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MEASURING ACHIEVEMENT AND SHORTFALL IMPROVEMENTS IN A CONSISTENT WAY

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In measuring improvements over time of bounded variables, one can focus on achievements or shortfalls. However, rankings of alternative social states in terms of achievements and shortfalls do not necessarily mirror one another. We characterize axiomatically different families of achievement and shortfall improvement indices, and present the necessary and sufficient conditions under which they rank social states in a consistent way. Empirical illustrations using child mortality data from South Africa suggest that consistency between achievement and shortfall improvements in standards of living is not only a matter of theoretical import but is also a problem that can be encountered in practice to a large extent.

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1. INTRODUCTION

When assessing the standard of living of our societies, it is important to look not only at the prevailing *levels* of different key socio-economic and development indicators but also to their *changes* over time. While several studies have explored the dynamics of well-being (e.g., Dasgupta and Weale, 1992; Dasgupta, 1993; Easterly, 1999; Mazumdar, 1999), only a few of them have attempted to define "improvement" or "progress" measures in a rigorous and satisfactory way—an important issue that, as will be argued below, has received insufficient attention from scholars. The only standard of living improvement measures proposed so far that we are aware of are those of Kakwani (1993), which were axiomatically characterized a few years later by Majumder and Chakravarty (1996) and extended to the multidimensional framework by Tsui (1996) and Chakravarty and Mukherjee (1999). Unfortunately, these measures do not address a relevant matter that has been ignored so far in the literature and which will be the main concern of this paper: the problem of "consistently measuring achievement and shortfall improvement."

Given the bounded nature of virtually all indices of standard of living (typical examples include health or education variables like life expectancy, child or adult mortality rates, literacy or school attendance rates, educational attainment and so

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forth), it is *a priori* possible to focus on the distribution of achievements or on the corresponding distribution of shortfalls with respect to the upper bound when measuring improvements over time. To illustrate: improvements in the coverage of public health plans could be assessed via the percentage of vaccinated children (an achievement indicator) or through the percentage of unvaccinated children (a shortfall indicator). While both approaches seem attractive in their own right, the few formal attempts to measure improvements in standards of living we are aware of have—somewhat strangely—only focused on the changes in the shortfall distributions (Kakwani, 1993; Majumder and Chakravarty, 1996; Tsui, 1996; Chakravarty and Mukherjee, 1999). However, we see *a priori* no reason to focus exclusively on those distributions and disregard their achievement counterparts. Apparently, both perspectives offer complementary views of the same problem, so it seems important to assess them in relation to each other. In this context, a natural question that might arise is: will improvements in shortfalls and improvements in achievements mirror each other or not? In this paper we will consider these complementary approaches simultaneously and present the conditions under which both classes of measures rank alternative states of affairs in a consistent way.

In the last few years, there has been a bourgeoning debate on the consistent measurement of achievement and shortfall *inequality* for bounded variables. The potential (and actual) mismatch between certain achievement and shortfall inequality measures was signaled by Clarke *et al*. (2002), and several authors have attempted to overcome that problem (Erreygers, 2009; Lambert and Zheng, 2011; Lasso de la Vega and Aristondo, 2012). Even if some of the results presented in this paper are inspired in the aforementioned works, there are important differences that are worth pointing out. On the one hand, we are not dealing with inequality but with improvements over time, so the functional forms of the indices we will be working with are completely different. On the other hand, while the notion of "inequality" is the same regardless of whether we are considering achievements or shortfalls (i.e., in both cases we are measuring the spread of a set of numbers), the notion of "improvement" can be considered as being, so to say, *directional* (as it depends on the end from which one stares at it), so it is important to distinguish between the two perspectives. Hence, while the same inequality index is used to measure attainment and shortfall inequality (the only thing that changes is the domain of the inequality index), we will need to define a specific improvement function for achievements and a specific improvement function for shortfalls. As a consequence, the conditions that are needed to satisfy the attainment–shortfall consistency test presented in this paper will differ with respect to the conditions used in the aforementioned papers.

In order to illustrate the usefulness of our proposal, we present the evolution over time of child health outcomes for the different municipalities in South Africa in the period 2001–07 using census microdata from the Integrated Public Use Microdata Series (IPUMS) project. The rest of the paper is organized as follows. In Section 2, we present the main axioms and characterize the improvement measures used in the paper. In Section 3, we explore the problem of consistently measuring achievement and shortfall improvements. Section 4 shows the empirical illustration and Section 5 concludes. The proofs are relegated to the Supporting Information.

2. Axioms and Improvement Measures

In this paper, the different units of analysis $\{1, \ldots, n\}$ ($n \in \mathbb{N}$) will be referred to as "individuals," even if in practice one might actually work with households, neighborhoods, municipalities, countries or any other group. The achievement level of a given individual will be measured with a certain standard of living indicator that will be tracked in two different moments in time (say, T_1 and T_2, with $T_1 < T_2$). We assume that such an indicator is measured in a continuous positive scale—an almost universal assumption in standards of living or well-being measurement—and that its values are naturally bounded from above and below. This last assumption is very common for most indicators that are typically incorporated in standard of living or well-being assessments. For instance: health or education variables (like life expectancy, mortality rates, educational attainment) can not increase or decrease indefinitely, so it is highly plausible to place a lower and upper bound on them. In this paper, the lower and upper bounds will simply be denoted by L and U respectively, with $0 \leq L < U$. It might be worth emphasizing that we are assuming that our achievement indicators can actually attain the values of the lower and upper bounds L, U. This is in contrast with the approach followed in other conceptually related studies, where the upper bound U is assumed to be unattainable (Kakwani, 1993; Majumder and Chakravarty, 1996; Tsui, 1996). Our choice has been motivated by the fact that many variables typically included in the assessment of the standard of living (e.g., literacy rates, enrolment ratios, gender gaps) do very often reach their upper bounds.[1] This apparently minor technicality will have important consequences in the derivation of the functional form of our improvement indices.

In order to measure improvements over time for a given individual we need to introduce some notation. We will denote by $x \in [L, U]$ the achievement level of a given unit of analysis in time T_1. Analogously, we will denote by $y \in [L, U]$ the achievement level of the same unit of analysis in time T_2. In this context, we can naturally define the shortfalls associated to achievements x and y as $p := U - x$ and $q := U - y$. Clearly, $p, q \in [0, U - L]$. When it comes to measuring the notion of "improvement," we should first decide whether the latter will be assessed through changes in achievements or in shortfalls. In this respect, the few formal attempts to measure improvements in standards of living we are aware of have only focused on the changes in shortfalls (Kakwani, 1993; Majumder and Chakravarty, 1996; Tsui, 1996; Chakravarty and Mukherjee, 1999). The fact that the corresponding achievements are disregarded in those papers is somewhat surprising, particularly because the existence of both approaches is acknowledged from the start. Since we consider that changes in both achievements and shortfalls are essentially measuring two sides of the same coin, in this paper we will incorporate them simultaneously and show the conditions under which they rank alternative states of affairs

[1]In case the underlying variable does not reach the upper bound (for instance, in the case of life expectancy), the results presented in this paper are equally valid.

in a consistent way. When improvements in standards of living are assessed through changes in achievements, we will introduce a so-called "achievement improvement index." Formally: an *achievement improvement index* ι^a is defined as a non-trivial real-valued function $\iota^a : [L, U]^2 \to \mathbb{R}$. The values of $\iota^a(x, y)$ should be interpreted as the improvement in standard of living of a given unit of analysis when the corresponding achievement changes from x to y. Analogously, a *shortfall improvement index* ι^s is defined as a non-trivial real-valued function $\iota^s : [0, U - L]^2 \to \mathbb{R}$ and its values $\iota^s(p, q)$ should also be interpreted as the improvement in standard of living observed when the shortfall changes from p to q.

Interestingly, the fact of introducing different indices to measure achievement and shortfall improvements (i.e., ι^a and ι^s) is in contrast with the approach followed in the measurement of achievement and shortfall inequality. In the latter case, the *same* inequality index D is used to measure both concepts; the only thing that changes is the domain (i.e., one compares inequality of a distribution of achievements $D(x_1, \ldots, x_n)$ *vis-à-vis* inequality of the corresponding distribution of shortfalls $D(U - x_1, \ldots, U - x_n)$). As mentioned in the introduction, the notion of "inequality" is the same regardless of whether we are considering one distribution or the other, but the notion of "improvement" depends on whether we use achievements or shortfalls.

Given the fact that the achievement and shortfall improvement measures are based on the same underlying ideas, our axioms will be presented in the following way. We will first present the general intuition behind the corresponding axiom and then formally show how this idea translates into certain restrictions for the ι^a and ι^s functions separately. Our first axiom reads as follows.

Continuity (CN): The improvement indices are continuous functions in their domains, that is: ι^a and ι^s are continuous.

This is an extremely common assumption in the literature of socio-economic indices. It requires that small changes in the achievements or shortfalls of individuals produce small changes in the corresponding improvement function. Stated otherwise: the change in standard of living does not abruptly change as individuals' achievements or shortfalls are slightly altered. Among other things, this property ensures that our measures will not be dramatically affected by measurement errors.

Monotonicity (MN): When a given unit of analysis sees its standard of living increasing from T_1 and T_2, then the corresponding improvement index should increase. In other words, $\iota^a(x, y)$ is increasing in y and $\iota^s(p, q)$ is increasing in p.

This assumption is quite unexceptionable for any index attempting to measure improvements in standard of living between any two moments in time.

Homotheticity (HM): When comparing and ranking two alternative states of affairs in terms of improvements, the results should remain unaltered when the corresponding achievements or shortfalls are changed by some multiple λ. Formally: for all $x_1, y_1, x_2, y_2 \in [L, U]$ and all $\lambda > 0$ such that $\lambda x_1, \lambda y_1, \lambda x_2, \lambda y_2 \in [L, U]$, one has that

$$(1) \qquad \iota^a(x_1, y_1) \geq \iota^a(x_2, y_2) \Leftrightarrow \iota^a(\lambda x_1, \lambda y_1) \geq \iota^a(\lambda x_2, \lambda y_2).$$

Analogously for ι^s: for all $p_1, q_1, p_2, q_2 \in [0, U - L]$ and all $\tilde{\lambda} > 0$ such that $\tilde{\lambda} p_1$, $\tilde{\lambda} q_1$, $\tilde{\lambda} p_2$, $\tilde{\lambda} q_2 \in 0, U - L]$, one has that

$$(2) \qquad \iota^s(p_1, q_1) \geq \iota^s(p_2, q_2) \Leftrightarrow \iota^s(\tilde{\lambda} p_1, \tilde{\lambda} q_1) \geq \iota^s(\tilde{\lambda} p_2, \tilde{\lambda} q_2).$$

It is important to note that Homotheticity should be interpreted as an ethical judgment with respect to improvement and *not* as a change in the measurement scale. Therefore, it should not be confused with other similar properties suggested in the literature, like "Scale Invariance" or "Unit Consistency."[2] These important differences have already been highlighted in Tsui (1996, p. 296).

Upward Sensitivity (US): Other things being equal, an improvement index should reward those improvements occuring at higher achievement levels. Formally: consider a hypothetical scenario with $L \leq z < w \leq U$. Take now any $\delta > 0$ such that $z + \delta$, $w + \delta \in [L, U]$. Then one has that

$$(3) \qquad \iota^a(z, z + \delta) < \iota^a(w, w + \delta).$$

Analogously for ι^s: consider a hypothetical scenario with $0 \leq z < w \leq U - L$. Take now any $\delta > 0$ such that $z - \delta$, $w - \delta \in [0, U - L]$. Then one has that

$$(4) \qquad \iota^s(z, z - \delta) > \iota^s(w, w - \delta).$$

Different authors have argued that, for certain standard of living indicators, improvement is much more difficult as the achievement level of the attribute becomes higher and higher (e.g., Sen, 1981, 1992; Dasgupta, 1993). Sen (1981), for instance, argues that it is not the same to increase life expectancy from 40 to 45 years (a case in which there is ample room for further improvement) as increasing it from 75 to 80 (an alternative scenario in which there is not much space for further improvement). In this line, Upward Sensitivity states that an increase of δ units in our improvement indicator is to be more valued when the initial achievement level is higher. Despite being a property that has been incorporated in all other improvement indices proposed in the literature so far, Upward Sensitivity is not entirely beyond dispute because it places lower value on improvements occuring at the bottom of the distribution. Since this collides frontally with a long stablished pro-poor tradition in poverty and inequality analysis that places *more* value at improvements occurring at the bottom of the distribution, it is also reasonable to introduce the opposite property.

[2]In the literature on income inequality measurement, an inequality index I is *scale invariant* if and only if $I(y) = I(\lambda y)$, $\lambda > 0$, where y is a vector of incomes. On the other hand, I satisfies *unit consistency* if for any two vectors of income distributions x, y and for any $\lambda > 0$, when $I(x) < I(y)$, then $I(\lambda x) < I(\lambda y)$ (Zheng, 2007). Observe that (HM) should be interpreted as an ethical judgment pertaining to the ranking of improvements, but not to the changes changes in the units of measurement as is the case with "scale invariance" and "unit consistency"—a subtle yet important difference.

Downward Sensitivity (DS): Other things being equal, an improvement index should reward those improvements occuring at lower achievement levels. Formally: consider a hypothetical scenario with $L \leq z < w \leq U$. Take now any $\delta > 0$ such that $z + \delta$, $w + \delta \in [L, U]$. Then one has that

$$\iota^a(z, z+\delta) > \iota^a(w, w+\delta). \tag{5}$$

Analogously for ι^s: consider a hypothetical scenario with $0 \leq z < w \leq U - L$. Take now any $\delta > 0$ such that $z - \delta$, $w - \delta \in [0, U - L]$. Then one has that

$$\iota^s(z, z-\delta) < \iota^s(w, w-\delta). \tag{6}$$

Even if this property has never been used in the improvement indices suggested so far, it places more value on improvements leading to more egalitarian distributions. To illustrate: if a donor agency successfully invests limited resources in increasing the life expectancy of a country from 40 to 45 rather than in increasing life expectancy for another country from 50 to 55, the resulting distribution will be less unequal and might therefore be more desirable in order to enhace overall welfare.

Additivity (AD): For any three periods, the improvement from period T_1 to T_3 may be expressed as the sum of the improvement from T_1 to T_2 and that from period T_2 to T_3. Formally: consider $x, y, z \in [L, U]$. Then

$$\iota^a(x, y) + \iota^a(y, z) = \iota^a(x, z). \tag{7}$$

Analogously for ι^s: consider $p, q, r \in [U - L]$. Then

$$\iota^s(p, q) + \iota^s(q, r) = \iota^s(p, r). \tag{8}$$

According to Additivity, the evaluation of the changes in living standards between two moments in time depends exclusively on these two moments in time and is not affected by the intermediate changes that might have occurred in between. This is a quite standard assumption that has already been introduced in the works of Kakwani (1993), Majumder and Chakravarty (1996), Tsui (1996) and Chakravarty and Mukherjee (1999).[3] One of the consequences of AD is that when no changes occur between T_1 and T_2, then our improvement measures should take the value of 0 (i.e., when $x = y = z$, AD implies that $\iota^a(x, x) + \iota^a(x, x) = \iota^a(x, x) = 0$).[4]

[3]In those papers the same axiom is named using alternative labels (e.g., "Subperiod Consistency" or "Period Consistency").

[4]Observe that this axiom together with Monotonicity imply that our improvement measures take positive (resp. negative) values when the underlying standard of living indicator increases (resp. decreases) over time—a property that Majumder and Chakravarty (1996) denote as "Range Subdivision."

Multiplicativity (MU): For any $x, y, z \in [L, U]$ one has that

(9) $$\iota^a(x, y)\iota^a(y, z) = \iota^a(x, z).$$

Analogously for ι^s: consider $p, q, r \in [0, U - L]$. Then

(10) $$\iota^s(p, q)\iota^s(q, r) = \iota^s(p, r).$$

According to Multiplicativity, if there is an α-fold improvement in standards of living from T_1 to T_2 and a β-fold improvement from T_2 to T_3, then there has been a $\alpha\beta$-fold improvement from T_1 to T_3. In contrast to Additivity, this axiom captures a complementary non-additive intuition that is also commonly used when assessing changes over time.

Normalization (NM): $\iota^a(L, U) = \iota^a(U - L, 0) = A$ for some real constant $A > 0$.

In order to render results easily interpretable, in some cases one normalizes the values of our improvement indicators between well-known bounds. Normalization stipulates that the improvement functions for a society with $(n =)1$ individual take their maximal value—equal to A—whenever the achievement indicator starts at its lowest level in T_1 and ends up at its highest level in T_2. While somewhat arbitrary and without universal application, the practice of bounding the values of socio-economic indicators is quite extensive in the literature. Standard and simple choices for such upper bounds could be 1, 10 or 100.

Combining these different axioms, we are able to characterize our achievement and shortfall improvement indices. We first present a quite general characterization result using a short list of "core axioms" that is further refined with the incorporation of the Upward and Downward Sensitivity and Normalization axioms.

Theorem 1. *An achievement improvement index ι^a satisfies the "achievement version" of the axioms CN, MN, HM and AD if and only if it can be written either as*

(11) $$\iota^a_{+1}(x, y) = C_1(y^\alpha - x^\alpha)$$

or as

(12) $$\iota^a_{+2}(x, y) = C_2(\ln(y) - \ln(x))$$

for all $x, y \in [L, U]$ ($x, y > 0$ for ι^a_{+2}) and for some real parameters $\alpha > 0$, $C_1 > 0$, $C_2 > 0$. Analogously, a shortfall improvement index ι^s satisfies the "shortfall version" of those axioms if and only if it can be written either as

(13) $$\iota^s_{+1}(p, q) = C_3(p^\beta - q^\beta)$$

or as

(14) $$\iota^s_{+2}(p, q) = C_4(\ln(p) - \ln(q))$$

for all $p, q \in [0, U-L]$ *(*$p, q > 0$ *for* ι^s_{+2}*) and for some real parameters* $\beta > 0$, $C_3 > 0$, $C_4 > 0$.

Proof: *See the Supporting Information.*

Theorem 2. *An achievement improvement index* ι^a *satisfies the "achievement version" of the axioms CN, MN, HM, and MU if and only if it can be written either as*

$$\iota^a_{\times 1}(x, y) = \left(\frac{y}{x}\right)^{\gamma} \tag{15}$$

or as

$$\iota^a_{\times 2}(x, y) = e^{E[y^{\zeta} - x^{\zeta}]} \tag{16}$$

for all $x, y \in [L, U]$ *(*$x \neq 0$ *for* $\iota^a_{\times 1}$*) and for some real parameters* $\gamma > 0$, $E > 0$, $\zeta > 0$. *Analogously, a shortfall improvement index* ι^s *satisfies the "shortfall version" of those axioms if and only if it can be written either as*

$$\iota^s_{\times 1}(p, q) = \left(\frac{p}{q}\right)^{\delta} \tag{17}$$

or as

$$\iota^s_{\times 2}(p, q) = e^{F[p^{\xi} - q^{\xi}]} \tag{18}$$

for all $p, q \in [0, U-L]$ *(*$q \neq 0$ *for* $\iota^s_{\times 1}$*) and for some real parameters* $\delta > 0$, $F > 0$, $\xi > 0$.

Proof: *See the Supporting Information.*

Remark 1. Interpretation of the indices. *Theorems 1 and 2 characterize axiomatically additive and multiplicative improvement indices respectively. The constants* C_i *from equations (11)–(14) are normalization parameters that can be chosen so that the range of values of our improvement indices fall within a pre-specified interval (see Remark 3 below for more details). When no change at all is observed between times* T_1 *and* T_2, ι^a_{+1}, ι^a_{+2}, ι^s_{+1} *and* ι^s_{+2} *take the value of 0. Therefore, values of the additive indices above (resp. below) 0 should be interpreted as an improvement (resp. worsening) in the standard of living of the corresponding unit of analysis between times* T_1 *and* T_2*. On the other hand, whenever standards of living improve (resp. deteriorate) over time, all multiplicative indices* $\iota^a_{\times 1}$, $\iota^a_{\times 2}$, $\iota^s_{\times 1}$ *and* $\iota^s_{\times 2}$ *take values above (resp. below) 1, and when no changes occur between times* T_1 *and* T_2*, then all of them are exactly equal to 1. As can be seen, even if all couples of indices* $(\iota^a_{+1}, \iota^s_{+1})$, $(\iota^a_{+2}, \iota^s_{+2})$, $(\iota^a_{\times 1}, \iota^s_{\times 1})$ *and* $(\iota^a_{\times 2}, \iota^s_{\times 2})$ *are highly related and have much in common—they are basically focusing on complementary aspects of the same phenomenon—their respective functional forms have some essential differences and they can not be derived from one another via trivial transformations. In order to clarify ideas, Table 1 classifies the eight improvement measures characterized in Theorems 1 and 2,*

TABLE 1

Classification of the Improvement Indices Characterized in Theorems 1 and 2

	Achievement	Shortfall
Additive	ι^a_{+1}, ι^a_{+2}	ι^s_{+1}, ι^s_{+2}
Multiplicative	$\iota^a_{\times 1}$, $\iota^a_{\times 2}$	$\iota^s_{\times 1}$, $\iota^s_{\times 2}$

TABLE 2a

Indices of Table 1 Satisfying the Upward Sensitivity Axiom

Upward Sensitivity	Achievement	Shortfall
Additive	ι^a_{+1} with $\alpha > 1$	ι^s_{+1} with $\beta < 1$, ι^s_{+2}
Multiplicative	$\iota^a_{\times 2}$ with $\zeta > 1$	$\iota^s_{\times 2}$ with $\xi < 1$, $\iota^s_{\times 1}$

TABLE 2b

Indices of Table 1 Satisfying the Downward Sensitivity Axiom

Downward Sensitivity	Achievement	Shortfall
Additive	ι^a_{+1} with $\alpha < 1$, ι^a_{+2}	ι^s_{+1} with $\beta > 1$
Multiplicative	$\iota^a_{\times 2}$ with $\zeta < 1$, $\iota^a_{\times 1}$	$\iota^s_{\times 2}$ with $\xi > 1$

depending on whether they are defined for achievements or shortfalls and whether they are additive or multiplicative.

Remark 2. Upward and Downward Sensitivity. *Given their controversial nature, the Upward and Downward Sensitivity axioms (US and DS) have not been used in the derivation of the improvement indices shown in Table 1. However, it is important to know what happens when these axioms are imposed on the improvement indices characterized in Theorems 1 and 2. In this respect, we have the following result.*

Proposition 1. *When the improvement indices of Table 1 are selected depending on whether they satisfy the Upward or Downward Sensitivity axioms, we obtain the tables of indices presented as Tables 2a and 2b.*

Proof: *See the Supporting Information.*

As can be seen in Tables 2a and 2b, parameters α, β, ζ and ξ regulate the extent to which improvements at higher or lower achievement levels are given more importance or not for some of the indices shown in Table 1 (i.e., they regulate whether the indices comply with the US and DS axioms or not). The farther away these parameters are from the value of 1, the more our measures will reward those improvements occuring in different parts of the distribution. At the other extreme, whenever $\alpha = \beta = 1$ or when $\zeta = \xi = 1$, our measures will *not* be sensitive to the places where improvements take place. Interestingly, when $\alpha = \beta = 1$ and $C_1 = C_3$ it turns out that $\iota^a_{+1}(x, y) = \iota^s_{+1}(p, q)$. Similarly, when $\zeta = \xi = 1$ and $E = F$ then $\iota^a_{\times 2}(x, y) = \iota^s_{\times 2}(p, q)$.

Remark 3. Bounds on the ranges. *Sometimes it is useful to normalize the values of our improvement indices between well-known bounds for different purposes. It is straightforward to check that when the Normalization axiom (NM) is imposed on* ι^a_{+1} *and* ι^s_{+1} *one obtains* $C_1 = A/(U^a - L^a)$, $C_3 = A/(U - L)^\beta$, *where A is the upper bound of the range of values we want our improvement indicators to have (i.e., after imposing Normalization,* ι^a_{+1} *and* ι^s_{+1} *range between −A and A). However, the unbounded nature of the logarithmic function does not render the improvement indices* ι^a_{+2} *and* ι^s_{+2} *easily amenable to normalization exercises. More specifically, whenever* $L = 0$, ι^a_{+2} *is unbounded, and the same can be said about* ι^s_{+2} *whenever U is assumed to be attainable—as is the case in this paper. The ranges of the multiplicative indices* $\iota^a_{\times 1}$, $\iota^a_{\times 2}$, $\iota^s_{\times 1}$ *and* $\iota^s_{\times 2}$ *are quite complicated because they depend on several parameters that are specific for each of the indices.*[5]

Remark 4. Relationship with other measures. *To our knowledge, the improvement indices characterized in Theorems 1 and 2 are the first measures of their kind that explicitly incorporate the achievement and shortfall perspectives in a common framework. When choosing A = 1, our shortfall improvement indices* ι^s_{+1}, ι^s_{+2} *partly coincide with the improvement indices suggested by Kakwani (1993, p. 314), which can be written as follows:*

$$
(19) \qquad f(U-x, U-y) := \left\{ \begin{array}{ll} \dfrac{(U-x)^r - (U-y)^r}{(U-L)^r} & \text{if } 0 < r < 1 \\ \dfrac{\ln(U-x) - \ln(U-y)}{\ln(U-L)} & \text{if } r = 0 \end{array} \right\}
$$

$$
= \left\{ \begin{array}{ll} \dfrac{p^r - q^r}{(U-L)^r} & \text{if } 0 < r < 1 \\ \dfrac{\ln(p) - \ln(q)}{\ln(U-L)} & \text{if } r = 0 \end{array} \right\} = f(p, q).
$$

Theorem 1 characterizes axiomatically Kakwani's improvement index f in a complementary way that differs from the characterization presented in Majumder and Chakravarty (1996). In addition, our characterization results widen the class of available improvement indices, including the preceding ones as particular cases.

3. Consistency between Achievement and Shortfall Improvement

Having defined four achievement and the corresponding four shortfall improvement indicators, it seems natural to ask whether these couples of measures

[5]The ranges of $\iota^a_{\times 1}$, $\iota^a_{\times 2}$, $\iota^s_{\times 1}$ and $\iota^s_{\times 2}$ are $[(L/U)^\gamma, (U/L)^\gamma], [e^{E(L^\zeta - U^\zeta)}, e^{E(U^\zeta - L^\zeta)}], (0, +\infty)$ and $[e^{-F(U-L)^\xi}, e^{F(U-L)^\xi}]$ respectively.

will provide consistent rankings when comparing alternative states of affairs. When dealing with the analogous problem in the context of inequality measurement, different authors have followed alternative approaches. Erreygers (2009) adopts a particularly strong interpretation of the consistency condition when he examines whether there exist inequality indices for which shortfall inequality is exactly equal to achievement inequality (i.e., if a generic inequality index is denoted by D, he imposes $D(x_1, \ldots, x_n) = D(p_1, \ldots, p_n)$, where $p_i = U - x_i$). Shortly after, Lambert and Zheng (2011) imposed a weaker consistency requirement according to which if a country A is ranked to be less unequal in attainments than country B, then country A should also exhibit less inequality in shortfalls than country B (formally: $D(x_1^A, \ldots, x_n^A) < D(x_1^B, \ldots, x_n^B) \Leftrightarrow D(p_1^A, \ldots, p_n^A) < D(p_1^B, \ldots, p_n^B)$). As we see the latter approach as quite natural when imposing consistency requirements, it is the one we have implemented in this paper in the context of improvements in standard of living. However, the fact of having achievement-specific and shortfall-specific improvement functions forces us to introduce some changes to our consistency condition, which reads as follows.

Achievement and Shortfall Consistency (AS): Let $x, y, z, w \in [L, U]$ be any achievements and let $p = U - x$, $q = U - y$, $u = U - z$, $v = U - w \in [0, U - L]$ be the corresponding shortfalls. Then

$$\iota^a(x, y) < \iota^a(z, w) \Leftrightarrow \iota^s(p, q) < \iota^s(u, v). \tag{20}$$

In words, AS imposes that if a country A is considered to have experienced less overall improvement in standard of living than another country B when measured with achievement indicators, then country A should also be considered to have experienced less overall improvement in standard of living than country B when measured with the corresponding shortfall indicators. When AS is satisfied, we will say that the couple of achievement and shortfall improvement indices (ι^a, ι^s) is *consistent*.

Our main result in this section is as follows:

Theorem 3. *For any $C_1 > 0$, $C_3 > 0$ the couple of achievement and shortfall improvement indicators $(\iota^a_{+1}, \iota^s_{+1})$ is consistent if and only if $\alpha = \beta = 1$. Analogously, for any $E > 0$, $F > 0$, $(\iota^a_{\times 2}, \iota^s_{\times 2})$ is consistent if and only if $\zeta = \xi = 1$. The couples $(\iota^a_{+2}, \iota^s_{+2})$ and $(\iota^a_{\times 1}, \iota^s_{\times 1})$ are not consistent.*

Proof: *See the Supporting Information.*

Given the regulating role of α, β, ζ and ξ regarding the compliance with the Upward or Downward Sensitivity axioms (see Tables 2a and 2b), we can conclude that giving more importance to the changes occurring in different parts of the distribution is at odds with the Achievement and Shortfall Consistency axiom. For the additive case, it is easy to check that whenever $\alpha = \beta = 1$, ι^a_{+1} and ι^s_{+1} are the same up to a multiplicative constant. Similarly, in the multiplicative case one has that whenever $\zeta = \xi = 1$, $\iota^a_{\times 2}$ and $\iota^s_{\times 2}$ can be obtained from one another via simple normalization transformations. Therefore, consistency between ι^a_{+1} and ι^s_{+1} and between $\iota^a_{\times 2}$ and $\iota^s_{\times 2}$ is obtained only when these couples of functions are very similar in nature. Interestingly, this result is reminiscent of the findings reported by Lambert and Zheng (2011), who find that the class of achievement and shortfall *inequality* indices that are consistent is quite restricted.

4. Empirical Illustration

In September 2000, the United Nations presented the Millennium Declaration, a milestone in international cooperation inspiring development efforts in order to improve the living conditions of millions of people around the world. As a result of the Millennium Declaration, all 193 United Nations member states agreed to achieve a series of time-bound targets—with a deadline of 2015—widely known as the "Millennium Development Goals" (MDGs, see www.un.org/millenniumgoals). One of those goals—MDG #4—prompts countries all over the world to reduce child mortality. Clearly, this is a health outcome that can *a priori* be approached from two angles: the shortfall perspective (i.e., reduce child mortality) or the attainment one (i.e., increase child survivorship). While the official MDG #4 is stated in terms of shortfalls,[6] one might legitimately wonder whether changes in children's health will be consistent when assessed via the shortfall and attainment perspectives, respectively. For this purpose, in this section we use census microdata from South Africa to assess the levels of child health improvement over time at the municipal level using the attainment and shortfall indices introduced in this paper.

4.1. *Data and Indicators*

In order to construct child health indicators at the municipal level we use census microdata samples from the Integrated Public Use Microdata Series database (IPUMS, see https://international.ipums.org/international) for South Africa. We have used the censuses samples from years 2001 and 2007. In both years, there were 225 municipalities in South Africa (these will be the units of analysis in this empirical illustration). In order to measure child health outcomes for municipality *i*, we simply compute the percentage of surviving children born to women in that municipality between ages 20 and 39, which will be denoted by P_i. This indicator is particularly suitable for small size populations and has been used among other things to describe the socio-demographic characteristics of indigenous populations in Latin America (ECLAC, 2010) and to explore the distribution of human development levels with high geographical detail (Permanyer, 2013). Clearly, P_i is an attainment indicator; its corresponding shortfall version is defined as $Q_i := 100 - P_i$ (i.e., the percentage of non-surviving children born to women between ages 20 and 39). In this context, one has that $L = 0$ and $U = 100$.

[6]MDG #4 prompts countries to reduce by two thirds, between 1990 and 2015, the under-five mortality rate.

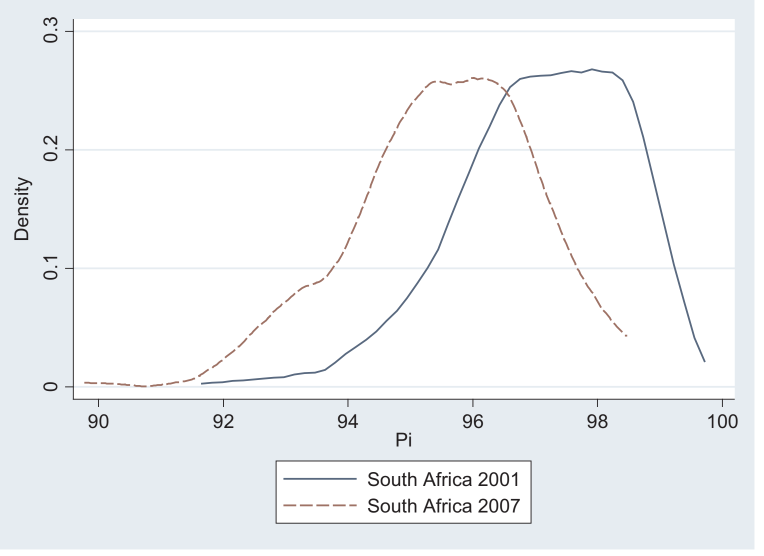

Figure 1. Density Functions of the Child Survivorship Index P_i for the case of South Africa
Source: Author's calculations using IPUMS data.

4.2. *Empirical Results*

Figure 1 shows the evolution of the distributions of child survivorship percentages of the corresponding municipalities for South Africa between 2001 and 2007. The clear deterioration[7] observed between these two years is in line with official figures of declining life expectancy reported in that country—which might be attributed to a large extent to the high prevalence of HIV/AIDS.

The density functions shown in Figure 1 only show the marginals of a distribution of paired data (that is: the achivement distributions in times T_1 and T_2: $\{(x_i, y_i)\}_{1\leq i\leq n}$). However, these marginal distributions are not informative on the specific patterns of change over time of the different municipalities we are working with.[8] In order to show these municipal-level improvements explicitly, Figures 2 and 3 plot the corresponding densities associated to the values of $(\iota_1, \ldots, \iota_{225})$ when $\iota_i = \iota^a_{+1}(x_i, y_i)$ with $\alpha = \beta = 1$, $A = 1$ (Figure 2) and when $\iota_i = \iota^s_{\times 1}(x_i, y_i)$ with $\delta = 1$ (Figure 3) for the 225 South African municipalities (the densities ensuing from the choice of the other improvement indices of Table 1 are quite similar, so they are not shown here to save space). For Figure 2, values of ι_i above (resp. below) 0 indicate that an actual improvement (resp. worsening) in child survivorship

[7]In this context, when we speak about overall "improvement" or "deterioration" we just refer to the general shape and position of the respective density functions, not to the specific changes observed for each municipality (which can not be inferred from that information only).

[8]To illustrate: assume, without loss of generality, that a distribution of achievements is ordered $(x_1 \leq \ldots \leq x_n)$. The hypothetical distributions of paired data $\{(x_i, y_i = x_i)\}_{1\leq i\leq n}$ and $\{(x_i, y_i = x_{n-i+1})\}_{1\leq i\leq n}$ have exactly the same marginal distributions but the individual-level improvements are completely different in the two cases (there are no changes whatsoever in the first one and extreme changes are observed in the second one).

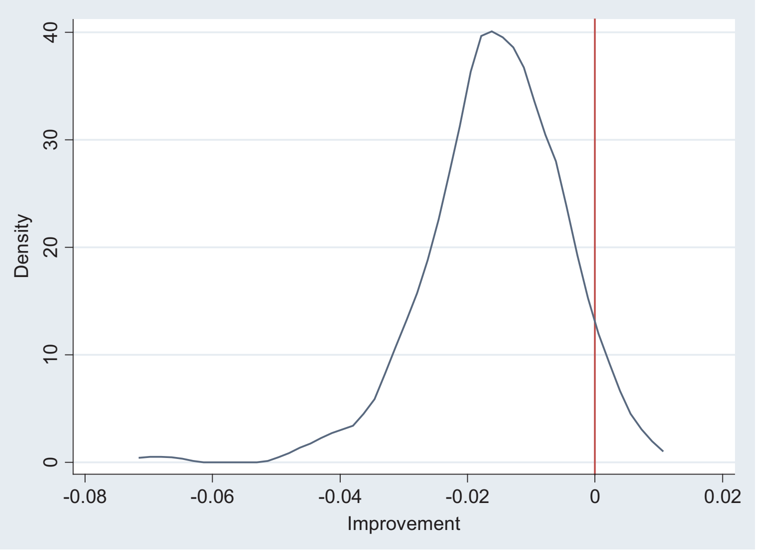

Figure 2. Density Function of the Additive Improvement Values ι_{+1}^{a} Using $\alpha = \beta = 1$, $A = 1$ for the 225 Municipalities in South Africa

Note: The vertical reference line indicates the value above which improvements over time take place.

Source: Author's calculations using IPUMS data.

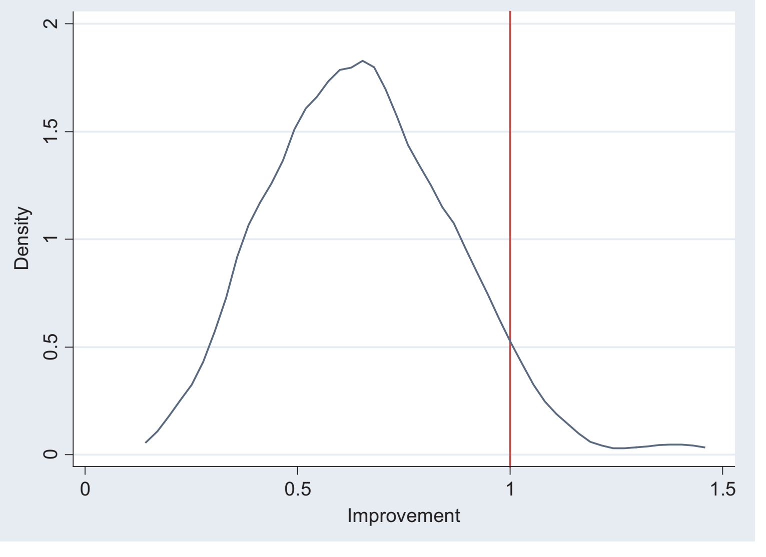

Figure 3. Density Function of the Multiplicative Improvement Values $\iota_{\times 1}^{s}$ Using $\delta = 1$ for the 225 Municipalities in South Africa

Note: The vertical reference line indicates the value above which improvements over time take place.

Source: Author's calculations using IPUMS data.

TABLE 3

Values of Kendall's Tau Associated to Countries' Achievement and Shortfall Improvement Distributions for Alternative Parameter Specifications

		$\tau(\iota^a_{+1}, \iota^s_{+1}),\ \tau(\iota^a_{\times 2}, \iota^s_{\times 2})$			
		$\beta = 1/2,\ \xi = 1/2$	$\beta = 1,\ \xi = 1$	$\beta = 2,\ \xi = 2$	$\tau(\iota^a_{\times 1}, \iota^s_{\times 1}),\ \tau(\iota^a_{+2}, \iota^s_{+2})$
South Africa, 2001–07	$\alpha = 1/2,\ \zeta = 1/2$	0.822	0.993	0.734	0.659
	$\alpha = 1,\ \zeta = 1$	0.828	1	0.727	
	$\alpha = 2,\ \zeta = 2$	0.839	0.989	0.717	

Source: Author's calculations using IPUMS data.

percentages has taken place in municipality *i*. Regarding Figure 3, the improvement threshold equals 1. As can be seen in Figures 2 and 3, most municipalities in South Africa have experienced deteriorations in child survivorship percentages, no matter whether we use additive or multiplicative improvement indices.

We will now explore empirically the extent to which the assessment of improvement levels is consistent when using the shortfall and achievement perspectives. For that purpose, we compare the municipalities ranking that is obtained using achivement improvement indicators with the ranking derived from the values of the corresponding shortfall improvement indicators. Table 3 shows the values of Kendall's tau coefficient[9] (henceforth τ) associated to the values of those achievement and shortfall indicators for alternative choices of the corresponding parameters. The definition of that statistic fits perfectly in our framework, since our consistency axiom (AS) precisely demands that the same set of municipalities is coherently ranked by alternative measures. The first columns of Table 3 show the values of τ for different parameter specifications of ι^a_{+1}, ι^s_{+1}, $\iota^a_{\times 2}$ and $\iota^s_{\times 2}$. When $\alpha = \zeta$ and $\beta = \xi$, ι^a_{+1} can be obtained from $\iota^a_{\times 2}$ via a monotonic transformation (and vice versa) and the same can be said about ι^s_{+1} and $\iota^s_{\times 2}$. Since Kendall's tau is invariant under monotonic transformations, the corresponding coefficients are the same for $(\iota^a_{+1}, \iota^s_{+1})$ and $(\iota^a_{\times 2}, \iota^s_{\times 2})$, so they are shown together in the table. By the same token, since $\iota^a_{\times 1}$ and $\iota^s_{\times 1}$ can be obtained from ι^a_{+2} and ι^s_{+2} respectively via monotonic transformations, their corresponding τ is also the same, so they are shown together in the last column of Table 3. It turns out that in virtually all comparisons considered in this paper, the municipality rankings that are obtained from the values of achievement and shortfall improvement indices are not *completely* consistent, that is, there exist couples of municipalities whose relative ranking is reversed when using one approach or the other. When this happens, the corresponding τ is strictly smaller than 1. For the case of South Africa there are certain choices of improvement measures for which $\tau \simeq 0.7$ or even lower, that is, many couples of municipalities (in some cases 30 percent or even more) are not ranked consistently according to ι^a and ι^s. Therefore, our assess-

[9]Let $(x_1, y_1), \ldots, (x_n, y_n)$ be a set of observations of the joint random variables X and Y. Assuming there are no ties, Kendall's tau is defined as $\tau := (C - D)/(n(n-1)/2)$, where C (resp. D) is the number of concordant (resp. discordant) pairs of observations and $n(n-1)/2$ is the total number of pair combinations. When all couples of observations are consistently ranked by X and Y, $\tau = 1$, and when all couples of observations are inconsistently ranked, $\tau = -1$.

ments of the child health improvements experienced in South African municipalities can differ to a great extent when using achievement or shortfall indicators.

5. Summary and Concluding Remarks

In this paper we have presented new indices of improvements in standards of living that address an important issue which has been ignored so far in the literature: the problem of "consistently measuring achievement and shortfall improvement." Given the bounded nature of virtually all standard of living indicators, it is possible to measure their improvements over time on the basis of the levels of achievement or on the basis of the corresponding shortfalls with respect to the upper bound. Integrating both approaches into a common framework, we have proposed the corresponding achievement and shortfall improvement indices and characterized them axiomatically. In a way, our improvement indices are reminiscent of the directional income mobility indices suggested by Fields and Ok (1999), adapted to the case where the variable of interest is bounded.

We argue that achievement and shortfall improvements are two sides of the same coin and that it is important to check whether both sides are measured in a consistent way. Such consistency can be imposed in different ways. A strong requirement to fulfill the consistency condition is to impose that both achievement and shortfall improvement indices take exactly the same values. A weaker requirement simply states that the orderings derived from the values of the indices have to be the same. It turns out that even when starting from the weaker requirement, the only achievement and shortfall improvement indices that rank alternative states of affairs in a consistent way are those that can be easily obtained from one another via straightforward monotonic transformations. Or the other way around: it is only when achievement and shortfall improvement indices are essentially the same functions (up to some normalization constants) that they are able to rank alternative states of affairs consistently. As shown in our analysis, the only way in which achievement and shortfall improvement indices can be consistent is to get rid of the Upward and Downward Sensitivity axioms—a couple of requirements that reward those improvements occurring at higher and lower achievement levels, respectively. This somewhat discouraging result is in line with the findings reported by Lambert and Zheng (2011) in the context of consistent achievement–shortfall inequality measurement. Given the fact that the achievement and shortfall approaches are not perfectly complementary (i.e., the orderings derived from one approach can not be deduced from the orderings of the other), it becomes necessary to give them separate and careful attention.

We have empirically illustrated our methodology exploring the evolution of child survivorship percentages in South African municipalities between the censuses of 2001 and 2007. Among other things, our results indicate that the corresponding municipality rankings derived from the values of our achievement and shortfall indicators are not always completely consistent (that is, there always exist some couples of municipalities which are inconsistently ranked with both kind of indicators). Even if both rankings tend to be highly correlated, in the case of South Africa there are as many as 30 percent of couples of municipalities that are inconsistently ranked by certain specifications of our achievement and shortfall improvement indicators. Therefore, consistency is not only a matter of theoretical import but is also a problem that can be encountered in practice to a large extent.

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Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's web-site:

Appendix. Proof of Theorem 1, Theorem 2, Proposition 1 and Theorem 3.

Figure A1. Level contours of $\iota^a(x, y)$ and $\iota^s(U\text{-}x, U\text{-}y)$ passing through (x_0, y_0). When $(z_0, w_0) \in C$ (resp. $(z_0, w_0) \in D$), (x_0, y_0) and (z_0, w_0) are ranked consistently (resp. inconsistently) in terms of ι^a and ι^s.