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A Note on α-Gini Measures

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This note provides a characterization of α-Gini inequality measures. These measures generalize the standard Gini index by including one sensitivity parameter α, which captures different value judgments. The α-Gini measures are shown to be weakly decomposable and unit consistent. Weak decomposition provides within-group and between-group inequalities. Unit consistency keeps unchanged the ranking of two income distributions when the income units vary. It is shown that the α-Gini measures are relevant with either “leftist” or“rightist” views.

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1. Introduction

The Gini index is one of the most popular indices employed in economics. It is expressed in various ways and its applications are concerned with concentration, income inequality, poverty, segregation, diversity, health, mobility, and many other fields. In recent decades, the Gini index has been generalized in order to bring out wider families of income inequality; for example, the S-Gini family due to Donaldson and Weymark (1980), the extended Gini index proposed by Yitzhaki (1983), and, among others, the $\mathcal{P}$-Gini family introduced by Gajdos (2002). The measures of those families may depend on a parameter of sensitivity toward inequality. The parametrization and the structure of those measures vary from one family to another; however, each member of these families inherits from the basic properties satisfied by the Gini index, highlighting either its connection with the Lorenz curve or with the dual social welfare function of Yaari (1987).

The aim of this note is to propose a characterization of the α-Gini measures, introduced by Chameni (2006) and Ebert (2010). A subgroup decomposition property (WDEC) is proposed and appears to be a generalization of the weak decomposition properties (DEC) of Ebert (2010) and $(\widehat{DEC}(\varepsilon))$. Such a property

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allows for computing within- and between-group inequalities. As mentioned in Ebert (2010), contrary to the traditional additive decomposition in which the between-group inequality term consists in comparing the mean income of the subgroups only, the weak decomposition outlines the comparison of each income pairwise between every pair of subgroups. On this basis, rich-to-poor transfers may occur between subgroups. This constitutes an important step for the implementation of redistributive actions targeting specific parts of the population in which inequality is concentrated.

Income redistribution is embodied by the value judgments included in any measure of inequality (see, e.g., Kolm, 1976a,1976b), such as "leftist" (absolute) or "rightist" (relative) views. Zheng (2007) proposes the *unit-consistency* property (UC) in order to capture and to enlarge such value judgments. It is an ordinal condition postulating that the ranking between two income distributions remains unchanged when income units differ. Thanks to basic axioms, combining (UC) with (WDEC), we provide a characterization of the α-Gini measures, which are consistent with either *leftist* or *rightist* value judgments.

The outline of the paper is as follows. In Section 2, the notation is introduced, as are the standard axioms usually employed in the literature on inequality measurement (2.1). The different formulations of the decomposition axioms are motivated (2.2). The main results are presented in Section 3, the interest being to combine weak decomposition and unit-consistency properties in order to characterize the α-Gini measures. Section 4 closes the paper.

2. Notation and Axioms

We first set out the notation and the usual axiomatic properties. Then, we present the various formulations underlying the concept of weak decomposition that has recently been introduced in the literature.

2.1. *Notation and Standard Axioms*

Let us consider a population of n individuals, $i=1,\dots,n$, with $n \in \mathbb{N}$, $\mathbb{N}$ being the set of positive integers. The income distribution is $\mathbf{x} := (x_1,\dots,x_n) \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ represents the n-dimensional non-negative Euclidean space ($\mathbb{R}^n_{++}$ being its positive part). The set of all admissible income distributions (with variable size n) is denoted by $\mathscr{X}_+ := \bigcup_{n\geq 1} \mathbb{R}^n_+$ with $\mathscr{X}_{++} := \mathscr{X}_+ \setminus \bigcup_{n\geq 1} \mathbf{0}^n$ ($\mathbf{0}^n$ being a vector of zeros of size n). The population may be partitioned into G exhaustive and exclusive subgroups of size n_g, $g=1,\dots,G$, such that $\mathbf{x}=(\mathbf{x}^1,\dots,\mathbf{x}^g,\dots,\mathbf{x}^G)$ and $\mathbf{x}^g \in \mathscr{X}_+$. The vector of subgroup population sizes is $\boldsymbol{n} := (n_1,\dots,n_G)$, such that $n=n_1+\cdots+n_G$. The vector of subgroup arithmetic means is $\boldsymbol{\mu} := (\mu(\mathbf{x}^1),\dots,\mu(\mathbf{x}^g),\dots,\mu(\mathbf{x}^G))$, such that $\mu(\mathbf{x}^g) > 0$ for all $g=1,\dots,G$, and $\mu(\mathbf{x})$ is the arithmetic mean of the population. The arithmetic mean between individual i's income and individual j's income is denoted by $\mu(x_i,x_j)$, such that $\mu(x_i,x_j) > 0$. The vector of ones of size $n(\mathbf{x}) \equiv n$ is denoted by $\mathbf{1}_{n(\mathbf{x})}$. A replication of an income distribution $\mathbf{x}$ by order k, for $k \geq 2$, is

$$\mathbf{x}^{[k]}=(\underbrace{x_1,\ldots,x_1}_{k\ times},\ldots,\underbrace{x_n,\ldots,x_n}_{k\ times}).$$

The inequality measure is a function $I:\mathscr{X}_+\rightarrow\mathbb{R}_+$, defined as a sequence of indices $I:\mathbb{R}^n_+\rightarrow\mathbb{R}_+$ such that there exists an inequality index for every n.

As a first requirement, the inequality measure is supposed to be continuous.

Axiom 2.1. [Continuity, (CN)]. $I(\mathbf{x})$ *satisfies continuity if, for all* $\mathbf{x}\in\mathscr{X}_+$, $I(\mathbf{x})$ *is a continuous function*.

The inequality measure remains unchanged if individuals are permuted within the income distribution. The inequality measure is not affected by a rearrangement of the components of vector $\mathbf{x}$.

Axiom 2.2. [Anonymity, (AN)]. $I(\mathbf{x})$ *satisfies anonymity if, for all* $\mathbf{x}\in\mathscr{X}_+$ *and all permutation matrices* Π *of size* $n\times n$, $I(\mathbf{x})=I(\Pi\mathbf{x})$.

The inequality measure is normalized, that is, the measure is null when incomes are identical.

Axiom 2.3. [Normalization, (NM)]. $I(\mathbf{x})$ *satisfies normalization if, for all* $\mathbf{x}\in\mathscr{X}_+$ *such that* $\mathbf{x}=\varepsilon\mathbf{1}_{n(\mathbf{x})}$ *with* $\varepsilon>0$, $I(\mathbf{x})=0$.

The inequality measure is invariant by replication. Such a property introduced by Dalton (1920), the Population Principle, allows comparisons of inequality measures for different population sizes.

Axiom 2.4. [Replication Invariance, (PP)]. $I(\mathbf{x})$ *satisfies replication invariance if, for all* $\mathbf{x}\in\mathscr{X}_+$ *and* $k\geq 2$, $I(\mathbf{x}^{[k]})=I(\mathbf{x})$.

By definition, inequality measures may be considered in absolute or relative terms. Absolute and relative inequality measures are characterized by the invariance properties recalled below.

Axiom 2.5. [Invariance by Translation, (INV)]. $J(\mathbf{x})$ *satisfies invariance by translation if, for all* $\mathbf{x}\in\mathscr{X}_+$ *and* $\theta>0$, $J(\mathbf{x}+\varepsilon\mathbf{1}_{n(\mathbf{x})})=J(\mathbf{x})$.

Axiom 2.6. [Scale Invariance, (SI)]. $I(\mathbf{x})$ *satisfies scale invariance if, for all* $\mathbf{x}\in\mathscr{X}_+$ *and* $\theta>0$, $I(\theta\mathbf{x})=I(\mathbf{x})$

The scale invariance property corresponds to a homogeneity of degree zero requirement. Note that in rescaling an absolute inequality measure by the mean, one obtains a relative inequality measure. For the sake of generality, it is assumed that the absolute measures may be rescaled by any given real-valued function of the mean income.

Definition 2.1. *Let* $J:\mathscr{X}_+\rightarrow\mathbb{R}_+$ *denote an absolute inequality measure and* $I:\mathscr{X}_+\rightarrow\mathbb{R}_+$ *a relative one. For any function* $f:\mathbb{R}_{++}\rightarrow\mathbb{R}_{++}$, *the relation between absolute inequality measures* $J(\mathbf{x})$ *and relative inequality measures* $I(\mathbf{x})$ *is as follows*:

$$I(\mathbf{x})=\begin{cases}\dfrac{J(\mathbf{x})}{f(\mu(\mathbf{x}))}, & \forall \mathbf{x}\in \mathscr{X}_{++},\\ 0, & \forall \mathbf{x}\in \bigcup_{n\geq 1}\mathbf{0}^n.\end{cases}$$

2.2. *Axioms of Weak Decomposition*

Subgroup decomposition properties are of interest in order to deal with a population composed of heterogeneous agents. In 2010, a weaker scheme of decomposition than the usual one, defined by Bourguignon (1979) or Shorrocks (1980), was axiomatized by Ebert (2010). This new property enables Pigou–Dalton transfers to be performed between precise individuals of distinct subgroups rather than the mean of the subgroups (see Ebert, 2010). The between-group inequality component is based on the comparison between each and every pair of incomes rather than the use of the mean incomes of the subgroups. The weakly decomposable measures are well suited for the study of the inequality between and within different subgroups of the population. In particular, they outline two components of inequality, which are relevant with regard to the aggregation principles of Kolm (1999) for pairwise inequality measures, such as the Gini indices, that capture envy between each and every pair of individuals. Ebert (2010) investigates two types of decomposition schemes. The first one exhibits weights that are functions of the subgroup sizes.

Axiom 2.7. [Decomposition, (DEC)]. $I(\mathbf{x})$ *satisfies weak decomposability if, for all income distributions* $\mathbf{x}=(\mathbf{x}^1,\mathbf{x}^2)\in\mathscr{X}_+$ *subdivided into two exhaustive and exclusive subgroups,*

$$I(\mathbf{x})=\alpha^1(\boldsymbol{n})I(\mathbf{x}^1)+\alpha^2(\boldsymbol{n})I(\mathbf{x}^2)+\beta(\boldsymbol{n})\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}I(x_i^1,x_j^2),$$

where x_i^1 $(x_j^2$ *) stands for the income of the ith (jth) individual in subgroup 1 (2) and* $\alpha^1(\boldsymbol{n})$, $\alpha^2(\boldsymbol{n})$, *and* $\beta(\boldsymbol{n})$ *are strictly positive weighting functions.*

Ebert (2010) also suggests an alternative version of the weak decomposition property (DEC), with weights embodied by functions of subgroup sizes and income shares.

Axiom 2.8. [Decomposition, $(\widehat{\mathbf{DEC}}(\varepsilon))$**].** $I(\mathbf{x})$ *satisfies weak decomposability if, for all income distributions* $\mathbf{x}=(\mathbf{x}^1,\mathbf{x}^2)\in\mathscr{X}_+$ *subdivided into two exhaustive and exclusive subgroups,*

$$I(\mathbf{x})=\alpha^1(\boldsymbol{n})\cdot\frac{\mu(\mathbf{x}^1)^\varepsilon}{\mu(\mathbf{x})^\varepsilon}\cdot I(\mathbf{x}^1)+\alpha^2(\boldsymbol{n})\cdot\frac{\mu(\mathbf{x}^2)^\varepsilon}{\mu(\mathbf{x})^\varepsilon}\cdot I(\mathbf{x}^2)$$
$$+\beta(\boldsymbol{n})\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\frac{\mu(x_i^1,x_j^2)^\varepsilon}{\mu(\mathbf{x})^\varepsilon}\cdot I(x_i^1,x_j^2),$$

where $\varepsilon > 0$, *and where* x_i^1 (x_j^2) *stands for the income of the ith (jth) individual in subgroup 1 (2), and* $\alpha^1(\boldsymbol{n})$, $\alpha^2(\boldsymbol{n})$, *and* $\beta(\boldsymbol{n})$ *are strictly positive weighting functions.*

In order to deal with a weaker axiom of decomposition, no functional form is imposed on the income shares. We assume that there exist strictly positive weighting functions denoted by $\alpha(n_1, n)$, $\alpha(n_2, n)$, $\beta(2, n)$, and $\mathbf{x}i(\mu(\mathbf{x}^g), \mu(\mathbf{x}))$, such that the functions $\alpha(\cdot, \cdot)$ have the same structure for all subgroups.

Axiom 2.9. [Weak Decomposition, (WDEC)]. $I(\mathbf{x})$ *satisfies weak decomposability if, for all income distributions* $\mathbf{x}=(\mathbf{x}^1, \mathbf{x}^2) \in \mathscr{X}_+$ *subdivided into two exhaustive and exclusive subgroups,*

$$I(\mathbf{x}) = \alpha(n_1, n) \cdot \xi(\mu(\mathbf{x}^1), \mu(\mathbf{x})) \cdot I(\mathbf{x}^1) + \alpha(n_2, n) \cdot \xi(\mu(\mathbf{x}^2), \mu(\mathbf{x})) \cdot I(\mathbf{x}^2)$$
$$+ \beta(2, n) \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \xi i(\mu(x_i^1, x_j^2), \mu(\mathbf{x})) I(x_i^1, x_j^2),$$

where $n = n_1 + n_2$ *such that* $n_1, n_2 \geq 1$, *with* $\alpha(\cdot, n)$, $\beta(2, n)$, *and* $\xi(\cdot, \mu(\mathbf{x}))$ *being strictly positive weighting functions.*

In the sequel, (WDEC) is used to characterize the family of weakly decomposable inequality measures that are unit consistent.[1]

3. The Family of α-Gini Measures

First, the unit-consistency requirement is introduced and combined with the weak decomposition property in order to demonstrate our main results.

Unit consistency is an ordinal and general property. According to Zheng (2007), a change in the units of the income distributions must preserve their ranking with respect to the inequality measure $I(\cdot)$.

Axiom 3.1. [Unit Consistency, (UC)]. $I(\mathbf{x})$ *satisfies unit consistency if, for all* $\mathbf{x}, \mathbf{y} \in \mathscr{X}_+$,

$$[I(\mathbf{x}) < I(\mathbf{y})] \Rightarrow [I(\theta\mathbf{x}) < I(\theta\mathbf{y})], \forall \theta \in \mathbb{R}_{++}.$$

An immediate consequence of (UC) is the following.

Proposition 3.1. [Zheng, 2007]. *An inequality index* $I(\mathbf{x})$ *is unit consistent if, and only if, for all* $\mathbf{x} \in \mathscr{X}_{++}$ *and* $\theta \in \mathbb{R}_{++}$, *there exists a continuous function* $f(\cdot, \cdot)$ *that is also increasing in the second argument such that*

$$I(\theta\mathbf{x}) = f[\theta, I(\mathbf{x})].$$

[1]See also Mornet (2016) for a characterization of weakly decomposable inequality measures, with a particular emphasis on the characterization of the weight functions $\alpha(\cdot)$ and $\beta(\cdot)$. In Mornet (2016), the link between unit consistency and weak decomposition is not investigated and the weight functions are more general.

On the one hand, it is noteworthy that Proposition 3.1 is valid for all $x \in \mathscr{X}_+$ as well. On the other, unit consistency (UC) can be interpreted as a *weak currency independence* property (see Zoli, 2012). Besides, it enables a wide spectrum of value judgments to be captured, from "leftist" points of view to "rightist" ones. The use of (UC) is of interest to derive a new class of inequality measures that are also weakly decomposable. Following Zheng (2007), when (UC) is combined with a proper subgroup decomposition property, it provides a homogeneity condition.

Lemma 3.1. *If an inequality measure* $I(\mathbf{x})$ *satisfies* (CN), (NM), (WDEC), (PP), *and* (UC), *then*

$$I(\theta\mathbf{x})=\theta^{\alpha}I(\mathbf{x}),\ \forall\theta \in \mathbb{R}_{++},\ \forall\alpha \in \mathbb{R}.$$

Proof. See the Appendix (in the Online Supporting Information). ■

Corollary 3.1. *The previous result holds true for* $(\widehat{\text{DEC}}(\varepsilon))$ *or* (DEC).

Proof. Use of the same approach as in the proof of Lemma 3.1 yields the desired result. ■

Measures that are both unit consistent and weakly decomposable are also homogeneous of degree α, for all $\alpha \in \mathbb{R}$.[2] A similar result has been obtained by Zheng (2007) with additive decomposition.

Now, adding anonymity (AN) yields the family of weakly decomposable and unit-consistent inequality measures.

Theorem 3.1. *An inequality measure* $I(\mathbf{x})$ *satisfies* (CN), (NM), (AN), (WDEC), (UC), *and* (PP) *if, and only if,*

$$I(\mathbf{x})=\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}h\left(\frac{\mu(x_i,x_j)}{\mu(\mathbf{x})}\right)I(x_i,x_j),$$

where $I(\cdot,\cdot):\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ *is homogeneous of degree* $\alpha \in \mathbb{R}$, *symmetric, and continuous such that* $I(z,z)=0$, *and where* $h:\mathbb{R}_{++}\to\mathbb{R}_{++}$ *is continuous.*

Proof. See the online Appendix. ■

From Theorem 3.1, several subclasses of inequality measures embodying specific points of view may be deduced. For instance, the class of absolute weakly decomposable inequality measures relevant with regard to the unit-consistency property is characterized to capture the *leftist* point of view.

[2]When the degree of homogeneity is null ($\alpha = 0$), the expression corresponds to scale invariance (SI).

Proposition 3.2. *An absolute inequality measure* $J(\mathbf{x})$ *satisfies* (CN), (NM), (AN), (WDEC), (UC), *and* (PP) *if, and only if,*

(1)
$$J(\mathbf{x}) = c \cdot \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|^{\alpha}, \; c > 0, \; \alpha \neq 0.$$

Proof. See the online Appendix. The logical independence of the characterizing axioms is also demonstrated in the Appendix. ■

This last expression corresponds to an extension of the Gini mean difference, that is, the absolute α-Gini measures. Ebert (2010) axiomatically derives those measures from (DEC), whereas we derive them by (WDEC), which is weaker than (DEC). The α parametrization is convenient in particular for the implementation of redistributive actions (rich-to-poor transfers) (see, e.g., Chameni, 2006, 2013). The larger α is, the more the measures are sensitive to transfers occurring at the tails of the income distribution (see Mornet *et al*., 2013).

Besides, since there exists a link between absolute and relative inequality measures, the latter may be directly deduced from the former in order to capture the *rightist* point of view.[3]

Corollary 3.2. *A relative inequality measure* $I(\mathbf{x})$ *that satisfies* (CN), (NM), (AN), (WDEC), (UC), and (PP) is given by

$$I(\mathbf{x}) = c \cdot \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{|x_i - x_j|^{\alpha}}{\mu^{\alpha}(\mathbf{x})}, \; c > 0, \; \alpha \neq 0.$$

Proof. See the online Appendix. ■

These measures are the relative α-Gini measures that respect $(\hat{\mathrm{DEC}}(\varepsilon))$.

4. Concluding Remarks

This note has introduced a characterization of the α-Gini measures as either absolute or relative. Instead of dealing with separate axioms of weak decomposition, as in Ebert (2010), a general axiom of weak decomposition has been used to derive absolute and relative measures. These "leftist" and "rightist" views are consistent with some well-known principles of transfers such as Pigou–Dalton ($\alpha \geq 1$) (see Chameni, 2006) and the principle of concentration ($\alpha > 0$) (see Ebert, 2010). The α-Gini measures are also relevant with regard to the strong principle of diminishing transfers ($\alpha > 2$), either at the top of the income distribution or at the bottom (see Mornet *et al*., 2013).

[3]By Lemma 3.1, a unit-consistent index is homogeneous of degree α; then (SI) inequality measures are included in the family of (UC) inequality measures. Therefore, invoking the *rightist* point of view (SI) will restrict the unit-consistent inequality measures to homogeneous functions of degree 0.

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Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Appendix