PROBLEMS IN WELFARE MEASUREMENT

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In its first part, this note is a summary of the French version of the discussion paper submitted to the present IARIW Conference.[1] The second part is devoted to theoretical aspects of aggregation, *as regards preference aggregation,* and gives necessary and sufficient conditions which are useful for our purposes.

PART ONE

1. INTRODUCTION

A great deal of attention is now being paid to national welfare measurement, the measurement by a unique number to be more precise. This has been reflected in the recent literature by attempts to measure national welfare by an adjusted GNP (e.g. Tobin and Nordhaus, Japanese NNW).

The purpose of our study is, first, to show that the measurement problem is only a problem of individual preference aggregation into a collective preference and nothing else and, secondly, to prove that there is no satisfactory solution to that aggregation problem. In other words: *national welfare regarded as a one-dimensional variable is an irrational concept* and so are the proxies for it (the "welfare oriented" aggregates).

We start with two remarks or questions as to why there is a long tradition in the national welfare literature of developing the subject in a strong national accounting climate and as to why GNP (with or without adjustments) is still thought of as being a good or pretty good welfare measure.

Turning then to the accounting point of view, it is easily seen that accounting methods do not provide any special means to solve the problem under consideration. Moreover, these methods may easily mislead, if applied to the matter in a forced way.

Turning thereafter to the price mechanism, one can quickly ascertain that the market (and, similarly, the social indicator approach) does not give any solution to the welfare problem despite appearances.

Thus, such approaches fall far short of what is expected by failing to refer to any explicit definition of national welfare. Hence, our next step will be the statement of a workable welfare definition in accord with the prevalent standpoint in this subject. We will recognize that the whole problem reduces to preference aggregation and nothing else.

We are therefore compelled to center the discussion upon aggregation analysis, before being able to make any progress. It will then appear that

[1] "Le bien-être national et la possibilité de sa mesure"—Oleg Arkhipoff—14ème Congrès de l'IARIW à Aulenko, Finlande, Août 1975. This French version will be published *in* "Peut-on mesurer le bien-être national?"—*Les Collections de l'I.N.S.E.E.*, série C, n° 41, mars 1976, Institut National de la Statistique et des Etudes Economiques, 18, Bd A. Pinard, 75675 Paris, Cedex 14.

preference aggregation may only be adequate with respect to the qualifications usually required, if and only if there exists one and only one dictator and no other (real) dictatorial group. In other terms, such a reservation is equivalent to the impossibility of aggregating satisfactorily, since any voting becomes meaningless: stated in a one-dimensional framework, the problem of national welfare has no answer.

Incidently, a conclusion hitherto largely neglected comes out: any aggregation is impossible—with or without a dictator—if individual and collective rationalities are of different species; and individual rationality must always be stronger than collective: an aggregation is always a loss of rationality.

2. The Accounting Approach

Much national welfare literature presents the welfare problem from an exclusively accounting point of view. We shall first criticize such an approach.

Economic history points out that GNP (or more precisely national income) was conceived in the beginning without any reference to an accounting framework. Similarly, one could expect that the national welfare problem should be first stated without accounting *a priori.*

Several reasons might explain why the discussion on national welfare measurement frequently focusses on accounting problems. In brief, many people think that if the items which should constitute the ideal national welfare aggregate are placed in an accounting framework, one may be sure that the analysis is consistent, exhaustive and neutral as regards economic theory.

Before discussing the reality of the qualities of accounting systems, it is necessary to define what an accounting system could be.[2]

Let us then suppose that the following terms or statements are intuitively known and meaningful: accounting period, the beginning and the end of this period, item t to be recorded, its value $v(t)$, the set T of such items, account i, the set I of such accounts, credit c_i and debit d_i of account i, opening balance s_i and closing balance s'_i of i, "to credit (or to debit) the account i with $v(t)$."

Let D (or C) be the sum of all d_i (or c_i) and S (or S') denote the sum of all s_i (or s'_i). Let us assume that $v(t)$, d_i, c_i are positive numbers, whereas s_i and s'_i may be also negative. Let S^+(or S'^+) denote the sum of all positive s_i (or s'_i) and let S^-(or S'^-) be the sum of all negative s_i (or s'_i) : $S = S^+ - S^-$); all these variables are defined with respect to a given accounting period.

Thereafter, let us state four axioms:

Double entry axiom: For any t, there exist one and only one account i debited with $v(t)$ and one and only one account $j (j \neq i$ or $j = i)$ credited with $v(t)$. As a result, the recording of T implies $D = C$.

Opening and closing balance axiom: For every account i, the equality below holds:

$$s_i - d_i + c_i = s'_i.$$

Balance sheet axiom: $S = 0$; in other words: $S^+ = S^-$.

[2]See O. Arkhipoff: "Finances publiques"—Ecole de Statistique, Abidjan 1972/73 and "Finance et comptabilité publiques"—Institut de Formation Statistique, Yaoundé, 1969.

Balance to next account axiom: For every account i, the closing balance of a period is equal to the opening balance of the following period.

By *balance sheet* is meant a two column table such that the first column records all negative s_i (Assets) and the second one registers all positive s_i (Liabilities). We shall distinguish an *opening balance sheet*, which records all s_i, from a *closing balance sheet*, which enters all s'_i. All these concepts are given with respect to a fixed accounting period.

The obvious main point is that the four axioms above are independent one from another. Therefore, if, say, a system satisfies the first axiom, this fact does not imply at all that the system must also satisfy the three remaining axioms.

Standard book-keeping relies upon the four axioms together but national accounting satisfies only the first one and even this statement may be criticized.[3]

As both systems and the like are termed accounting, it is then obvious that the notion of an accounting system itself is indefinite and so are its qualities.

The second point is that consistency and exhaustivity are not properties solely related to the four axioms aforesaid: consistency and exhaustivity (in a limited sense) depend on the existence of relations in these axioms but do not rely at all upon the *nature* of these ones (in an "accounting" system, the relations are linear with parameters equal to ± 1).

Exhaustivity is equivalent to consistency if by that one means that accounting prevents one from wrong recording (e.g. forgetting contra-entries). But if by exhaustivity one means that ability to correctly define the set T—and this is very important in the national welfare measurement problem—accounting exhaustivity is deceptive: this fact is particularly obvious when one establishes a subsistence production account in the national accounts of an underdeveloped country.

It may be also noticed that, as a framework for co-ordination of statistics to be registered in its accounts, national accounting provides nothing more or less than what can give consistency and exhaustivity which are related to it. Moreover, this co-ordination problem is certainly premature in the present state of the national welfare measurement question.

The last quality frequently related to national accounting is neutrality as regards economic theory. On philosophical grounds, this is certainly a moot point and it has to be emphasized that any identification of an accounting system to a given domain of reality always entails some assumptions and approximations, even when one handles ordinary book-keeping! All that could be said about neutrality is that there is a large consensus in favour of using national accounting for special purposes, but nobody can prove that this evidence will remain when considering the national welfare measurement problem.

Thus and *a priori*, there is no evidence of any accounting necessity when trying to measure national welfare. On the contrary, the analysis of the problem points out many nonaccounting components in national welfare (e.g. income

[3]See: "Les limitations de la comptabilité nationale"—*Revue Economique*, n° 5, September 1967, Paris; and: "Le problème de la synthèse économique—un exemple: le traitement sur ordinateur de la compatibilité nationale camerounaise"—*Revue Economique de Madagascar*, n° 6, 1971, Editions Cujas, Paris—by Oleg Arkhipoff. See also: "The synthesis and reliability of national accounts by means of electronic data processing"—2nd International Conference on Dynamic Modelling, Vienna, January 1977 (IFAC/IFORS/IIASA).

inequalities). And if one encounters "flows" which are easily recorded in an accounting way, sometimes it is not obvious at all that the "stocks" related to such flows satisfy the second axiom: that is certainly one of the most dangerous linguistic traps!

As a conclusion, it can be asserted that the accounting approach is at best premature and certainly dangerous if it contributes to the distortion of facts for accounting purposes and to the postponement of the very problem which presently matters: what is meant by national welfare?

3. The Economic Theory Trap—The Social Indicator Illusion

The time has now come to try to understand why GNP is sometimes, not to say often, thought of as a measurement of national welfare, or at least a proxy for it.

The core of such an irrational belief is a trivial sophism: "when I get richer, I am happier;[4] an increase in GNP enables everybody to get richer; therefore, the nation is happier."

A more sophisticated version of the same sophism is the so-called Pareto ordering on which is based the theory of economic equilibrium and optimum: the present economic state (read: aggregated consumption) is preferred to the old one, if every consumer prefers his own new consumption.

The second illusion is the belief that the market mechanism solves *without any dictator* the difficult problem of preference aggregation. Thereby, many deeply believe there is solely an academic connection between the national welfare problem and preference aggregation!

Let us examine briefly why the market mechanism does not solve any aggregation at all. The central point is the so-called Pareto ordering: it is obvious that this ordering is not a "collective" aggregate ordering, because it is an ordering *provided by a dictator*: the economist himself. In addition to what has been said, the ballot is certainly unfair, since the consumer is questioned on his *own* consumption, whereas the final result deals with *aggregate consumption.* Therefore, such an aggregation does not fit, *not because the ordering is not complete* as frequently asserted, but only because it is dictatorial. As a result, one can conclude that the social optimum theory does not provide any satisfactory aggregation and if one wishes to prove that in certain conditions GNP measures national welfare variations, one must *assume* the existence of a genuine collective preference.

The social indicator approach is certainly much more subtle, since it seems to take into consideration items (such as interdependency) which classic consumer theory drops and, thus, to solve the welfare definition problem by means of adequate social indicators and since it seems to settle the aggregation problem by showing that there is a positive correlation between the GNP and appropriate indicators, whose increase is always (?) experienced as a positive contribution to national welfare; in addition, the utility problem seems to be equally solved, since GNP is equivalent to the collective ordering obtained in such a statistical way.

[4]The author has not yet encountered anybody who looked unhappy when speaking of an increase of his personal income he got or hoped to get.

The weakness of this second approach remains nevertheless the same and a similar argument, as above, applies to it: let us assume for a moment that it is possible to get a general concensus about a suitable set of indicators providing an adequate definition of national welfare, and that *all* these indicators are correlated with GNP as desired; the fact remains that, as above, each voter is "questioned" on each indicator separately, whereas the collective answer (economist's answer!) itself proceeds on the plane of all these indicators considered together simultaneously.

Apart from the possible doubts about statistical evidence, we are confronted once more with an unfair and dictatorial ballot.

In conclusion, neither market theory nor the social indicator approach gives any solution to the welfare measurement problem, at least as far as one-dimensional welfare is concerned. (For further details see "A propos d'agrégation, d'utilité collective et de bien-être national"—Revue de Science Financière n°-1, janvier/mars 1977, LGDJ, Paris).

4. A Definition of National Welfare

Apart from the fact that either the above welfare approaches are premature, if not irrelevant, or they depend on a crude sophism, all these attempts usually fall short of clearly defining what national welfare should be.

Turning now to this important part of the problem, we shall try to provide a general definition of national welfare with the following unique reservation: the point under examination is that of a one-dimensional welfare concept, *as usually asked for.* We think that the definition set out here formalizes the prevalent ideas about the matter.

Our definition rules as follows:

Welfare states. The best way to define a welfare state is probably to proceed from the standpoint of social indicators. Consider then several given social indicators: $(x_1), (x_2), \dots, (x_k)$. Set $(x_i) = x_i$, $i = 1, 2, \dots, k$, x_i being appropriate given numbers: the sequence $x = (x_1, x_2, \dots, x_k)$ will be termed a *welfare state* and X will denote the set of all possible x.

An indicator (x_i) could be a value (say, GNP *per capita*, for example), an index (as a price index), a number (e.g., a number of physicians, an average life expectancy), a ratio (e.g. a percentage of low incomes), etc.

Preference. Let i be any member of a collectivity N and let us assume that any member or voter i is able to order X. In other words, he is able to say whether he prefers x to y or not, for every x and y taken in X. Let R_i denote the ordering or *individual preference* chosen by i.

A *collective preference* Q is defined as a preference or an ordering which is a function F in all R_i ($i = 1, 2, \dots, n$ [5]) : $Q = F(R_1, R_2, \dots, R_n)$.

Hence *national welfare* is defined as any pair (x, Q). Such a definition is not necessarily one-dimensional and is open.

By *preference* will be meant any *transitive* binary relation; that is to say: for any x, y, z, if one prefers x to y and y to z, one necessarily prefers x to z.

[5] n is the number of voters.

Indeed, we cannot and do not wish to justify such a preference conception; but we may notice that this is how preference is customarily thought of. Defining nontransitive "preference" is certainly a fruitful idea, which however may sometimes lead to misleading attitudes. To underline the fact that people are not rational and their preferences never transitive does not cancel out the other fact that a collective preference is always assumed rational and, here, rational is always translated by "transitive". This device, when conceived as a possible clue to an impossible aggregation problem, is but a mishit, for intransitive individual preferences can never then be aggregated into a transitive collective preference, as will be established in the second part of the present paper. To define intransitive "preference" is to remove the difficulty without solving the aggregation problem: the aggregation impossibility is not due to transitivity in particular *but is due to complexity*. Thus, one losses intelligibility without any profit. Finally, a numerical conception of welfare entails transitivity at least.

As a matter of fact, the choice of preference types is not at all free, as soon as national welfare is defined as a number. This demands at least a *total* preference: for any x and any y, necessarily, x is preferred to y or y to x. And it will be shown further on that beyond this level aggregation is impossible without an absolute dictator. Something more must be pointed out: even if a voting procedure admits an absolute dictator, arbitrariness is not avoided. Indeed, a numerical ordering necessitates such complex properties that even an absolute dictator cannot order satisfactorily the set X wihout the use of mathematical algorithms, which have have to be chosen arbitrarily.

This is the place to mention the problem of social choice: it is a standard way to study preference aggregation, as choice function is a more general concept than preference.

The social choice problem is basically different from the social preference problem, as far as aggregation is concerned, and the methods used in the second part of this paper do not apply; for this reason the problem will not be considered here.

Nonetheless, it seems obvious that if preferences cannot be directly aggregated in a satisfactory way, the same problem may certainly not be solved by an *indirect* method, even if an adequate way is found for aggregating individual choice functions. One may expect to find again the same argument as that of market objection: a revealed preference is indeed an economist's definition and not the adequate result of a fair voting (more precisely: of any *possible* free voting!).

What is a Satisfactory Preference Aggregation?

Any preference cannot deserve the adjective "collective", if the aggregation function F is given without particular prescribed qualifications. An aggregation could only be termed satisfactory when certain conditions are fulfilled.

Suitable conditions on F will be stated in a more precise way in the second part of the present paper. The spirit of these qualifications is the following:

1. All R_i must be taken into account: the existence of an absolute dictator (see further) or a "collective" preference given outside the ballot contradicts this first qualification.

2. The type of R_i being given, any voter can vote in any possible way as regards the fixed kind of individual preference. Even if one votes, or *may vote*, in a manner which seems improbable or contrary to ethics or common sense, a satisfactory voting procedure must taken such ballots into account, that is to say it must not exclude them, and *must always give an unambiguous collective answer.*[6]

3. If the collective answer is that of an unanimous group of voters, a larger unanimous set of voters including the proceding one will carry the same collective answer (that is the definition of dictatoriality—see Part Two).

These qualifications are surprisingly few, "reasonable" and, therefore, hard to remove. One possible way to alter successfully the conditions on F is to choose satisfactory voting procedures as regards the above qualifications with a small number of slightly probable incoherent ballots: if an incoherent ballot occurs, then the voting procedure is changed. Unfortunately, such permissive rules lead to wavering algorithms for national welfare.

The well-known Arrow Theorem states that for a particular class of voting procedures (individual preferences are transitive and total and collective preference is negatively transitive and asymmetric), if there are never inconsistent collective answers, then there exists a dictator, and it is usually argued that this is an impossibility theorem. Such a constraint is certainly too drastic and political science analyzes forms of government in which a dictator or a directory is checked by another dictator or directory: this point will be explored in the next pages.

In conclusion and by anticipating the results established in the next part of this paper, we can say that preference aggregation is strictly impossible (there must exist one and only one dictator without any other dictator or directory), as regards the national welfare problem: if a one-dimensional and satisfactory national welfare concept is introduced in a discourse, such a discourse is necessarily inconsistent and hence one can prove anything from these premises.

PART II

THE AGGREGATION PROBLEM

1. INTRODUCTION

The problem of welfare measurements, as has been shown, is clearly related to the possibility that individual preferences should be aggregated into a satisfactory "collective" preference.

If one is interested in getting a collective utility—and this is the problem analyzed in this paper—it is clear that collective preference should have very special properties. We can nevertheless get equally interested in any collective preference, provided it is transitive and suitable. We must also be able to answer any question such as: is it possible to aggregate transitive individual preferences into a transitive collective preference? (the answer is "yes")—or, is it possible to aggregate negatively transitive preferences into a transitive collective preference? (the answer is "no", always "no")—or is it possible to aggregate individual utilities

[6]For example, a Pareto voting procedure is unambiguous: if x and y are optimums, the collective answer is "x and y are not compared" (Q is not total).

into a collective utility, if we admit the existence of a dictator, checked by another dictator or/and a dictatorial group of voters not including a dictator itself (say, a Directory)? In other words: we wish to have at our disposal a list of necessary *and* sufficient conditions for *every* case which may be relevant to our problem, with the possibility, at each time, to exactly appreciate the quality of the voting procedure then prescribed. Therefore, it is necessary to explore completely the field of all types of preference applicable to our aim. That was undertaken in [1] in an Arrowian way and more directly in [2]. Here, we shall recall all these preceding results and some new theorems established *in* "Peut-on mesurer le bien-être national?", already cited. We shall begin by analyzing the aggregation problem at a high level of generality and prove some results, which were just quoted in [2].

2. General Aggregation: Some Results

At a very high level of abstraction, an aggregation μ can be considered as a mapping: $S \to S'$, where S (individual or micro-level) and S' (collective, aggregate or macro-level) are sets of elements which are built up from propositional functions, with constraints put on both levels: an individual constraint or rationality D and a collective one E.

It is not necessary, in this paper, to provide a general classification of the various possible types of aggregations μ. We shall only focus out attention on the class of aggregations μ which are entirely defined by D, E and a certain parameter δ: $\mu = \langle D, E, \delta\rangle$, where D is a propositional function of a variable x (and, maybe, of variables $x, y, z, \ldots$ chosen in an auxiliary set X) and where E is a propositional function too of variables r, δ (and, maybe, $x, y, z, \ldots$). More concretely, r will be thought of as a "ballot" and μ as "voting procedure".

Let us consider the following propositional function $H(\delta)$, which means: "the voting procedure $\mu = \langle D, E, \delta\rangle$ is *coherent*",

$$H(\delta) \Leftrightarrow (\forall r)(D(r) \Rightarrow E(r, \delta)).$$

If not, that is to say if non $H(\delta)$ is true, μ will be termed *incoherent*.

If $D(r)$ is true, r is said to be D-possible. If non $E(r, \delta)$ is true, r is called E-critical (according to procedures of δ-type).

It is also illuminating to introduce the following sets

$$\Delta(D, E) = \{\delta; H(\delta)\},$$

$$\mathcal{D}(D) = \{r; D(r)\}$$

$$\mathcal{C}(E, \delta) = \{r; \text{non } E(r, \delta)\},$$

$$\mathcal{H}(D, E, \delta) = \mathcal{D} \cap \mathcal{C}^C = \{r; D(r) \,\&\, E(r, \delta)\},$$

We shall first note that *if D does not imply E (for every r), then the voting procedure μ is never coherent, with or without dictators.* In other terms: any aggregation always entails a loss of rationality, as already noticed above.

Theorem 01: $D_1 \Rightarrow D_2$ is equivalent to $\mathcal{D}(D_1) \subset \mathcal{D}(D_2)$ and $D \Leftrightarrow D_1 \,\&\, D_2$ is equivalent to $\mathcal{D}(D) = \mathcal{D}(D_1) \cap \mathcal{D}(D_2)$.

Theorem 02: $\delta \in \Delta \Leftrightarrow \mathcal{D} \cap \mathcal{C} = \emptyset$, $\delta \in \Delta \Leftrightarrow \mathcal{H} = \mathcal{D}$.

Write now

$$\mu_1 = \langle D_1, E_1, \delta\rangle, \quad \mu_2 = \langle D_2, E_2, \delta\rangle, \text{ and define}$$

$$\mu = \mu_1 \times \mu_2 = \langle D_1 \;\&\; D_2, E_1 \;\&\; E_2, \delta\rangle,$$

$$\mu = \mu_1 + \mu_2 = \langle D_1 \text{ or } D_2, E_1 \text{ or } E_2, \delta\rangle,$$

$$\mu_1 \to \mu_2 \quad \text{iff} \quad (D_1 \Rightarrow D_2) \;\&\; (E_1 \Rightarrow E_2)$$

Thereby, we can state

Theorem 03: Let C_1 (resp. C_2) be the necessary and sufficient condition for μ_1 (resp. μ_2) to be coherent. Then C_1 & C_2 is the sufficient condition for $\mu_1 + \mu_2$ (or $\mu_1 \times \mu_2$) to be coherent with respect to E_1 or E_2 (or E_1 & E_2).

Theorem 04: With the same notations, suppose that for any δ taken outside $\Delta_1 = \Delta(D_1, E_1)$(resp. $\Delta_2 = \Delta(D_2, E_2)$), there exists a ballot r taken in $\mathscr{D}(D_1 \;\&\; D_2)$ and E_1-critical (resp. E_2-critical). Then, the necessary and sufficient condition for $\mu = \mu_1 \times \mu_2$ to be coherent is C_1 &C_2.

This theorem is constantly and implicitly used in the proofs of [1], [2], and, here, of § 3: it is easy to verify that the additional assumptions always hold in all cases considered both here and there.

Theorem 05: If $\mu_1 \to \mu_2$ and $\delta \notin \Delta(D_2, E_2) \Rightarrow (\exists r)(r \in \mathscr{D}(D_1) \;\&\; \text{non } E_2)$, for any δ, then $\delta \in \Delta(D_1, E_1) \Rightarrow \delta \in \Delta(D_2, E_2)$ that is to say, $C_1 \Rightarrow C_2$.

The proof is immediate and the only interesting point is that the proof only relies upon the assumption $E_1 \Rightarrow E_2$, when considering $\mu_1 \to \mu_2$: what is the meaning of the assumption $D_1 \Rightarrow D_2$?

Consider then any δ such that $(\exists r)(D_2 \;\&\; \text{non } E_2)$ and $(\forall r)(D_1 \Rightarrow E_2)$. Suppose also D_1. Then for any r, E_2 and D_2 are true. Therefore $D_1 \Rightarrow D_2 \;\&\; E_2$, that is to say $\mathscr{D}(D_1) \subset \mathscr{H}(D_2, E_2, \delta)$: if $\mu_1 \to \mu_2$ and if C_1 does not imply C_2, then for any δ, $\mathscr{D}(D_1)$ is included in $\mathscr{H}(D_2, E_2, \delta)$ or, in other words, all ballots r taken in $\mathscr{D}(D_1)$ are not critical with respect to μ_2. This is the general result quoted in [2, Theorem 20].

3. Dictatorial Aggregations

The aggregation theory, though already severely restricted, remains still too general. It is necessary to indicate how to get the more familiar theory used in [1] or [2] which proceeds from dictatorship–majority conceptions.

Thereafter, we shall briefly summarize the definitions and results stated in the papers just quoted above: these theorems settle the national welfare problem, when considering transitive preferences. This is not the place to investigate nontransitive "preferences": all we can say here is that conclusions are then the same and so are possibility conditions . . .

3a. *QD-voting Procedures*

We shall henceforth restrict our analysis to a particular class of voting procedures, which will be termed *QD*-procedures for the sake of simplicity. Three procedures will be defined as follows:

Consider a given list $\mathscr{L}$ of "questions" or statements Q_j ($j=1,2,\ldots,s$)[7] and an auxiliary set N (the set of voters i)—Define $r(Q_j)=\{i, Q_j\}$ as the set of all i who vote "Q_j" ("yes" or "Q_j is true"), when the ballot is r. Let a two element set $\mathscr{E}=\{d, nd\}$ be given; let us introduce a mapping $\delta: \mathfrak{P}(N)\to\mathscr{E}$ and define the following procedure: the aggregate level will consist of questions $Q_1,\ldots,Q_s$ and so does the individual level; at aggregate level

$$Q_j \text{ is true} \quad \text{iff} \quad \delta(r(Q_j))=d$$

$$Q_j \text{ is false} \quad \text{iff} \quad \delta(r(Q_j))=nd$$[8]

To clarify the definitions we shall say, for any set S taken in $\mathfrak{P}(N)$:

$$S \text{ is dictatorial (or "}d\text{")} \quad \text{iff} \quad \delta(S)=d,$$

$$S \text{ is non dictatorial (or "}nd\text{")} \quad \text{iff} \quad \delta(S)=nd.$$

Moreover, we introduce the following conditions on δ:

Condition $4b$: δ is a mapping of $\mathfrak{P}(N)$ *onto* $\mathscr{E}$,

Condition $1b$: Given a suitable number of any sets A_j taken in $\mathfrak{P}(N)$, there exists an r such that:

$$r(Q_1)=A_1, \qquad r(Q_2)=A_2,\ldots, \qquad r(Q_j)=A_j,\ldots$$

We need something more, a "definition" of dictatoriality "d" and nondictatoriality "nd".

Condition $2b$:

$$V\subset W \;\&\; \delta(V)=d \Rightarrow \delta(W)=d,$$

$$V\subset W \;\&\; \delta(W)=nd \Rightarrow \delta(V)=nd,$$

for any sets V and W taken in $\mathfrak{P}(N)$.

Finally, the fact that we choose $\mathscr{E}$ as a two element set *can be expressed as condition* $6b$.

3b. *The Traditional Problem of Preference Aggregation*

We have established some results which entirely settle the question of preference aggregation in a negative way in [1] and [2]. These results and then introduced definitions will be briefly recalled here and one can easily appreciate the close connection between the general approach provided just above in § 3a and the present one which is obviously a QD-procedure.

First some notations and terminology will be recalled:

N—set of voters $i, j,\ldots$

X—set of alternatives $x, y, z,\ldots$

$\mathscr{R}$ denotes the class of binary relations of the type R defined on X.

$\mathscr{R}^n$—set of ballots $r=(R_1, R_2,\ldots, R_n)$, n being the number of voters.

$R_r^{xy}=\{i; xR_iy\}$ is the set of all i which order x and y in the same way; R_i denotes the binary relation chosen by voter i in $\mathscr{R}$.

$|E|$ stands for the cardinal power of the set E; $|N|=n$, $|X^2|=m$.

E^c is the complement of the set E.

[7] Q_j can depend on variables $x, y, z,\ldots$ see [3].

[8] $\mathfrak{P}(N)$ denotes the family of all subsets of N, including N itself.

F is the mapping of $\mathscr{R}^n$ into $\mathscr{Q}$, satisfying several conditions which will be stated further on; F is called a Social Welfare Function. We shall note $Q = F(r)$, $Q' = F(r')$, . . . as usual.

Our basic problem is this: given $\mathscr{R}$ and $\mathscr{Q}$ and a set of conditions on F, does F exist? Namely, the voting procedure $(\mathscr{R}, \mathscr{Q}, F)$ will be said *coherent* or *consistent* if and only if for every r taken in $\mathscr{R}^n$, $Q = F(r)$, belongs to $\mathscr{Q}$; the procedure will be considered *incoherent* or *inconsistent* otherwise; if $F(r)$ does not belong to $\mathscr{Q}$, r will be termed *critical.*

Like Fishburn in [4], we shall number the next classic binary relations in the following terms:

(1) Reflexivity	xRx;
(2) Antireflexivity	non xRx;
(3) Symmetry	xRy implies yRx;
(4) Asymmetry	xRy implies non yRx;
(5) Antisymmetry	xRy & yRx implies $x = y$;
(6) Transitivity	xRy & yRz implies xRz;
(7) Negative transitivity	non xRy & non yRz implies non xRz;
(8) Totality	xRy or yRx;
(9) Completeness	$x \neq y$ implies xRy or yRx;

for every x, y and z.

$R(6)$ will stand for a relation R entirely defined by the poperty of transitivity, etc.; and we shall write $R \geq R(6)$ if R is transitive (R can have other properties), etc. R will stand for a relation with undefined properties.

A voting procedure is partly characterized by R and Q and we shall often denote it by RQ or $R(6)$ $Q(4, 6)$, etc.

It is perhaps not entirely unfruitful to note that transitivity *does not* imply negative transitivity and *vice versa*. The proofs are both simple and quite similar. For instance, let us prove that negative transitivity does not imply transitivity: it will be sufficient to exhibit a counter-example such that R is negatively transitive but not transitive:

non xRy,	yRz,	zRz.
non yRx,	xRz,	
non xRx,	zRx,	
non yRy,	zRy,	

Changing "no" into "yes" and vice versa, we get an example of a relation R which is transitive without being negatively transitive.

In conclusion, we can never aggregate negatively transitive (resp. transitive) preferences into a transitive (resp. negatively transitive) collective preference.

Now, one can easily establish the next theorem

Theorem 06: The necessary and sufficient condition for R to be

(3) symmetric is $R_r^{xy} = R_r^{yx}$

(4) asymmetric is $R_r^{xy} \cap R_r^{yx} = \varnothing$

(5) antisymmetric is $x \neq y$ implies $R_r^{xy} \cap R_r^{yx} = \varnothing$

(6) transitive is $R_r^{xy} \cap R_r^{yz} \subset R_r^{xz}$

(7) negatively transitive is $R_r^{xz} \subset R_r^{xy} \cup R_r^{yz}$

(8) total is $R_r^{xy} \cup R_r^{yx} = N$

(9) Complete is $x \neq y$ implies $R_r^{xy} \cup R_r^{yx} = N$,
for any x, y, z and r.

Let V be any set of voters. V will be called *decisive for x against y*, or *(x, y)-decisive*, if $V \subset R_r^{xy}$ implies xQy ($Q = F(r)$). V is called *(x, y)-nondecisive* if $R_r^{xy} \subset V$ implies non xQy. V is *(x, y)-mixed* if there exists a ballot r such that $V \subset R_r^{xy}$ implies non xQy and a ballot r' such that $R_{r'}^{xy} \subset V$ implies $xQ'y$. If, for every x and y, V is (x, y)-decisive or (x, y)-nondecisive V will be said to be *dictatorial* or *nondictatorial*. Sometimes we shall set d for *dictatorial* and *nd* for *nondictatorial*.

V will be said to be *mixed* if it is not dictatorial nor nondictatorial (see [2]).

If V is a dictatorial singleton, V will be called a *dictator*. If V is dictatorial and contains no dictator and is different from N, V will be called a *real dictatorial college* (a directory).

D is the family of all dictatorial sets and D' is the family of all nondictatorial sets.

Now, the following theorem is immediate:

Theorem 07: Whenever W is included in V: W (x, y)-decisive (dictatorial) implies V (x, y)-decisive (dictatorial); V (x, y)-nondecisive (nondictatorial) implies W (x, y)-nondecisive (nondictatorial); to conclude: if U is included in V and includes W and if V and W are (x, y)-mixed (mixed), U is (x, y)-mixed (mixed).

The condition 2b of § 3a is included in that statement.

The following symbols will stand for the below statements:

$H(i)$ the voting procedure is consistent with critical configuration i (see below),

$K(k)$ there exist at least k dictatorial sets whose intersection is nondictatorial,

$Y(k)$ there exist at least k nondictatorial sets whose union is dictatorial,

$J(k)$ there exist at least k disjoint dictatorial sets,

$C(k)$ there does not exist any partition of N into k nondictatorial nonvoid sets,

$R(k)$ there does not exist any covering of N into k nondictatorial (nonvoid) sets,

K, Y, C, R, J, will be sometimes put when $k = 2$. When k is infinite, we shall write K^*, Y^*, C^*, R^*, J^* respectively,

$T!$ there exists one and only one dictator,

$T(k)$ there exists at least k dictators,

$G(1)$ there exists at least one real dictatorial college,

Z there exists one and only one dictator and no real dictatorial college:

$Z \Leftrightarrow T!$ & non $G(1)$,

$F \Leftrightarrow D$ is a filter,

$L \Leftrightarrow D$ is an ultrafilter,

An *absolute dictator* is a dictator which satisfies condition Z.

It is particularly easy to establish the next results:

Theorem 08:

$$\text{non } J(k) \Rightarrow \text{non } J(k-1), \qquad C(k) \Rightarrow C(k-1),$$

$$\text{non } K(k) \Leftrightarrow \text{non } K, \qquad \text{non } Y(k) \Leftrightarrow \text{non } Y,$$

$$\text{non } Y \Rightarrow \text{non } C(k), \qquad \text{non } K(k) \Rightarrow \text{non } j(k),$$

$$C(k) \,\&\, \text{non } J(k-p) \Leftrightarrow C(k-q) \,\&\, \text{non } J(k),$$

$$\text{with } k-p, k-1 \geq 2 \text{ and } p, q \geq 0,$$

$$L \Leftrightarrow \text{non } Y \,\&\, \text{non } J,$$

$$\Leftrightarrow \text{non } K \,\&\, C,$$

$$\Leftrightarrow C(p) \,\&\, \text{non } J(q), \text{ with } p \geq 3 \text{ or } q \geq 3.$$

Theorem 09:

$$\text{non } J \Leftrightarrow (Vd \Rightarrow V^c nd),$$

$$C \Leftrightarrow (Vnd \Rightarrow V^c d).$$

Theorem 10: In the present theory, $F \Leftrightarrow \text{non } K$ and $C(k) \Leftrightarrow R(k)$.
Theorem 11: When the number n of voters is finite,

$$L \text{ is equivalent to } Z\ (T! \,\&\, \text{non } G(1)).$$

$$\text{non } Y \text{ is equivalent to } T(1) \,\&\, \text{non } G(1).$$

For any n, $T(1)$ & non $G(1) \Rightarrow$ non Y and $Z \Rightarrow L$
Theorem 12:

$$\text{non } Y^* \Rightarrow \text{non } Y,$$

$$\text{non } Y^* \Rightarrow T(1),$$

$$\text{non } Y^* \,\&\, \text{non } J \Leftrightarrow Z.$$

3c. *Conditions on F. Possible and Critical Configurations.*

We shall assume that the Social Welfare Function F satisfies the three conditions given below.

Condition 1*a*: given a sequence of m sets of voters: $A_1, A_2, \ldots, A_m$—there always exists a ballot r such that: $R_r^{x_1y_1} = A_1$, $R_r^{x_2y_2} = A_2, \ldots, R_r^{x_my_m} = A_{m'}$ for every $x_1, y_1, x_2, y_2, \ldots, x_m, y_m$ taken in a subset S containing at least k elements (or alternatives).

In this paper we set $k = 3$.

This condition is similar to Arrow's Condition 1, see [6] and the connection between Condition 1a and Condition 1b is obvious.

Condition 4*a*: the families D and D' are not empty.

This is the condition 4b.

Condition 6*a*: any set belonging to $\mathfrak{P}(N)$ is in an exclusive manner either dictatorial or nondictatorial (in particular, there does not exist any mixed set).

This is the condition 6b, when we handle QD-procedures in general.

We now shall describe four basic critical configurations (more precisely: classes of configurations). These configurations involve no more than three distinct alternatives and that is the reason why in Condition 1ak was put equal to 3.

Configuration I or Condorcet Configuration:

$$*Vd, \qquad V'd, \qquad V''nd*$$

If we can find in $\mathfrak{P}(N)$ such a sequence of three sets, Condition 1a implies that such a configuration is critical: let us assume that there exists a possible ballot r such that:

$$R_r^{xy} = V, \qquad R_r^{yz} = V', \qquad R_r^{xz} = V''$$

and that the voting result Q is transitive. The rules of voting imply: xQy, yQz and non xQz, whereas the transitivity of Q implies xQz, whenever xQy and yQz: the procedure is inconsistent.

Configuration II or Arrow Configuration

$$*Vd, \qquad V'nd, \qquad V''nd*$$

Let us assume $Q(7)$ and suppose that there exists a ballot such that:

$$R_r^{xy} = V', \qquad R_r^{yz} = V'', \qquad R_r^{xz} = V$$

Then non xQy and non yQz and xQz. This leads to a contradiction since $Q(7)$ implies non xQz.

In the same way, are also critical:

Configuration III or Antisymmetric Configuration

$$*Vd, \qquad V'd*$$

Configuration IV or Totality Configuration

$$*Vnd, \qquad V'nd*$$

We have shown in [1] and [2], that for a voting procedure $R(4,7)Q(4,7)$ (the one investigated by Arrow), the present system of conditions is equivalent to Arrow's Conditions 1, 2′ and 4. Incidently, a condition analog to Arrow's Condition 3 (Independence or Irrelevant Alternatives) is not necessary to our purposes, as one will appreciate in the next pages.

From inspection of the three proceeding conditions, it is obvious that *Corollary*: N is dictatorial (Arrow's Consequence 3) and the empty set $\varnothing$ is nondictatorial.

The proof is trivial and depends upon Condition 4a and Theorem 07.

By inspection of Theorem 06 it is obvious that the nature of R determines the class of all *possible* ballots r(i.e. $\mathscr{R}^n$) and moreover the class of all *possible* configurations. A possible configuration A_i will be termed *critical* if there exists a critical ballot r such that:

$$R_r^{x_1y_1} = A_1, \ldots, R_r^{x_my_m} = A_m.$$

3d. *Basic Possibility and Impossibility Theorems for Transitive Preferences.*

We can now recall different necessary *and* sufficient conditions for voting procedures to be consistent (Possibility Theorems).

Here will be investigated a large and classic range of binary relations; we shall only consider reflexive, irreflexive, transitive, negatively transitive, symmetric, antisymmetric, asymmetric, total and complete relations and their combinations up to $R(5, 6, 8)$, $R(5, 6, 9)$, $R(4, 7, 9)$. When we term R or Q to be undetermined, R or Q will be any relation taken in the class above.

To begin with, we formulate the most important basic possibility theorem:

Theorem 13: For every n (finite or no), when R is at least transitive, the necessary and sufficient condition to aggregate R into a transitive collective Q is that D be a filter (that is to say, here, non$K$).

$$R \geq R(6) \Rightarrow (F \Leftrightarrow H(I))$$

It must be once more emphasized that *if R is no more transitive, the aggregation of individual preferences into a (at least) transitive collective preference is impossible.*

It must also be noted that the aggregation of transitive individual preferences into a transitive collective preference is possible *without any dictator.* To put it in another way: if one admits a multi-dimensional National Welfare the aggregation problem has solutions. But, unfortunately, it is usually required a stronger concept for National Welfare.

Corollary (*Condorcet Paradox*): the majority decision procedure is inconsistent with respect to Condorcet Configuration.

Corollary: A unanimity decision procedure is consistent with respect to Condorcet Configuration.

The second important configuration is the one studied by Arrow and gives place to the following theorem:

Theorem 14: For every n, when R is at least negatively transitive, the necessary and sufficient condition to aggregate R *into a negatively transitive collective Q* is that two nondictatorial sets with a dictatorial union cannot be found:

$$R \geq R(7) \Rightarrow (\text{non } Y \Leftrightarrow H(II))$$

Theorem 15: for every *finite*, n, every Q and every negatively transitive R, the necessary and sufficient condition for a procedure RQ to be consistent with respect to Arrow Configuration is that there exist one dictator and no real dictatorial college:

$$R \geq R(7) \Rightarrow (T(1) \,\&\, \text{non } G(1) \Leftrightarrow H(II)).$$

Corollary (*Arrow Impossibility Theorem*): If n is finite, a sufficient condition for a procedure $R(4,7)Q(4,7)$ to be inconsistent with respect to configuration II is that there exists no dictator: non $T(1)$.

Corollary (*Arrow General Possibility Theorem*): If n is finite, a necessary condition for a procedure $R(4,7)Q(4,7)$ to be consistent with Arrow Configuration is that there exists a dictator.

In fact, Configuration II was the one implicitly studied by Arrow, though apparently the procedure recorded there was $R(4,7)Q(4,7)$. If one admits a procedure with several dictators checking one another, *the voting procedure is possible* (with respect to Configuration II only). Let us also say that this result is not "obvious" at all.

Corollary: Majority decision method and unanimity decision procedures are inconsistent with respect to Arrow Configuration.

We now state the following other basic theorem:

Theorem 16: For every n, when R is at least antisymmetric or asymmetric, the necessary and sufficient condition to aggregate R into an antisymmetric or asymmetric collective Q is that there do not exist two disjoint dictatorial sets:

$$R \geq R(4)\,(\text{or } R(5)) \Rightarrow (\text{non } J \Leftrightarrow H(III))$$

Corollary: Majority and unanimity decision procedures are consistent with Antisymmetry Configuration, whenever R is antisymmetric or asymmetric.*

Our last basic possibility theorem is:

Theorem 17: For any n, when R is at least total or complete, the necessary and sufficient condition to aggregate R into a total or complete collective Q is that no partition of N into two nondictatorial sets exists.

$$R \geq R(8)\,(\text{or } R(9)) \Rightarrow (C \Leftrightarrow H(IV)).$$

Corollary: The unanimity decision procedure is always inconsistent with respect to Totality Configuration.*

Corollary: The majority decision procedure is always inconsistent with respect to Totality Configuration, whenever n is even. It is consistent, whenever R is total or complete and n is odd (naturally the majority decision method is meaningless when n is infinite).

We shall at present consider all mixed procedures combining two or more properties among $R(6)$, $R(7)$, $R(4)$, $R(5)$, $R(8)$ or $R(9)$. But in order to be able to go further on, we must recall the theorem below:

Theorem 18: If a procedure is not critical for a relation Q, entirely defined by k properties $P_1, P_2, \ldots, P_k$, it is equally not critical for any property P implied by $P_1, P_2, \ldots$ and P_k.

The proofs of the following theorems rely upon Theorem 04: it is easy to verify that in all cases considered here the assumptions of Theorem 04 hold.

We can state

Theorem 19: For any n, when $Q(4, 6)$ or $Q(5, 6)$ and R such that $R \geq R(4, 6)$ (or $R(5, 6)$), the necessary and sufficient possibility condition is that D forms a filter:

$$R \geq R(4, 6)\,(\text{or } R(5, 6)) \Rightarrow (F \Leftrightarrow H(\text{I, III})).$$

Theorem 20: For every n, when $Q(7, 8)$ or $Q(7, 9)$ and R such that $R \geq R(7, 8)$ (or $R(7, 9)$), the necessary and sufficient possibility condition is that there do not exist two nondictatorial sets whose set-union is dictatorial:

$$R \geq R(7, 8)\,(\text{or } R(7, 9)) \Rightarrow (\text{non } Y \Leftrightarrow H(\text{II, IV})).$$

Theorem 21: For any n, when $Q(4, 9)$ or $Q(5, 8)$ or $Q(5, 9)$ and R such that $R \geq R(4, 9)$ (or $R(5, 8)$ or $R(5, 9)$), the necessary and sufficient possibility condition is that there does not exist any partition of N into two nondictatorial sets and there do not exist two dictatorial disjoint sets:

$$R \geq R(4, 9)\,(\text{or } R(5, 8) \text{ or } R(5, 9)) \Rightarrow (C\ \&\ \text{non } J \Leftrightarrow H(\text{III, IV})).$$

All other combinations entail the same necessary and sufficient condition: D must be an ultrafilter and, if n is finite, D must verify condition $Z: T!$ & non $G(1)$.

i.e. there exists an absolute dictator (one and only one).

Namely, we have

Theorem 22: For every n and Q and *ad hoc* R, the necessary and sufficient condition for a procedure RQ to be consistent is:

$$
\begin{array}{ll}
R \geq R(6,7) & \Rightarrow (L \Leftrightarrow H(\mathrm{I}, \mathrm{II})) \\
R \geq R(6,8) \text{ (or } R(6,9)) & \Rightarrow (L \Leftrightarrow H(\mathrm{I}, \mathrm{IV})) \\
R \geq R(4,7) \text{ (or } R(7,5)) & \Rightarrow (L \Leftrightarrow H(\mathrm{III}, \mathrm{II})) \\
R \geq R(4,6,7) \text{ (or } R(5,6,7)) & \Rightarrow (L \Leftrightarrow H(\mathrm{I}, \mathrm{II}, \mathrm{III})) \\
R \geq R(4,7,8) \text{ (or } R(4,7,9)) & \Rightarrow (L \Leftrightarrow H(\mathrm{II}, \mathrm{III}, \mathrm{IV})) \\
R \geq R(5,7,8) \text{ (or } R(5,7,9)) & \Rightarrow (L \Leftrightarrow H(\mathrm{II}, \mathrm{III}, \mathrm{IV})) \\
R \geq R(4,6,9) \text{ (or } R(5,6,8) \text{ or } R(5,6,9)) & \Rightarrow (L \Leftrightarrow H(\mathrm{III}, \mathrm{IV}, \mathrm{I})) \\
R \geq R(6,7,8) \text{ (or } R(6,7,9)) & \Rightarrow (L \Leftrightarrow H(\mathrm{IV}, \mathrm{I}, \mathrm{II})) \\
R \geq R(4,5,7,9) \text{(or } R(5,6,7,9) \text{ or } R(5,6,7,8)) & \Rightarrow (L \Leftrightarrow H(\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}))
\end{array}
$$

$$L \Rightarrow (Vd \Leftrightarrow V^c nd)$$

When n is finite: $L \Leftrightarrow T!$ & non $G(1)$;

Corollary (Fishburn Family P^*) [5]: In the case of an infinite number of voters the Fishburn procedure is consistent for $R(4,7)$ $Q(4,7)$.

It is possible to give a more general result by considering relations which imply either $Q(5,6,8)$ or $Q(5,6,9)$ or $Q(4,7,9)$:

Theorem 23: For any R such that R implies either $R(4,7,9)$ or $R(5,6,8)$ or $R(5,6,9)$ and for any n and any Q implying either $Q(4,7,9)$ or $Q(5,6,8)$ or $Q(5,6,9)$, a necessary condition for a procedure RQ to be consistent is that D forms an ultrafilter. If n is finite, that necessary condition becomes Z: there must be an absolute dictator.

Let us come back to the finite case. Can we find a numerical collective utility (or welfare) function by satisfactory means? The passage from a collective preference to a utility function involves for Q to be more particularized than $Q(4,7,9)$ or $Q(5,6,8)$; moreover and clearly a consistent aggregation entails that $R \geq Q$. So Theorem 23 holds and in that case it is necessary to have an absolute dictator . . . and quite useless to put the question to the vote: it is sufficient to ask the dictator's opinion. The dictator cannot deal, humanly speaking, with an infinite number of alternatives and formulate an adequate preference without using an utility function *a priori* given. *Therefore, any solution to these problems is always doubly arbitrary*. Therefore, Theorem 23 settles the problem of utility aggregation, which is that of National Welfare.

We can go further and state a stronger result:

Theorem 24: For every finite and *infinite* n, the necessary and sufficient condition for the procedure $RQ(4,7,9)$ or $RQ(5,6,8)$, with $R \geq R(4,7,9)$ or $R \geq R(5,6,8)$ respectively, to be coherent is that there exists an absolute dictator (Condition Z):

Thus, we cannot have a collective utility without an absolute dictator who makes any voting procedure meaningless: *National Welfare cannot be defined as a one-dimensional concept.*

In conclusion, let us come back to the nondictatorship condition and let us also prove that Single-Peaked Preference Possibility Theorem as proved by Arrow *does not hold*: this theorem, therefore, cannot contradict our Theorem 23.

First, let us investigate again the nondictatorship condition: Arrow's Condition 5′, which is non $T(1)$, may appear too drastic. We can reformulate that condition in more realistic terms:

Condition 5*a*: There is no dictator *or* there are two dictators *or* there is a real dictatorial college.

This nondictatorship condition is much more general than Arrow's and clearly is the negation of condition Z.[9] It means that if we cannot avoid dictatorial elements, we can hope to balance each dictatorial group by introducing similar groups into the voting procedure.

Unfortunately, in the most interesting cases, such a reasonable compromise would lead nevertheless to inconsistency.

Secondly, one may argue that Arrow's Possibility Theorem for Single-Peaked Preferences gives a consistent procedure with an R which is not only $R(6, 8)$ but also verifies xRy & $B(x, y, z) \Rightarrow$ non zRy (see [6]). Naturally, Condition 1a is changed in that case into the Single-Peaked Preference Condition. But one may come back to condition 1a by considering not $R(6, 8)$ but an adequate binary relation, more particularized than $R(6, 8)$. Theorem 23 therefore, holds. But there is no dictator!

This point is doubly important: first because it seems to contradict Theorem 23 and, secondly, because it is an example of attempts to solve the aggregation problem by pointing out regularities, that is to say by combining $R(6, 8)$ with some special properties.

As regards Arrow's Theorem, we may note that, naturally, in Theorem 23 one must not combine contradictory properties, as it was done in the Single-Peaked Preference Theorem, since, there, the transitivity of $R(6, 8)$ contradicts the Betweenness relation $B(x, y, z)$ (see [2] and [6]): hence Arrow's proof is false, since it depends on the very assumption that R *is* transitive.

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[9] A general demonstration of non $T! \Leftrightarrow$ non $T(1)$ or $T(2)$ is given in [2, Appendix I].