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MEASURING EARNINGS INEQUALITY: AN ECONOMIC ANALYSIS OF THE BONFERRONI INDEX

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This paper studies the economic content of the Bonferroni index. The most remarkable property of the Bonferroni index is that it overweights income transfers among the poor, and the weights are higher the lower the transfers occur on the income distribution. Hence, it is a good measure of inequality when changes in the living standards of the poor are concerned. There are many problems—especially in labor economics—that fall into this category. Using a version of the assignment model, we show that the Bonferroni index can be formulated endogenously within a mechanism featuring efficient assignment of workers to firms. This formulation is useful in evaluating the interactions between the distribution of skills and earnings inequality with a special emphasis on the lower tail of the earnings distribution. Moreover, it allows us to think about earnings inequality by separately analyzing the contribution of each economic parameter.

JEL Codes: D31, D63, O15

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1. INTRODUCTION

Study of inequality has relied upon analyses of certain statistical objects including the Lorenz curve, the Gini coefficient, the Bonferroni curve, and other commonly used inequality measures such as Theil's and Atkinson's.[1] Our research is motivated by the observation that the Bonferroni curve and the associated inequality index have received little attention in the economics discipline despite their attractive features. First, unlike the Gini coefficient, the Bonferroni index is more sensitive at the lower tail of the income distribution (Nygård and Sandström, 1981; Giorgi and Crescenzi, 2001c; Aaberge, 2007). More precisely, it is more sensitive to an income transfer from the richer to the poorer the lower the transfer occurs over the income scale. This feature makes it a useful statistical measure in the analysis of the lower end of the income distribution. Second, it is possible to attribute interesting and precise interpretations to what the Bonferroni index measures, i.e. attitudes including envy, deprivation, subjective evaluation of distributive justice (see below for a detailed explanation). Third, the Bonferroni curve

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[1]For the original articles, see Lorenz (1905), Gini (1914), Bonferroni (1930), Theil (1967), and Atkinson (1970).

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is naturally associated with several well-known distribution functions—such as Pareto, exponential, Gamma, uniform, and Weibull—which are commonly used in the economics discipline, especially in the study of human capital. Finally, there are many interesting problems in labor economics concerning the interactions between the distribution of skills and the lower end of the earnings distribution. These problems include minimum wages, unemployment insurance, low-skilled migration, residential segregation, early childhood education, etc. In all of these examples, a considerable bulk of transfers occur within the lower portion of the income distribution. Hence, in studying these problems, using standard measures of inequality—such as the Gini coefficient—will zoom out the changes in inequality. For this reason, and other technical reasons we summarize above, the Bonferroni index is a potentially useful tool and should be given more attention in the study of economic inequality.

Pyatt (1976) interprets the Gini coefficient as the average gain expected by an individual who is given the option of being someone else in the population divided by average income. A similar interpretation holds for the Bonferroni index with one nuance. For the Bonferroni index, this average gain is calculated by systematically assigning higher weights for the poor. Another interpretation of the Bonferroni index is that it is a measure of "deprivation" in society (Chakravarty, 2007).[2] Individuals with lower income experience deprivation. If this deprivation is proportional to the income gap, the Bonferroni index can be considered as the weighted average of all such deprivations in all possible pairwise comparisons, where the weights are higher at the lower end of the distribution. This interpretation comes from its formulation. It forms an average of lower averages (Chakravarty, 2007; Cowell, 2009), a concept that we develop in Section 2. In this sense, it measures the income differentials between subgroups, highlighting the tension among neighbors at the lower end of the income distribution. Therefore, it can also be interpreted as a measure of envy. Additionally, it can be related to the concept of "transferring poverty" from rural to urban areas, across the borders, etc. These interpretations open up a wide array of research opportunities in various fields including labor economics, development economics, social interactions and networks, urban economics, international economics, and econometrics literatures.

Our ultimate goal in this paper is to show that the Bonferroni index can be formulated as an endogenous object within a fully-specified economic model. Our focus is on "earnings." Thus, what we analyze is the inequality in labor income. There is a large literature—known as the CEO-pay literature—investigating the interactions between labor markets and the upper portion of the earnings distribution (Murphy, 1999; Piketty and Saez, 2003; Gabaix and Landier, 2008; Terviö, 2008). Although the empirical literature on the analysis of poverty is rich (Gottschalk and Danziger, 2005), theoretical work linking the labor market pricing literature to the analysis of inequality with a special emphasis on the interactions between the distribution of skills and the lower tail of the earnings distribution is non-existent. There are many examples that can be associated with a theoretical framework featuring the link between skills and earnings of the poor. For example, public policies related to expanding preschool education for disad-

[2]Sen (1973) and Yitzhaki (1979) provide similar interpretations for the Gini coefficient.

vantaged children are found to be effective in improving labor market outcomes of the children in the treatment group (Heckman et al., 2010). Such a policy intervention is certainly most effective at the lower end of the earnings distribution. Another example is low-skilled migration. When there is a large inflow of low-skilled immigrants, increased competition among native and foreign low-skilled workers alters the earnings distribution negatively, especially for the native low income earners (Davis, 1992; Levy and Murnane, 1992; Blackburn and Bloom, 1995). Interestingly, this has a "deprivation" interpretation in the sense that opposition to political authority is non-negligibly stronger for individuals who are exposed to risks, such as low-skilled workers in countries that receive low-skilled immigrants (Mayda et al., 2007; Bertola, 2010). Minimum wages also affect the lower end of the earnings distribution. For example, Autor et al. (2008) document that the abrupt decline in mid-1980s in U.S. real minimum wages explains the widening wage gap in the lower tail of the distribution rather than that in the upper tail. Dustmann et al. (2009) report that the same result holds for Germany. We *do not* claim that our framework is capable of explaining the stylized facts related to all the examples we provide above. This paper suggests that the analysis of inequality at the lower tail of the earnings distribution may be more effective when the tools specialized on this task are used. The Bonferroni index is one such tool. We construct analytical links between the Bonferroni index and some of the main economic variables affecting earnings inequality.

Understanding the sources of earnings inequality—paying special attention to the interactions between these sources—is at least as important a task as documenting the extent of inequality. With this in mind, we associate the Bonferroni curve and the Bonferroni index to a particular theory of the demand and supply of labor: Sattinger's assignment model.[3] Using a fairly standard version of the assignment model, we derive explicit formulas for the Bonferroni curve and the Bonferroni index. In particular, we show that the Bonferroni curve and the Bonferroni index can be formulated as functions of the following elements: (i) the distribution of skills across workers; (ii) the distribution of productive capital across firms; (iii) the characteristics of the production technology employing these two inputs; and (iv) the properties of the mechanism ensuring the optimal resource allocation in the economy. Such a setup allows us to evaluate the Bonferroni index by analytically separating out the individual effects of each sub-component. This paper closely follows Sattinger (1979, 1993) in that the structure of the labor market features a one-to-one match between workers and firms. The matching procedure yields an efficient assignment through which the best workers are assigned to the best firms. We extend this framework by establishing that the Bonferroni curve and the associated index of inequality are endogenous objects in the assignment model. This is a novelty in the inequality literature.

Our paper is related to the literature on the co-movement between the skills and earnings distributions. Major papers in this literature include Banerjee and Newman (1993), Galor and Zeira (1993), Durlauf (1996), and Acemoglu (1997). Specifically, Banerjee and Newman (1993) study the interactions between occupational choice and the wealth distribution. They distinguish between poor and

[3]See Sattinger (1979, 1993).

wealthy agents, and they argue that the occupational structure in the economy depends on various features of the distribution of skills. Galor and Zeira (1993) provide a cross-country analysis of income distribution. They show that the initial wealth distribution (including the factors affecting human capital investment) affects the steady state outcomes, and therefore differences in initial conditions can partly explain the persistence in cross-country inequality. Durlauf (1996) studies the co-evolution of human capital investment in children and neighborhood choices of parents. He shows that the neighborhood location affects children both through local public finance of education as well as through social influences. Acemoglu (1997) shows that increasing wage inequality is more likely to arise in economies where the extent of skills mismatch is smaller. The common feature of all these papers is that, like our paper, they relate inequality to the evolution of the distribution of skills in the society. In fact, these papers are only conceptually linked to our paper. They neither focus on income transfers among the poor nor mention the Bonferroni index. What happens at the lower end of the distribution is kept in the background as part of the general environment in these models. The distinctive feature of our analysis is its focus on the lower end of the earnings distribution, since at the end, we will be able to measure earnings inequality in terms of the magnitude of the Bonferroni index.

The plan of the paper is as follows. The next section gives a definition of the Bonferroni index and reminds the reader of its basic properties. Section 3 presents the assignment framework we employ and gives a full formulation of the resulting earnings functions. We provide an explicit formula for the earnings distribution and, using this formula, we derive the Bonferroni curve and the Bonferroni index. We discuss the predictions of our model on a numerical example. Section 4 concludes.

2. The Bonferroni Curve and Index

The literature defines the Bonferroni index as the average of lower averages (Chakravarty, 2007; Cowell, 2009). To make this definition concrete, we set up the following simple example. Suppose that we know the earnings of N workers. Let $n = 1, \ldots, N$ be the rank of each worker over the income scale, with N being the name of the worker with the highest earnings. In other words, $y_{n+1} \geq y_n$, $n \geq 1$, where y_n defines the earnings of the n-th worker. Let μ_n denote the n-th partial mean of the earnings distribution which is defined by

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

The sample average is, therefore, defined as $\mu_N = \mu$. Following Giorgi (1998), we conclude that the Bonferroni index, in this example, is equal to

$$\mathcal{B} = 1 - \frac{1}{N-1}\sum_{n=1}^{N-1}\frac{\mu_n}{\mu}. \quad (2.1)$$

It takes values in the unit interval [0,1]. $\mathcal{B} = 1$ means that worker N earns all the income, whereas $\mathcal{B} = 0$ means $y_{n+1} = y_n$, for all $n \geq 1$, i.e. perfect equality. It is obvious from equation (2.1) that the Bonferroni index systematically overweights the low-income workers. It is a measure of inequality that attaches more importance to the income transfers among the poor and this importance goes up as we move down the income scale.

Formally, the Bonferroni index is defined as the area lying between the Bonferroni curve and the line of perfect equality. We ignore the discussion on the shape and curvature of the Bonferroni curve and we guide the interested readers to Giorgi (1998) and Giorgi and Crescenzi (2001b) for further information on this. Next we extend our formulation to the real line and define the Bonferroni curve and index using absolutely continuous distribution functions for earnings. Suppose that the earnings of each individual is defined by the non-negative random variable Y and that the realizations of this random variable are denoted with y. Let the cumulative distribution function for Y be

$$F_Y(y) = \mathbb{P}[Y \leq y] = \int_0^y f_Y(s)\,ds,$$

where f_Y is the probability density function corresponding to F_Y. Let the first-moment distribution function be

$$F_{Y,1}(y) = \frac{1}{\mu}\int_0^y s f_Y(s)\,ds,$$

where $\mu = \int_0^\infty s f_Y(s)\,ds$ is the overall mean of the earnings distribution. Following Giorgi and Crescenzi (2001b), we formulate the Bonferroni curve as the ratio of the first-moment distribution function and the cumulative distribution function for earnings:

$$B(y) = \frac{F_{Y,1}(y)}{F_Y(y)} = \frac{\mu_Y}{\mu}. \tag{2.2}$$

The parameter μ_Y is the partial mean value for $Y \leq y$ and it is formulated as

$$\mu_Y = \frac{\int_0^y s f_Y(s)\,ds}{\int_0^y f_Y(s)\,ds}.$$

In words, the Bonferroni curve measures the ratio of this partial mean to the overall mean at every point along the earnings distribution. Therefore, by definition, it takes values in the unit interval [0,1]. In its standard representation, the Bonferroni curve is defined in the unit box by rewriting it as a function of F_Y instead of y. To do this, we first define $F_Y(y) = p$. Then,

$$B(p) = \frac{F_{Y,1}[F^{-1}(p)]}{p}. \quad (2.3)$$

The values that the Bonferroni curve takes are lined up on the vertical axis and the cumulative distribution of earnings is on the horizontal axis. As we mention above, the Bonferroni index is defined by the area lying between this curve and the line of perfect equality, i.e. where $B(p) = 1$ for all p. Therefore, the general formula for the Bonferroni index is

$$\mathcal{B} = 1 - \int_0^1 B(p)\,dp. \quad (2.4)$$

The behavior of the Bonferroni curve depends on the properties of the distribution function F_Y studied. In the rest of the paper, we abstract from these technicalities and, instead, focus on the economic meaning of the Bonferroni curve. In the next section, we motivate the basics of our assignment framework and we derive the associated earnings functions.

3. The Assignment Model

3.1. *Basics*

Our starting point is Sattinger's (1979) assignment model. It is a "differential rents" model—sort of a hedonic model—with continuous distributions of workers and firms. There are no consumer preferences in this economy. There is a one-to-one match between workers and firms. The aggregate output is the sum of the production from each match. The efficient assignment of workers to the firms will be the one that maximizes this aggregate output. There are no frictions, no information imperfections.

This is a model with two-sided heterogeneity. Workers are indexed by the level of their productive skills, which we denote by ℓ. We assume that there is only one type of labor skill so that we can rank workers using a single measure. Firms are indexed by a single quality measure k, the productive capacity. Following Akerlof (1969), we interpret this capacity as the level of productive capital each firm owns. Let F_ℓ be the cumulative density of ℓ and F_k be the cumulative density of k in the economy. We assume that both densities are monotone, strictly increasing, and have positive support.

The efficient assignment of workers to the firms promotes positively assortative matching, i.e. high-skill workers are employed in firms with greater productive resources. For each k, the assignment planner optimally chooses ℓ to maximize the surplus that the corresponding match produces. Formally, the problem is

$$\max_{\ell}[y(k, \ell) - \omega(\ell)], \quad (3.1)$$

for given k, where $y(k, \ell)$ is the amount of output that worker ℓ produces in firm k and $\omega(\ell)$ denotes what the worker is paid. We assume that $y(k, \ell)$ has the same functional form for all firms in the economy, which implies that output is homogeneous across firms. The first-order condition for this problem is

$$\frac{\partial y(k,\ell)}{\partial \ell} = \omega'(\ell), \tag{3.2}$$

where the term on the right hand side is called the wage differential, i.e. $\omega'(\ell) = d\omega(\ell)/d\ell$. The wage differential depends on the name of the firm. In other words, the magnitude of $\omega'(\ell)$ depends on the given k that the problem is being solved for. This dependency defines a functional relationship, $k(\ell)$, which we formally characterize below. This relationship is also known as the "sorting rule."

The second-order condition for a maximum is

$$\frac{\partial^2 y(k,\ell)}{\partial \ell^2} < \omega''(\ell). \tag{3.3}$$

Totally differentiating (3.2) yields

$$\frac{\partial^2 y(k,\ell)}{\partial \ell^2} d\ell + \frac{\partial^2 y(k,\ell)}{\partial \ell \partial k} dk = \omega''(\ell) d\ell, \tag{3.4}$$

which can be rearranged as

$$\frac{\partial^2 y(k,\ell)}{\partial \ell \partial k} \frac{dk}{d\ell} = \omega''(\ell) - \frac{\partial^2 y(k,\ell)}{\partial \ell^2}. \tag{3.5}$$

The right hand side of this differential equation is positive by (3.3). This implies that the left hand side must also be positive. Thus, to have $dk/d\ell > 0$ as the optimal solution, the sign of the cross partial $\partial^2 y(k,\ell)/\partial \ell \partial k$ has to be positive. More precisely, to match the best workers with the best firms, we need to assume complementarity between skills and capital. To pursue this goal, we assume for the rest of the paper that the production technology is of the canonical Cobb–Douglas form

$$y(k,\ell) = k^{\alpha} \ell^{1-\alpha}, \tag{3.6}$$

where $0 < \alpha < 1$ is the share of productive capital that each firm is endowed with and $1 - \alpha$ is the share of worker skills. The constant returns to scale assumption greatly simplifies the algebra. Our calculations so far prove that to ensure positive sorting between workers and firms, the sign of the cross partial for the production technology has to be positive. The Cobb–Douglas assumption serves exactly for this purpose. Next we derive an explicit sorting rule. Positive sorting requires that the top n workers should be assigned to the top n firms. Formally, we need to have

$$L \int_{\ell}^{\infty} f_{\ell}(i)\, di = K \int_{k(\ell)}^{\infty} f_k(j)\, dj, \tag{3.7}$$

where L and K are the total number of workers and firms, and f_ℓ and f_k are the probability densities of workers and firms, respectively. The intuition is the following. Suppose that $\%q$ of the workers are above some certain skill level $\tilde{\ell}$. This means that $\%q$ of the firms will have productive resources greater than $k(\tilde{\ell})$. Following Sattinger (1979), and for practical purposes that will become obvious soon, we assume that both the workers and firms are Pareto distributed as

$$f_\ell(\ell) = (\sigma_\ell - 1)\ell^{-\sigma_\ell} \text{ and } f_k(k) = (\sigma_k - 1)k^{-\sigma_k}, \tag{3.8}$$

respectively, where we assume that $\sigma_\ell > 2$ and $\sigma_k > 2$ to ensure finite variances, and that $\ell \geq 1$, $k \geq 1$.[4] Solving out the sorting equation using these two Pareto densities gives

$$k(\ell) = \mathcal{K}^{\left(\frac{1}{\sigma_k - 1}\right)} \ell^{\left(\frac{\sigma_\ell - 1}{\sigma_k - 1}\right)}, \tag{3.9}$$

where $\mathcal{K} = K/L$. We assume $\mathcal{K} \in (0,1)$, which means that the number of workers in this economy is greater than the number of firms. The interpretation is the following. The sorting rule yields a functional relationship between ℓ and k. Obviously, $k'(\ell) > 0$. How fast k increases as ℓ increases depends on the size of the worker population, the number of firms in the economy, and the distributions of workers and firms. Substituting $k(\ell)$ into the first-order condition yields the following key formula for the slope of the hedonic line:

$$\omega'(\ell) = (1 - \alpha)\mathcal{K}^{\left(\frac{\alpha}{\sigma_k - 1}\right)} \ell^{\phi}, \tag{3.10}$$

where ϕ is a constant and

$$\phi = \alpha\left(\frac{\sigma_\ell - \sigma_k}{\sigma_k - 1}\right). \tag{3.11}$$

Let $\hat{\ell}$ be the marginal worker employed. Integrating out the slope function with respect to ℓ gives us the following hedonic earnings function:

$$\omega(\ell) = \left[\frac{(1 - \alpha)\mathcal{K}^{\left(\frac{\alpha}{\sigma_k - 1}\right)}}{1 + \phi}\right] \ell^{1+\phi} + \omega_r, \tag{3.12}$$

where ω_r is the constant of integration—also interpreted as the reserve price of workers—and needs to be calculated. From $\mathcal{K} \in (0,1)$, we know that some of the workers are unemployed. Hence, it has to be the case that $\omega(\hat{\ell}) \geq \omega_r$. In fact, for the assumption of the competitive exhaustion of resources, the marginal worker

[4]There is strong evidence that the Pareto distribution is a good approximation for the distribution of firm productivity in the United States (Simon and Bonini, 1958; Axtell, 2001; Helpman *et al.*, 2004; Luttmer, 2007; Chaney, 2008). Like our paper, Helpman *et al.* (2010) use Pareto distributions in both worker skills and firm productivity dimensions.

gets the reservation wage, i.e. $\omega(\hat{\ell}) = \omega_r$. From $k \geq 1$, we must have $k(\hat{\ell}) = 1$, since all firms operate by the assumption $\mathcal{K} \in (0,1)$. Therefore, using equation (3.9), we obtain

$$\hat{\ell} = \mathcal{K}^{\left(\frac{1}{1-\sigma_\ell}\right)}.$$

Plugging this expression into the earnings function (3.12) and using the assumption of competitive utilization of resources, we get

$$\omega_r = \left(\frac{1-\alpha}{1+\phi}\right)\mathcal{K}^{\left(\frac{\alpha}{\sigma_k - 1} + \frac{1+\phi}{1-\sigma_\ell}\right)}, \tag{3.13}$$

which pins down the reservation wage as a function of the model parameters only. This completes the derivation of the earnings function.

Notice that the shape and the curvature of the earnings function depend on the dispersions of skills and capital. If $\sigma_\ell < \sigma_k$, then, from (3.10), $\phi < 0$, which implies that the earnings function is concave in worker skills. If $\sigma_\ell > \sigma_k$ ($\phi > 0$), then it is convex. The intuition is simple. When $\sigma_\ell > \sigma_k$, the right tail is fatter for the distribution of capital than it is for the distribution of skills. This implies that the demand for top skills is high, which is responsible for the convexity of the earnings function. Scarcity of top talent relative to top firms generates this result. The opposite insight holds when $\sigma_\ell < \sigma_k$. The share of productive capital (α) magnifies the curvature of the earnings function.

Recent studies in the CEO-pay literature document that small changes in talent result in large compensating differentials at the top of the income distribution.[5] Moreover, in an influential study, Piketty and Saez (2003) find that the top earners have experienced enormous gains over the last three decades. Using these two insights, we infer that $\phi > 0$, i.e. the earnings function is convex in worker skills (at least at the top of the earnings distribution). Hence, for the rest of the paper, we assume that $\sigma_\ell > \sigma_k$. Next we derive the earnings distribution that our model implies.

3.2. *Distribution of Earnings and Wage Inequality*

As a general rule, if $h(x)$ is the continuous, twice differentiable probability density of the random variable x and if there exists some invertible function, $y = g(x)$, of x, then the probability density of y is given by the formula

$$f_y = h\left(g^{-1}(y)\right)\frac{dg^{-1}(y)}{dy}. \tag{3.14}$$

We use this formula to derive the earnings distribution that our model implies.

[5]See, for example, Terviö (2008) and Gabaix and Landier (2008) who also use versions of Sattinger's assignment model. See Murphy (1999) for an excellent survey of the earlier work in the CEO-pay literature.

Equation (3.12) defines labor earnings as a function of worker skills. By definition, it is an invertible function. We invert the earnings equation to get

$$\ell = \left[\frac{(1+\phi)\mathcal{K}^{\left(\frac{\alpha}{1-\sigma_k}\right)}}{1-\alpha} \right]^{\left(\frac{1}{1+\phi}\right)} [\omega - \omega_r]^{\left(\frac{1}{1+\phi}\right)}. \tag{3.15}$$

We know the distribution of ℓ and, from (3.14), we know how to convert it into an earnings distribution. We set $W = \omega - \omega_r$. Using (3.14), the "translated" earnings distribution can be formulated as

$$f_W(w) = \frac{\sigma_\ell - 1}{1+\phi} \left[\frac{1-\alpha}{(1+\phi)\mathcal{K}^{\left(\frac{\alpha}{1-\sigma_k}\right)}} \right]^{\left(\frac{\sigma_\ell - 1}{1+\phi}\right)} w^{-\left(1+\frac{\sigma_\ell - 1}{1+\phi}\right)}, \tag{3.16}$$

where w defines the values that W takes. This formula yields the result that earnings are Pareto distributed. To see this more clearly, suppose that there is a Pareto random variable Z living in the interval $[z_m, +\infty)$, where $z_m > 0$ is a constant. The probability density of Z is

$$f_Z(z) = (\sigma_z - 1)\, z_m^{\sigma_z - 1} z^{-\sigma_z},$$

where σ_z characterizes the shape of the density function.[6] By analogy, the earnings distribution that we derive above fits perfectly into this formulation, where

$$z_m = \frac{1-\alpha}{(1+\phi)\mathcal{K}^{\left(\frac{\alpha}{1-\sigma_k}\right)}}$$

and

$$\sigma_z = 1 + \frac{\sigma_\ell - 1}{1+\phi}. \tag{3.17}$$

Having a Pareto earnings structure helps us to match an important stylized fact reported in the empirical earnings inequality literature. Data reveal that between-group inequality and within-group (residual) inequality move in the same direction (see, for example, Lemieux, 2006). This stylized fact is roughly matched when the distribution of earnings is Pareto. That is, when σ_z goes down the overall dispersion of earnings increases. At the same time, the dispersion at different sub-regions of the density also increases. This implies that both the overall and residual earnings inequality increase when σ_z decreases.

[6]We need to have $\sigma_z > 2$ to get an earnings distribution with finite variance.

3.3. *Formulating the Bonferroni Index*

It is possible to analytically derive the Bonferroni curve and the associated Bonferroni index from a Pareto distribution. From equation (2.3) and using the first moment function given by Quandt (1966), the Bonferroni curve for a Pareto earnings distribution can be formulated as

$$B(p) = \frac{1-(1-p)^{\frac{\sigma_z - 2}{\sigma_z - 1}}}{p}. \tag{3.18}$$

From Giorgi and Crescenzi (2001a) and using equation (2.4), the Bonferroni index can be written as

$$\mathcal{B} = \Psi(2) - \Psi\left(1 + \frac{\sigma_z - 2}{\sigma_z - 1}\right), \tag{3.19}$$

where $\Psi(\cdot)$ is the Digamma function and $\sigma_z > 2$ describes the dispersion of the earnings distribution.[7] The Digamma function is a strictly increasing function of σ_z, which implies that the Bonferroni index is a strictly decreasing function of σ_z. In other words, perhaps not surprisingly, wage inequality, as it is measured by the Bonferroni index, increases as the wages become more dispersed. Using the definition of σ_z, we conclude that the Bonferroni index can be formulated as a function of the dispersion of worker skills, σ_ℓ, and the curvature of the earnings function, ϕ, which itself is a function of σ_ℓ, the dispersion of productive capital across firms, σ_k, and the share parameter, α. Next we perform a basic comparative statics analysis and show how the changes in these parameters affect the Bonferroni index.

3.4. *Comparative Statics and the Interpretation of the Results*

In the earnings inequality literature, the composition of worker skills in the society is considered as one of the major determinants of the earnings structure. Our model confirms this point. We also show that the distribution of earnings is affected by other economic factors such as the shares of capital and labor in the production process and the dispersion of productive capital across firms. The following proposition describes how these parameters affect the distribution of earnings, and therefore the Bonferroni index of earnings inequality. Before stating the proposition, we would like to note that $\sigma_\ell > \sigma_k$—that the earnings function is convex in skills—will be our maintained assumption.

Proposition 1. *The Bonferroni index of earnings inequality increases—equivalently, σ_z decreases—when:*

[7]The Digamma function is equal to the first derivative of the natural logarithm of the Gamma function $\Gamma(\cdot)$, i.e. $\Psi(x) = \Gamma'(x)/\Gamma(x)$. See Giorgi and Crescenzi (2001a) for a formal statement of this result.

(a) *the distribution of worker skills becomes more dispersed, i.e.* σ_ℓ *decreases;*
(b) *the distribution of productive capital across firms becomes more dispersed, i.e.* σ_k *decreases; and*
(c) *the share of labor in the production technology increases, i.e.* α *decreases.*

Proof: Since σ_z completely parameterizes the Bonferroni index, and therefore the disperison of earnings, it is sufficient to look at what happens to σ_z to understand how the Bonferroni index reacts to the changes in parameters. We start with part *a*. Differentiating equation (3.17) with respect to σ_z and σ_ℓ, we obtain the following expression:

$$d\sigma_z = \frac{1}{1+\phi} d\sigma_\ell - \frac{\sigma_\ell - 1}{(1+\phi)^2} d\phi.$$

We know that ϕ is a function of σ_ℓ. Thus, we also need to know how it changes when σ_ℓ changes. Differentiating (3.11) with respect to σ_ℓ yields

$$d\phi = \frac{\alpha}{\sigma_k - 1} d\sigma_\ell.$$

As a result, we have

$$\frac{d\sigma_z}{d\sigma_\ell} = \frac{1}{1+\phi} - \frac{\sigma_\ell - 1}{(1+\phi)^2} \frac{\alpha}{\sigma_k - 1}.$$

We need to show that $\frac{d\sigma_z}{d\sigma_\ell} > 0$. We start with the expression

$$1 \gtreqless \frac{\sigma_\ell - 1}{1+\phi} \frac{\alpha}{\sigma_k - 1} \Rightarrow 1 \gtreqless \left(\frac{\sigma_\ell - \sigma_k}{1+\phi} \frac{\alpha}{\sigma_k - 1} + \frac{\sigma_k - 1}{1+\phi} \frac{\alpha}{\sigma_k - 1} \right)$$

which can be rewritten as

$$1+\phi \gtreqless \phi + \alpha \Rightarrow 1 \gtreqless \alpha.$$

Thus, everything comes down to whether α is less than or greater than 1. Obviously it is less than 1 in our model, which directly implies that $\frac{d\sigma_z}{d\sigma_\ell} > 0$. This completes part *a*. Proving parts *b* and *c* is simpler. Differentiating equation (3.17) with respect to σ_z and σ_k, we get

$$d\sigma_z = -\frac{\sigma_\ell - 1}{(1+\phi)^2} d\phi.$$

How ϕ changes as a response to a change in σ_k is given by

$$d\phi = -\alpha \frac{\sigma_\ell - 1}{(\sigma_k - 1)^2} d\sigma_k,$$

and therefore we have

$$\frac{d\sigma_z}{d\sigma_k} = \alpha \frac{(\sigma_\ell - 1)^2}{(1+\phi)^2 (\sigma_k - 1)^2} > 0.$$

This completes part *b*. For part *c*, we first calculate how ϕ changes when α changes:

$$d\phi = \frac{\sigma_\ell - \sigma_k}{\sigma_k - 1} d\alpha.$$

Thus,

$$\frac{d\sigma_z}{d\alpha} = -\frac{\sigma_\ell - 1}{(1+\phi)^2} \frac{\sigma_\ell - \sigma_k}{\sigma_k - 1} < 0, \text{ as required.} \quad \blacksquare$$

This proposition displays the links between the economics of the problem and its reflection on the Bonferroni index. Part α states that as the dispersion of skills increases, firms start having access to a larger set of relatively skilled workers. This enlarges the earnings horizon and wages become more dispersed. How this change is translated to an increase in the Bonferroni index is given by equation (3.19). Part *b* has a similar conclusion. Given the distribution of worker skills, an increase in the dispersion of productive capital across firms increases the competition for skilled workers at the top of the firm distribution. The incremental cost of buying an extra unit of skill becomes more expensive for the firms. Therefore, wage inequality increases and the Bonferroni index goes up. Part *c* says that, under convexity, the marginal product of labor increases when the labor share goes up. Acquiring one more unit of skill becomes more expensive and inequality rises. Next we provide a numerical application to illustrate some of the basic predictions of our model.

3.5. *Numerical Application*

At the heart of the human capital theory is the interrelationship between the accumulation of skills and the evolution of earnings distribution. Schooling is one of the major determinants of human capabilities and trends in schooling achievement affect the degree of earnings inequality in an economy. Over the last few decades, the society in the United States has faced a polarization in schooling achievement in the sense that the high school graduation rates have fallen—after correcting for the General Educational Development (GED) certificate holders—and college enrollment among high school graduates has increased (see Figure 1). This points to an increased dispersion of skills in the society (Heckman and Masterov, 2007). In terms of the language of our paper, σ_ℓ has declined in the United States. Figure 2 illustrates how such a movement is represented over a Pareto distribution of skills. An increase in the fraction of higher educated workers leads to a fatter tail in the distribution of skills.

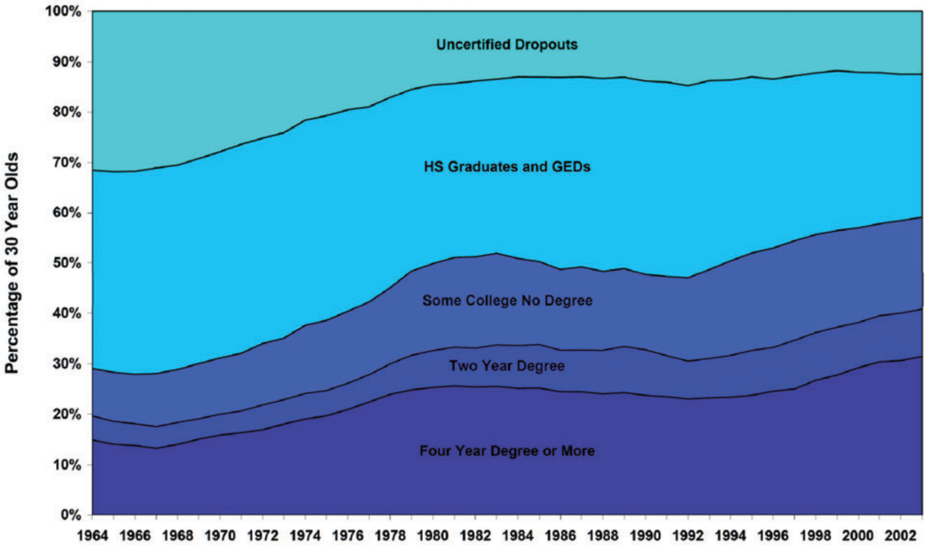

Figure 1. Trends in Educational Attainment in the United States

Note: The figure is based on CPS data from 1964 to 2003 and reports schooling achievement in percentages among 30-year-old workers.

Source: Heckman and Masterov (2007).

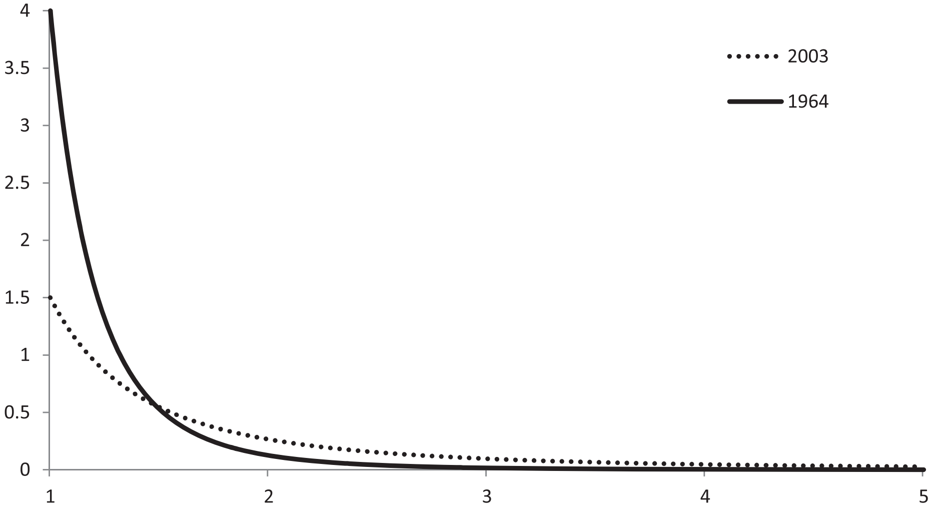

Figure 2. Examples of Pareto Densities Illustrating the Difference in the Dispersion of Schooling Achievement Between 1964 and 2003

Note: We characterize the year 1964 with $\sigma_\ell = 5$ and the year 2003 with $\sigma_\ell = 2.5$. The choice of these numbers and the scale of the horizontal axis—1 is the minimum skill level and 5 is the maximum—are arbitrary and for illustrative purposes only. The dispersion of schooling achievement among the 30-year-old workers has increased—which means that σ_ℓ has declined—over time.

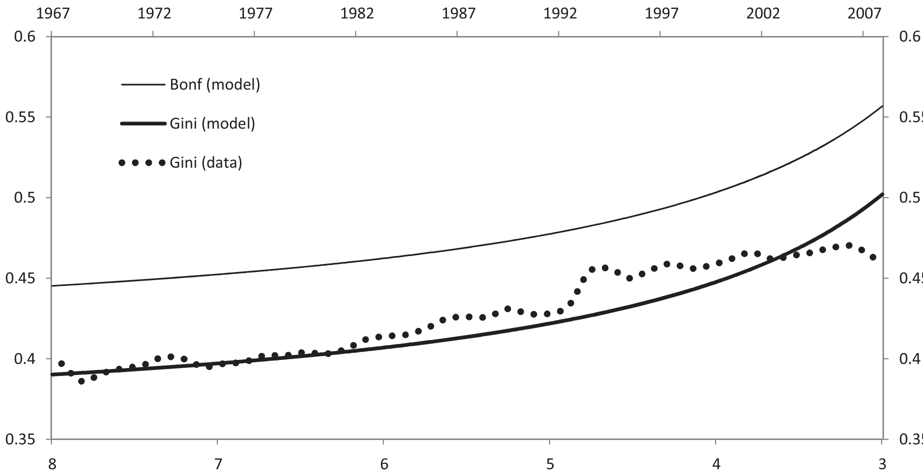

Figure 3. Data and Model Predictions

Note: The lower horizontal axis characterizes the increase in the dispersion of skills, i.e. the decrease in σ_ℓ, during this period. Bonf (model) and Gini (model) denote respectively the Bonferroni index and the Gini coefficient estimates that our theoretical model yields. The parameters $\sigma_k = 2.37$ and $\alpha = 0.7$ are chosen to match our model's estimates of the Gini coefficient to the data. Notice that we restrict $\sigma_\ell > \sigma_k$ parallel to our theoretical results.

In this subsection, we examine how an increase in the dispersion in skills affects the Bonferroni index versus the Gini coefficient.[8] The United States Census Bureau calculates the Gini coefficient for the U.S. economy from 1967. We first calibrate our model to match the theoretical Gini coefficient that our model produces to the empirical Gini coefficient for the U.S. economy. The parameters $\sigma_k = 2.37$ and $\alpha = 0.7$ joined with an increased dispersion of skills—roughly characterized by a decline in σ_ℓ from 8 to 3—yield the fit that Figure 3 displays. Using this parameterization, we derive the corresponding Bonferroni index from the model. We report two findings. First, we find that earnings inequality in the United States is substantially higher when it is measured in terms of the Bonferroni index. Second, it seems from Figure 3 that there is only a level shift when one switches from Gini to Bonferroni. However, the relative movement between the Bonferroni index and the Gini coefficient has a non-linear nature. Specifically, the Bonferroni index rises faster than the Gini coefficient when σ_ℓ moves from 8 down to 4.3. The Bonferroni index rises slower than the Gini coefficient after $\sigma_\ell = 4.3$ until it reaches 3.[9] The intuition is simple. When σ_ℓ is high, a lot of workers are low skilled and the right tail is thin. The effect of the transfers among the poor on inequality will become more important

[8]Since the Gini coefficient is naturally associated with the Pareto distribution, our model yields the following formula for the Gini coefficient:

$$\mathcal{G} = \frac{1}{2(\sigma_z - 1) - 1}.$$

[9]The threshold $\sigma_\ell = 4.3$ is specific to this example and may change depending on the configuration of the parameters.

in such a scenario. Conversely, when σ_ℓ is low, the right tail will be fat and the interactions among the poor will become less important. The parameter value $\sigma_\ell = 4.3$ is the inflection point in this relationship. This proves that, within the context of our model, the difference between these two measures is more complicated than what Figure 3 reveals. This numerical example suggests that, within the period analyzed, the Gini coefficient and the Bonferroni index differ in terms of both levels and changes. The difference comes from the fact that the Bonferroni index systematically focuses on the lower tail. We therefore conclude that, for the problems concerning the lower tail of the earnings distribution, using the Bonferroni index may lead to a more effective analysis of inequality.

This simple exercise highlights the importance of the choice of the inequality measure depending on the nature of the problem at hand. For example, if the research question is about the effect of low-skilled migration on earnings inequality, then the Bonferroni index might be a better candidate to measure inequality than the standard measures—such as the Gini coefficient—since it specializes on what happens on the lower tail. This paper suggests a potentially useful framework to employ the Bonferroni index in the study of earnings inequality with an emphasis on human capital. It is possible to extend this framework by relaxing or altering some of our assumptions. Uncovering the link between the Bonferroni index and some of the basic economic parameters was the major goal of this paper. We believe that this goal is achieved. Given its characteristics, the Bonferroni curve and the associated inequality index can be useful in both empirical and theoretical work in the economics discipline and our paper provides a first attempt in this direction.

4. Concluding Remarks

Our paper bridges the labor market pricing literature, the literature investigating the lower end of the earnings distribution, and the literature on the statistical indices of income inequality. We demonstrate that it is possible to formulate the Bonferroni index using an economic model with two-sided heterogeneity. Our model predicts that the interactions between the dispersion of worker skills, the dispersion of firm capital, and the share parameter determine the magnitude of the Bonferroni index.

The Bonferroni index is useful in answering questions about the effect of income transfers among the agents at the lower end of the income distribution. Many problems in public policy and development studies fall into this category. By proposing a theoretical link between the assignment model and the Bonferroni index, we show that it is possible to separately identify the effects of skills, capital, and factor shares on inequality when the focus of the analysis is the lower tail of the distribution. One caveat however is that since we focus on the "efficient" assignment of workers to firms, the applications of this framework should primarily be directed to countries with developed labor markets. We leave the task of studying the effects of certain frictions and labor market imperfections on the formulation of inequality indices to future research.

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