding rates of growth for investment and capital stock prices, quantities and values. Comparing equations (18) with equations (7), it can be seen that the rates of growth for capital services prices, quantities and values for the one hoss

[14]In Diewert and Lawrence (2000) and Diewert (2005), index number methods were used to aggregate the various vintages of the one hoss shay capital stock. This is not necessary in the present situation due to our assumptions about the persistence of growth rates of investment prices; i.e. under our assumptions, the vintage asset prices, the p_{Ki}^t, will all vary proportionally to the variations in the new asset prices. Under these conditions, all standard index number formulae will lead to aggregate prices of capital that move proportionally to the p_K^t. In the Appendix where our simplifying assumptions are not satisfied, we will use index number techniques to aggregate across vintages.

shay depreciation model are equal to the corresponding rates of growth for capital services prices, quantities and values for the geometric depreciation model. However, *the period 0 (and period 1) values of capital services for the geometric model are in general not equal to the corresponding period 0 (and period 1) values of capital services for the one hoss shay model.*[15]

We conclude this section by working out the prices, quantities and values for the one hoss shay model when the length of life of the asset is L = 3. Equations (1) still describe prices and quantities for investments in the asset. Equations (15) are replaced by the following equations which define the vintage asset prices for period 0 and the relative asset weights f_1 and f_2:

$$\text{(19)}\qquad \begin{aligned} p_K^0 &\equiv (1+r)^{-1}u^0(1+\beta+\beta^2) && = 1; \\ p_{K1}^0 &\equiv (1+r)^{-1}u^0(1+\beta) && = (1+\beta)/(1+\beta+\beta^2) \equiv f_1; \\ p_{K2}^0 &\equiv (1+r)^{-1}u^0 && = 1/(1+\beta+\beta^2) \qquad \equiv f_2. \end{aligned}$$

The counterparts to equations (17) are equations (20) which list the one hoss shay capital stock price, quantity and value data for L = 3:

$$\text{(20)}\qquad \begin{aligned} &p_K^0 = 1; && p_K^1 = (1+i)p_K^0 = 1+i \\ &q_K^0 = (1+g)^{-1}[1+f_1(1+g)^{-1}+f_2(1+g)^{-2}]; && q_K^1 = (1+g)q_K^0; \\ &v_K^0 = q_K^0; && v_K^1 = (1+i)(1+g)v_K^0. \end{aligned}$$

The L = 3 counterparts to the L = 4 service flow equations (18) are equations (21) which list the one hoss shay capital services price, quantity and value data for the 3 period length of asset life:

$$\text{(21)}\qquad \begin{aligned} &p_S^0 = (1+r)/(1+\beta+\beta^2); && p_S^1 = (1+i)p_S^0; \\ &q_S^0 = (1+g)^{-1}+(1+g)^{-2}+(1+g)^{-3}; && q_S^1 = (1+g)q_S^0; \\ &v_S^0 = p_S^0 q_S^0; && v_S^1 = (1+i)(1+g)v_S^0. \end{aligned}$$

In the following section, we will look for conditions which will reconcile the geometric depreciation model to a one hoss shay model.

4. Reconciling The Geometric Model of Depreciation to a One Hoss Shay Model

Suppose that the one hoss shay model of depreciation is the "truth" for L = 4. Then under our stationary growth rate assumptions, the geometric model

[15]Thus when forming input aggregates for a sector or the economy, the choice of depreciation model will in general lead to different estimates for aggregate input growth even under our somewhat restrictive assumptions. Although the geometric and one hoss shay depreciation models generate identical rates of growth of prices and quantities for a capital services component under our assumptions on stationary growth rates, the alternative depreciation models will in general generate different *weighting* of these component growth rates which will lead to different overall input growth rates.

of depreciation will generate exactly the same capital stocks, provided that the geometric capital stock for period 0, q_K^0, defined in equations (4) is equal to the corresponding period 0 one hoss shay capital stock defined in equations (17). This leads to the following equation:

$$(22)\qquad 1/(g+\delta)=(1+g)^{-1}[1+f_1(1+g)^{-1}+f_2(1+g)^{-2}+f_3(1+g)^{-3}]\equiv\gamma_4$$

where the f_i are defined in equations (15). Equation (22) can be solved for the geometric depreciation rate δ^* that will make the capital stocks in the two models identical:

$$(23)\qquad \delta^*\equiv\gamma_4^{-1}-g.$$

Using Australian data on investment in computers for the past 25 years, we find that the average real growth rate of investment over this period was $g^*\equiv 0.20378$ so that real investment in computers grew at an annual average (geometric) rate of 20.4 percent. The corresponding (geometric) average rate of change in investment prices was $i^*\equiv -0.14096$.

For an approximation to the beginning of the year cost of capital r, we chose the average yield on 5 year Australian government bonds at the beginning of each year. The geometric average of these rates over the past 25 years was $r^*\equiv 0.06627$.[16] With these values for the parameters in our model, we find that the depreciation rate that solves equation (22) for the Australian data is $\delta^*\equiv\gamma_4^{-1}-g^*=0.32055$. This rate is considerably below the average of the official real depreciation rates over the past 25 years, which was $\delta_{ABS}\equiv 0.39220$.[17]

Instead of choosing a geometric depreciation rate that makes the one hoss shay and geometric capital stocks at the beginning of period 0 equal, we could choose the geometric rate δ_S that makes the period 0 geometric value of capital services equal to the corresponding one hoss shay value of capital services. Using equations (7) and (18), this leads to the following equation:

$$(24)\qquad \begin{aligned}&[r^*-i^*+(1+i^*)\delta_S]/(g^*+\delta_S)\\&=[(1+g^*)^{-1}+(1+g^*)^{-2}+(1+g^*)^{-3}+(1+g^*)^{-4}](1+r^*)/(1+\beta^*+\beta^{*2}+\beta^{*3})\equiv\phi_4\end{aligned}$$

where $\beta^*\equiv(1+i^*)/(1+r^*)$. Equation (24) can be solved for the geometric depreciation rate δ_{S^*} that will make the value of capital services in the two models identical:

$$(25)\qquad \delta_S^*\equiv[r^*-i^*-g^*\phi_4]/[\phi_4-(1+i^*)].$$

Again using the Australian data on investment in computers for the past 25 years, we find that the depreciation rate that solves equation (24) for the

[16]The calculation of g*, i* and r* is explained in more detail in the Appendix.

[17]The precise method for computing this average ABS depreciation rate is explained in the Appendix. It should be noted that the ABS does not use the geometric model of depreciation.

Australian data is $\delta^*_S \equiv 0.32055$, which is precisely equal to δ^*, the solution to (22) which equated the quantities (and values) of period 0 geometric and one hoss shay capital stocks.

Now suppose that the one hoss shay model of depreciation is the "truth" for L = 3. Again, we equate the period 0 geometric capital stock to the period 0 one hoss shay capital stock with length of life equal to three years. This leads to the following counterpart to equation (22):

$$(26) \qquad 1/(g^*+\delta)=(1+g^*)^{-1}[1+f^*_1(1+g^*)^{-1}+f^*_2(1+g^*)^{-2}] \equiv \gamma_3$$

where $f^*_1 \equiv (1+\beta^*)/(1+\beta^*+\beta^{*2}), f^*_2 \equiv 1/(1+\beta^*+\beta^{*2})$ and $\beta^* \equiv (1+i^*)/(1+r^*)$. Equation (26) can be solved for the geometric depreciation rate δ^{**}_S that will make the capital stocks in the two models identical:

$$(27) \qquad \delta^{**} \equiv \gamma_3^{-1}-g^*.$$

The depreciation rate that solves equation (26) for the long run Australian data is $\delta^{**}_S = 0.43240$, which is 10.2 percent above the official average depreciation rate of 0.39220.

Instead of choosing a geometric depreciation rate that makes the one hoss shay and geometric *capital stocks* at the beginning of period 0 equal, we could choose the geometric rate that makes the period 0 geometric value of *capital services* equal to the corresponding one hoss shay value of capital services. Using equations (7) and (21), this leads to the following equation:

$$(28) \qquad \begin{aligned}&[r^*-i^*+(1+i^*)\delta_S]/(g+\delta_S)\\ &=[(1+g^*)^{-1}+(1+g^*)^{-2}+(1+g^*)^{-3}+(1+g^*)^{-4}](1+r^*)/(1+\beta^*+\beta^{*2}+\beta^{*3}) \equiv \phi_4\end{aligned}$$

Equation (28) can be solved for the geometric depreciation rate δ^{**}_S that will make the value of capital services in the two models identical:

$$(29) \qquad \delta^{**}_S \equiv [r^*-i^*-g^*\phi_4]/[\phi_4-(1+i^*)].$$

The depreciation rate that solves equation (28) for the Australian data is $\delta^{**}_S \equiv 0.43240$, which is precisely equal to δ^{**}, the solution to (26) which equated the quantities (and values) of period 0 geometric and one hoss shay capital stocks.

The above results suggest that if the true depreciation model is a one hoss shay model with length of life half way between L = 3 and L = 4 years, then a geometric depreciation model that sets δ equal to the average of $\delta^* \equiv 0.43240$ and $\delta^{**} \equiv 0.32055$ (which is 0.37648 which in turn is reasonably close to the 0.39220 geometric depreciation rate which best approximates the official ABS depreciation rates over the past 25 years) will approximate the Australian computer capital stock data fairly well. This is an encouraging result; it shows that if the growth rate of investment in an asset and the rate of constant quality price change and the nominal discount rate are reasonably constant, then an appropriate geometric model of depreciation can approximate a one hoss shay model of depreciation fairly well.[18] This is an important result because geometric models of depreciation are very easy to implement; one does not need to keep track of separate vintages of investment and depreciate each vintage separately and then aggregate the vintage capital stocks and flows.

A remaining puzzle is: why are the δ^* solutions to equations (22) and (24) exactly the same when the equations look very different? And why are the δ^{**} solutions to equations (26) and (28) exactly the same? In the following section, we will consider a general family of depreciation models that contain one hoss shay models as a special case and show that for this class of models, a similar "puzzling" result occurs. In later part of Section 6, we will show why the exact equality holds for this class of models in equations (50) to (53).

5. The CEP Depreciation Model

In this section, we consider a generalized version of the one hoss shay model of depreciation and in the following section, we show that under our constant growth rate assumptions, a geometric model of depreciation can provide an exact approximation to this more general model.

The more general model that we will consider here is the *Constant Efficiency Profile* (CEP) model of depreciation. This model makes two main assumptions:

The length of life of the asset under consideration is L periods (a finite number greater than 2) and

- The relative efficiency of an asset that is i periods old relative to a new asset remains fixed over time.

Denote the end of period 0 rental price for a new unit of the asset by u^0. We assume that the end of period 0 rental price for units of the asset that are i periods old at the beginning of period 0 is u_i^0 for $i = 1,2,\ldots,L-1$. The *relative efficiency* or utility, e_i, of an older asset of age i relative to a new asset in period 0 is defined as the ratio of the older asset rental price u_i^0 to the period 0 rental price of a new asset u^0:

$$e_i \equiv u_i^0/u^0; \quad i=1,2,\cdots,L-1. \tag{30}$$

We assume that this pattern of relative efficiencies will persist through all future periods.[19]

Denote the beginning of period 0 price of a new unit of the asset by p_K^0 and the period 0 price of the same asset that is i periods old by p_{Ki}^0 for $i = 1,2,\ldots,L-1$. As usual, these asset values are set equal to the discounted stream of expected rentals that they are expected to generate. Again assuming a constant nominal cost of capital equal to r and a constant expected asset price inflation rate of i,

[18] As will be seen in the Appendix, while the i^t and g^t for Australia do not have definite trends over the past 25 years, the nominal interest rates r^t have a very strong downward trend from about 14.5 percent in 1989 to 2.5 per cent in 2013. It will be shown in the Appendix that our geometric approximation method works well for capital services but it does not work so well for the capital stocks.

[19] Of course, this model contains the one hoss shay model as the special case where all of the e_i are equal to one.

the sequence of period 0 asset prices by age of asset at the beginning of period 0 are defined as follows:

$$(31)\quad \begin{aligned} p_K^0 &\equiv (1+r)^{-1}[u^0+\beta u_1^0+\beta^2 u_2^0+\cdots+\beta^{L-1}u_{L-1}^0];\\ p_{K1}^0 &\equiv (1+r)^{-1}[u_1^0+\beta u_2^0+\beta^2 u_3^0+\cdots+\beta^{L-2}u_{L-1}^0];\\ &\cdots\\ p_{KL-1}^0 &\equiv (1+r)^{-1}u_{L-1}^0 \end{aligned}$$

where $\beta \equiv (1+i)/(1+r)$ as usual. Now set $p_K^0 = 1$ and substitute equations (30) into (31) in order to obtain the following system of equations which define the period 0 asset values by age in terms of the CEP user cost for a new asset at the beginning of period 0, u^0, and the relative efficiencies of the assets by their ages, the e_i:

$$(32)\quad \begin{aligned} p_K^0 &\equiv (1+r)^{-1}u^0[1+\beta e_1+\beta^2 e_2+\cdots+\beta^{L-1}e_{L-1}] &&=1;\\ p_{K1}^0 &\equiv (1+r)^{-1}u^0[e_1+\beta e_2+\beta^2 e_3+\cdots+\beta^{L-2}e_{L-1}] &&=f_1;\\ &\cdots\\ p_{KL-1}^0 &\equiv (1+r)^{-1}u_{L-1}^0 &&=f_{L-1}; \end{aligned}$$

where the fractions f_i are defined as follows:[20]

$$(33)\quad \begin{aligned} f_1 &\equiv [e_1+\beta e_2+\beta^2 e_3+\cdots+\beta^{L-2}e_{L-1}]/[1+\beta e_1+\beta^2 e_2+\cdots+\beta^{L-1}e_{L-1}];\\ f_2 &\equiv [e_2+\beta e_3+\beta^2 e_4+\cdots+\beta^{L-3}e_{L-1}]/[1+\beta e_1+\beta^2 e_2+\cdots+\beta^{L-1}e_{L-1}];\\ &\cdots\\ f_{L-1} &\equiv e_{L-1}/[1+\beta e_1+\beta^2 e_2+\cdots+\beta^{L-1}e_{L-1}]. \end{aligned}$$

Under our assumptions on the constancy of r, i and the efficiency profile parameters (the e_i), it can be seen that the f_i are also constant. Under these assumptions, it can also be seen that the sequence of asset prices by age for period 1 are equal to (1+i) times the period 0 counterparts; i.e., the following equations will be satisfied:

$$(34)\quad \begin{aligned} p_K^1 &= (1+i)p_K^0;\\ p_{Ki}^1 &= (1+i)p_{Ki}^0 = f_i p_K^1 = (1+i)f_i \quad i=1,2,\cdots,L-1. \end{aligned}$$

Note that given r, i and the e_i, u^0 and the f_i are determined by equations (32). Note also, that given r, i and the f_i (or equivalently, given the sequence of cross

[20]These fractions f_i can be used to define the sequence of one period *cross sectional depreciation rates* for the assets as they age. Define these deprecation rates δ_i using the following equations: $1-\delta_1 \equiv p_{K1}^0/p_K^0$ and $1-\delta_i \equiv p_{Ki}^0/p_{Ki-1}^0$ for $i=2,3,\ldots,L-1$. Then $1-\delta_1=f_1$, $(1-\delta_1)(1-\delta_2)=f_2$, $\cdots$, $(1-\delta_1)(1-\delta_2)\ldots(1-\delta_{L-1})=f_{L-1}$.

sectional depreciation rates δ_i), then equations (32) can be used to determine the relative efficiency parameters e_i and the CEP user cost u^0 for a new unit of the asset at the beginning of period 0. Thus under our constant rates of growth assumptions, the CEP model of depreciation is consistent with both an arbitrary (but finite) pattern of asset relative efficiencies as well as well as with an arbitrary pattern of cross sectional depreciation rates.

Using our steady growth of investments at the rate (1+g) both going forward and backward, the sequence of asset quantities that are available at the beginning of period 0 are given by $(1+g)^{-1}$, $(1+g)^{-2}$, …, $(1+g)^{-L}$. Using these quantities and the asset prices defined in equations (32), we can calculate the beginning of period 0 aggregate asset value for the capital stock, v_K^0, as follows:

$$(35) \qquad v_K^0 = (1+g)^{-1} + f_1(1+g)^{-2} + \cdots + f_{L-1}(1+g)^{-L} \equiv \gamma.$$

Equation (35) is the CEP counterpart to the corresponding one hoss shay capital stock valuation equation (16). As usual, we will define the period 0 price of the capital stock, p_K^0, to be equal to the corresponding investment price for a new asset which we have normalized to equal 1. Thus as in Section 3 above, we can define the vintage quantity components of the period 0 capital stock in terms of equivalent units of new capital stock components using the f_i as relative weights. Thus we define $q_K^0 \equiv v_K^0$ with $p_K^0 \equiv 1$.

Repeating the above analysis for period 1 shows that $p_K^1 = (1+i)p_K^0 = 1+i$, $q_K^1 = (1+g)q_K^0$ and $v_K^1 = (1+i)(1+g)v_K^0$. Using the above analysis, the CEP capital stock price, quantity and value data are summarized in equations (36) below.

$$(36) \qquad \begin{array}{ll} p_K^0 = 1; & p_K^1 = (1+i); \\ q_K^0 = (1+g)^{-1} + f_1(1+g)^{-2} + \cdots + f_{L-1}(1+g)^{-L} \equiv \gamma; & q_K^1 = (1+g)q_K^0; \\ v_K^0 = q_K^0; & v_K^1 = (1+i)(1+g)v_K^0; \end{array}$$

where the f_i were defined in equations (33). Note that the rates of growth for the CEP capital stock prices, quantities and values are equal to the corresponding rates of growth for investment prices, quantities and values and these rates of growth are also equal to the corresponding rates of growth for investments and stocks for the geometric model of depreciation. However as was the case for the one hoss shay model, *it is not the case that the period 0 capital stock quantities and values necessarily coincide for the geometric and one CEP models.*

Note that v_K^1 is the value of the capital stock at the beginning of period 1. This value is made up of two components:

- v_K^{1*}, the beginning of period 1 value of the capital stocks that were in place at the beginning of period 0;
- The quantity of investment during period 0 (which is 1) but valued at the beginning of the period price of investment made in period 1, which is (1+i). Thus this value is also equal to (1+i).

Thus v_K^{1*} and v_K^1 are equal to the following expressions:

$$(37)\qquad v_K^{1*} = (1+i)[f_1(1+g)^{-1} + f_2(1+g)^{-2} + \cdots + f_{L-1}(1+g)^{-(L-1)}];$$

$$(38)\qquad v_K^1 = (1+i) + v_K^{1*}.$$

We turn now to the determination of the value of capital services for the CEP model for period 0. The sequence of period 0 user costs or rentals by age of asset is u^0, u_1^0, u_2^0, …, u_{*L-1}^0, Under our constant growth rate assumptions, the corresponding quantities are $(1+g)^{-1}, (1+g)^{-2}, \ldots, (1+g)^{-L}$. Thus the period 0 value of capital services for the CEP model, v_S^0, is defined as follows:

$$(39)\qquad \begin{aligned} v_S^0 &\equiv u^0(1+g)^{-1} + u_1^0(1+g)^{-2} + \cdots + u_{L-1}^0(1+g)^{-L} \\ &= u^0[(1+g)^{-1} + e_1(1+g)^{-2} + \cdots + e_{L-1}(1+g)^{-L}] \quad \text{using (30)} \end{aligned}$$

where u^0, the period 0 user cost for a new unit of the asset, can be defined as follows using the first equation in (32):

$$(40)\qquad u^0 \equiv (1+r)/[1+\beta e_1 + \beta^2 e_2 + \cdots + \beta^{L-1} e_{L-1}].$$

We define the aggregate price and quantity of period 0 CEP capital services, p_s^0 and q_s^0, as follows:

$$(41)\qquad p_S^0 \equiv u^0; \quad q_S^0 \equiv (1+g)^{-1} + e_1(1+g)^{-2} + \cdots + e_{L-1}(1+g)^{-L}.$$

To determine u^1, we use the following equations, which are period 1 counterparts to the first equations in (32):

$$(42)\qquad p_K^1 = (1+r)^{-1} u^1 [1+\beta e_1 + \beta^2 e_2 + \cdots + \beta^{L-1} e_{L-1}] = 1+i.$$

It can be seen that the u^1 solution to (42) satisfies $u^1 = (1+i)u^0$ where u^0 is defined by (40). It is easy to see that under our steady growth rate assumptions, the aggregate quantity capital services in period 1 is $(1+g)q_s^0$ and the value of CEP capital services in period 1, v_S^1, is equal to $(1+i)(1+g)v_S^0$.

The CEP capital services price, quantity and value data are summarized in equations (43) below.

$$(43)\qquad \begin{aligned} &p_S^0 = (1+r)/[1 + \beta e_1 + \beta^2 e_2 + \cdots + \beta^{L-1} e_{L-1}]; && p_S^1 = (1+i)p_S^0; \\ &q_S^0 = (1+g)^{-1} + e_1(1+g)^{-2} + \cdots + e_{L-1}(1+g)^{-L}; && q_S^1 = (1+g)q_S^0; \\ &v_S^0 = p_S^0 q_S^0 \equiv \phi && v_S^1 = (1+i)(1+g)v_S^0. \end{aligned}$$

In the following section, we will approximate the CEP model defined in the present section by a geometric model of depreciation.

6. Approximating a CEP Depreciation Model by a Geometric Depreciation Model

From equations (4) in Section 2, we know that the period 0 starting capital stock for a geometric model of depreciation (under our regularity conditions on growth rates) is $1/(g+\delta)$. Using the analysis presented in the previous sections, we know that we can obtain a geometric depreciation model with depreciation rate, δ^*, that will generate exactly the same capital stock prices, quantities and values that the CEP model generates[21] provided that δ^* is the solution to the following equation:

$$(44) \qquad 1/(g+\delta)=(1+g)^{-1}+f_1(1+g)^{-2}+f_{L-1}(1+g)^{-L} \equiv \gamma$$

where the f_i are defined by equations (33). Thus the solution to (44) is[22]

$$(45) \qquad \delta^* \equiv \gamma^{-1}-g.$$

Assume that the depreciation rate for the geometric model of depreciation is defined by (45). Denote the value of the geometric capital stock at the beginning of periods 0 and 1 by V_{KG}^0 and V_{KG}^1 respectively and denote the corresponding values of the CEP capital stocks by v_K^0 and v_K^1. Then for the geometric depreciation rate δ^* defined by (45), we have the following equalities:

$$(46) \qquad v_{KG}^0=v_K^0;\ v_{KG}^1=v_K^1.$$

Note that V_{KG}^1 is the value of the geometric capital stock at the beginning of period 1. This value is made up of two components:

- V_{KG}^{1*}, the beginning of period 1 value of the geometric capital stock that was in place at the beginning of period 0;
- The quantity of investment during period 0 (which is 1) but valued at the beginning of the period price of investment made in period 1, which is $(1+i)$. Thus this value is also equal to $(1+i)$.

Thus V_{KG}^{1*} and V_{KG}^1 are equal to the following expressions:

$$(47) \qquad v_{KG}^{1*}= (1+i)(1-\delta)q_{KG}^0=(1+i)(1-\delta)v_{KG}^0$$

$$(48) \qquad v_{KG}^1= (1+i)+v_{KG}^{1*}.$$

Now compare equations (38) and (48). Since $v_{KG}^1 = v_K^1$, it can be seen that the following equality must also hold:

[21]See equations (36) above for the CEP capital stock prices, quantities and values for periods 0 and 1.

[22]In order to ensure that $\delta^*$is between 0 and 1, it is necessary that γ satisfy the following inequalities: $1/(1+g) < \gamma < 1/g$.

$$v_{KG}^{1*} = v_K^{1*}. \tag{49}$$

Thus the end of period 0 value of the depreciated beginning of the period 0 capital stocks coincide for the geometric and CEP models provided that the geometric depreciation rate δ^* is defined by (45).

We now turn our attention to the possible equality of capital services for the geometric and CEP models of depreciation. Using equations (7) and (39), it can be seen that we want the value of geometric capital services for period 0, V_{SG}^0, using the geometric depreciation rate defined by (45), to equal the value of CEP capital services, v_S^0, defined by (39); i.e. we want the following equation to hold:

$$v_{SG}^0 = [r - i + (1+i)\delta]/(g+\delta) = u^0(1+g)^{-1} + u_1^0(1+g)^{-2} + \cdots + u_{L-1}^0(1+g)^{-L} = v_S^0. \tag{50}$$

At this point, it is necessary to develop some alternative expressions for V_{SG}^0 and v_S^0. Recall equations (31) which relate the sequence of period 0 CEP asset prices by age, p_K^0 and the p_{Ki}^0, to the period 0 CEP user costs by age, u^0 and the u_i^0. These equations can be differenced to provide the following expressions for the sequence of CEP user costs in terms of CEP asset prices:[23]

$$\begin{aligned}
u^0 &= (1+r)p_K^0 - (1+r)\beta p_{K1}^0 &&= (1+r)p_K^0 - (1+i)f_1 p_K^0;\\
u_1^0 &= (1+r)p_{K1}^0 - (1+r)\beta p_{K2}^0 &&= (1+r)f_1 p_K^0 - (1+i)f_2 p_K^0;\\
u_2^0 &= (1+r)p_{K2}^0 - (1+r)\beta p_{K3}^0 &&= (1+r)f_2 p_K^0 - (1+i)f_3 p_K^0;\\
&\ldots\\
u_{L-1}^0 &= (1+r)p_{KL-1}^0 &&= (1+r)f_{L-1} p_K^0
\end{aligned} \tag{51}$$

where we have used equations (32), $p_{Ki}^0 = f_i p_K^0$, to derive the second set of equations in (51). Now set $p_K^0 = 1$ and substitute equations (51) into the first equation in (39) in order to obtain the following expression for the value of CEP capital services in period 0:

$$\begin{aligned}
v_S^0 &= u^0(1+g)^{-1} + u_1^0(1+g)^{-2} + \cdots + u_{L-1}^0(1+g)^{-L}\\
&= (1+r)[(1+g)^{-1} + f_1(1+g)^{-2} + f_2(1+g)^{-3} + \cdots + f_{L-1}(1+g)^{-L}]\\
&\quad - (1+i)[f_1(1+g)^{-1} + f_2(1+g)^{-2} + \cdots + f_{L-1}(1+g)^{-(L-1)}]\\
&= (1+r)v_K^0 - v_K^1
\end{aligned} \tag{52}$$

where we have used equations (35) and (37) in order to derive the last equation in (52). Equation (52) says that the value of period 0 CEP capital services, v_S^0, is equal to $(1+r)$ times the period 0 CEP value of the beginning of the period capital stock, v_K^0, minus the beginning of period 0 value of the depreciated period 0 starting capital stock, v_K^{1*}.

[23]See Diewert (2005) for similar expressions.

We need to obtain a counterpart to the CEP equation (52) for the geometric model of depreciation. Using equations (6) and (7), we find that the period 0 value of capital services for the geometric model, v_{SG}^{0}, is equal to the following expression:

$$
\begin{aligned}
v_{SG}^{0} &= [r-i+(1+i)\delta]/(g+\delta) \\
&= [(1+r)-(1+\delta)(1+i)]/(g+\delta) \qquad (53)\\
&= (1+r)v_{KG}^{0}-v_{KG}^{1*}
\end{aligned}
$$

where the last equation follows using equations (4) and (47). Recall that V_{KG}^{1*} was defined as the beginning of period 1 value of the geometric capital stock that was in place at the beginning of period 0.

All the pieces that are necessary to establish the equivalence of the CEP model to the geometric model of depreciation under our constant growth rate assumptions are in place. Choose the geometric depreciation rate δ^* equal to $\gamma^{-1} - g$ where γ is defined in (44). This will ensure that the geometric capital stock prices, quantities and values are equal to their CEP counterparts and it will also ensure that geometric value of the depreciated capital stock at the beginning of period 1, v_{KG}^{1*}, is equal to its CEP counterpart value, v_{K}^{1*}. Using (52) and (53), it can be seen that (50) is also satisfied; i.e., the value of capital services will be the same in periods 0 and 1 for the two models provided that we choose the geometric depreciation rate defined by (45).

7. Conclusion

What conclusions can we draw from the above computations? For computers, the geometric model of depreciation is a priori implausible. The one hoss shay model of depreciation is much more plausible with an expected length of life of 3 or 4 years.

However, under the assumption that the rate of growth of investments in computers is constant, the rate of decline in constant quality computer prices is constant and the nominal discount rate is constant, then by choosing the "right" geometric depreciation rate, the geometric model of depreciation can closely approximate the price and quantity behavior of the one hoss shay model of depreciation. Somewhat surprisingly, when the "right" geometric depreciation rate is chosen, then the geometric and one hoss shay values of capital services and stocks are exactly matched under our stationarity assumptions. A similar result holds for the CEP model of depreciation.

Unfortunately, the above results do not justify the use of the geometric model of depreciation under all circumstances. The equivalence of the geometric and CEP models will fail if:

- Rates of investment in the asset are far from being constant.
- Rates of change in the price of constant quality investment are far from being constant.

- The (nominal) cost of capital for users of the asset is far from being constant over time.

In the Appendix, using Australian data on computer investment over the past 25 years, we find that while the first two assumptions listed above are approximately satisfied, the third assumption is not justified: Australian interest rates have had a very strong downward trend during the past 25 years. Nevertheless, when we use our "best" geometric approximations to one hoss shay models of depreciation with length of life equal to either 3 or 4 years, we find that the approximating geometric model generates capital services data that are quite close to the corresponding one hoss shay model. However, the approximating geometric capital stocks are not nearly as close to their one hoss shay counterparts. These results suggest that national statistical agencies should consider moving to one hoss shay models of depreciation for computers. These models are not that difficult to implement but they do have the disadvantage that they may be a bit difficult to explain to users.

Finally, it is not only computers where one hoss shay models of depreciation are more plausible than geometric models of depreciation: many long lived infrastructure assets could be better approximated by one hoss shay models. Such assets include pipelines, sewers, electricity and telecommunication networks, railway lines, docking facilities and some commercial structures. Even more assets could be better described by CEP models, which are just as easy to implement as one hoss shay models, provided one has reasonable estimates for the efficiency profiles.

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Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Appendix A: Approximating a One Hoss Shay Model for Computers with a Geometric Model: The Case of Australia

Table A1: ABS Price and Quantity (Volume) of Computer Investment, Capital Stocks and Depreciation, 1985–2013, Millions of 1985 Dollars

Table A2: Real Investment in 1988 Dollars q_I^t, Beginning of the Year Price of Capital p_K^t, Growth Rate of Investment over Previous Year g^t, Annual Inflation Rate for the Price of Capital i^t, Nominal 5 Year Government Bond Yield at the Beginning of the Year r^t, Smoothed Inflation Rate i_S^t and Smoothed Bond Rate r_S^t

Table A3: Comparison of Geometric and One Hoss Shay (L = 4) Capital Stock Prices, Quantities and Values

Table A4: Comparison of Geometric and One Hoss Shay (L = 4) Capital Service Prices, Quantities and Values

Table A5: Comparison of Geometric and One Hoss Shay (L = 3) Capital Stock Prices, Quantities and Values

Table A6: Comparison of Geometric and One Hoss Shay (L = 3) Capital Service Prices, Quantities and Values

Table A7: Comparison of Geometric and One Hoss Shay (L = 4) Capital Stock Prices, Quantities and Values using Smoothed Asset Inflation and Interest Rates i_S^t and r_S^t

Table A8: Comparison of Geometric and One Hoss Shay (L = 4) Capital Service Prices, Quantities and Values using Smoothed Asset Inflation and Interest Rates i_S^t and r_S^t