Review of Income and Wealth
Series 64, Number 3, September 2018
DOI: 10.1111/roiw.12310

SECOND-ORDER DISCRIMINATION AND GENERALIZED LORENZ DOMINANCE

BY RAFAEL SALAS

Departamento de Análisis Económico, Universidad Complutense de Madrid

JOHN A. BISHOP AND LESTER A. ZEAGER

Department of Economics, East Carolina University

We propose a definition of second-order discrimination that does not require the reference distribution to first-order dominate the comparison one, and allows rankings of discrimination patterns when both the reference and the comparison distributions differ. It involves comparing the probabilities that randomly selected individuals in the reference and comparison distributions belong to subgroups having the same cumulative mean income, yields orderings of distributions equivalent to those from generalized Lorenz dominance, and allows orderings of discrimination patterns, partial or complete, across pairs of distributions. We compare discrimination against U.S. seniors (inter-distributional inequality between seniors and non-seniors) by ethnicity.

JEL Codes: D31 (personal income, wealth, and their distributions), I3 (welfare, well-being and poverty), C1 (econometric and statistical methods and methodology: general)

Keywords: discrimination measurement, economic advantage, generational income comparison, inter-distributional inequality, stochastic dominance

1. INTRODUCTION

Responding to increasing social concerns about racial discrimination and gender inequality in recent decades, statisticians and economists created methods for making inter-distributional inequality comparisons—inequality *between* two populations, not within a single population—with Gastwirth (1975), Dagum (1980), Shorrocks (1982), Ebert (1984), and Butler and McDonald (1987), among others making pioneering contributions to the literature.[1] In these comparisons of incomes or wages for males and females, whites and nonwhites, or younger and

Notes: The authors thank Eugene Eisenstein, Juan Gabriel Rodriquez, Angel Salas, two anonymous referees, and participants in the Department of Economics seminar at East Carolina University for useful comments and suggestions. Rafael Salas gratefully acknowledges funding from the Ministerio de Economía y Competitividad of Spain under project ECO2013-46516-C4-4-R and Comunidad de Madrid (Spain) under project S2015/HUM-3416-DEPOPORCM.

*Correspondence to: Rafael Salas, Departamento de Análisis Económico I, Universidad Complutense de Madrid, Campus de Somosaguas, 28035 Madrid, Spain (r.salas@ucm.es).

[1]See Dagum (1987), Gastwirth *et al.* (1989), Butler and McDonald (1989), Yitzhaki (1994), and Deutsch and Silber (1999) for further discussion of the proposed methods.

older persons, the essential data reside in a *pair* of distributions rather than a single distribution, as with a Lorenz curve.

More recently, Le Breton *et al.* (2012) proposed a dominance approach to defining discrimination patterns between two distributions and ordering discrimination patterns, which involves two pairs of distributions. They also proposed refinements of first-order discrimination orderings by integration, yielding first- and second-order discrimination curves. The dominance approach has two advantages. First, it links discrimination orderings to welfare theory. Second, dominance relations are more general—less sensitive to differences in value judgments—than a discrimination index, and therefore achieve broader agreement on the resulting orderings over the pairs of distributions.

We support the spirit of the Le Breton *et al.* (2012) approach, but we propose an alternative formulation of the second-order discrimination measure with desirable features. We integrate with respect to the same variable in both the first- and second-order measures and find a formulation that allows us to *quantify* the relative discrimination between two pairs of income (or wage) distributions as the areas between the relevant generalized Lorenz curves. It follows from this connection that our rankings of discrimination patterns are equivalent to those obtained from all S-convex discrimination indices. By these measures, a narrowing of the income gap at lower incomes combined with a mean-preserving widening of the income gap at higher incomes results in a reduction in second-order discrimination. We also demonstrate that comparisons based on our second-order discrimination curves (SDCs) are equivalent to comparisons of the truncated mean incomes for the t poorest proportions of families across pairs of reference and comparison distributions for population subgroups.

The alternative formulation advances discrimination comparisons in two ways. First, it can be applied when the reference distribution does not first-order dominate the comparison one. Second, it permits rankings of discrimination patterns in situations where both the reference and comparison distributions differ, which expands the range of possible discrimination comparisons. The price we pay for these gains is that our approach, being a second-order stochastic dominance notion, can be applied only to variables that contain cardinal information, such as incomes or wages. As an illustration, we compare the discrimination (or inter-distributional income inequality) between seniors and non-seniors across ethnic groups.

Section 2 presents our alternative formulation, derives its implications, and explores connections to the related literature. Section 3 illustrates our approach by an application to the inter-distributional inequality between U.S. seniors and nonseniors across ethnic groups, using data from the Current Population Survey. Section 4 offers concluding remarks.

2. Methods

2.1. *Previous Approaches*

We compare the income distributions for two populations with right-continuous, non-decreasing cumulative distribution functions, $F_c(m) = \int_0^m f_c(y)dy$

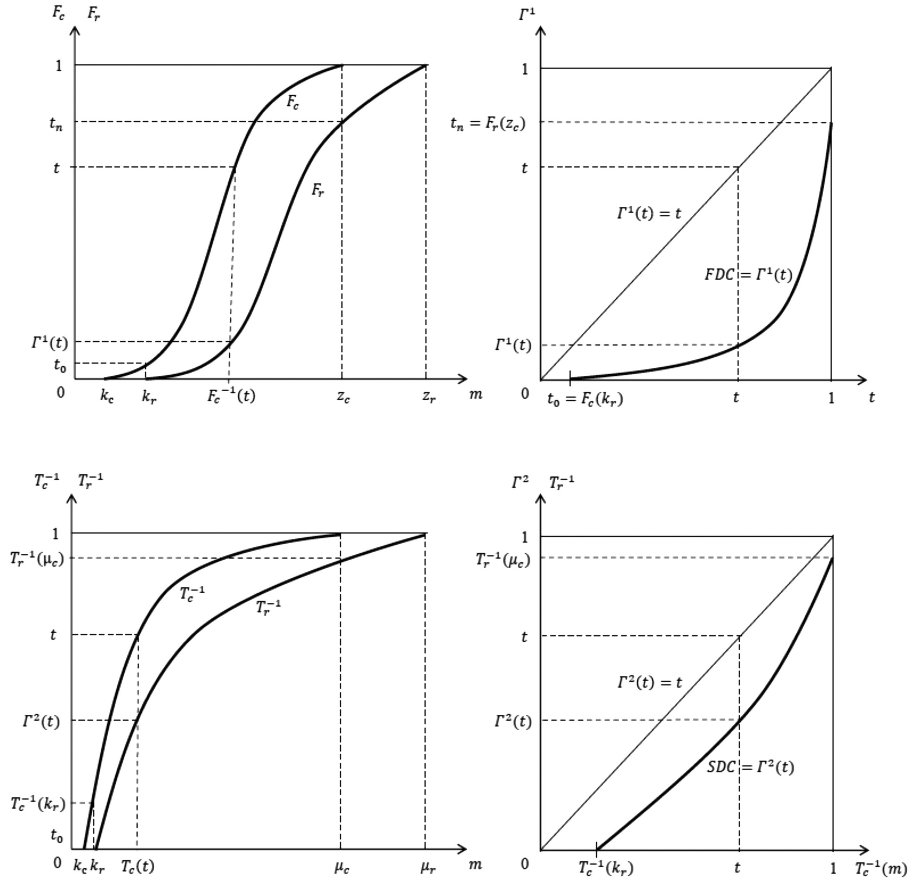


Figure 1. Construction of First-Order Discrimination Curve (FDC) and Second-Order Discrimination Curve (SDC)

and $F_r(m) = \int_0^m f_r(y)dy$, defined over non-negative income values, $m \in [0, \infty)$ where c denotes the comparison population, r denotes a reference population, and $f_c(y)$ and $f_r(y)$ denote the respective density distribution functions over the income values. Let $F_c(z_c)=F_r(z_r)=1$ for some $z_c, z_r < \infty$ and assume that the lowest income in each population is $k_c > 0$ and $k_r > 0$. The top-left panel of Figure 1 illustrates these functions. Le Breton *et al.* (2012) define the *first-order discrimination curve* (FDC), $\Gamma^1(t)=F_r\left[F_c^{-1}(t)\right]$, where t is the proportion of the comparison population with income less than or equal to m, and $F_c^{-1}(t)$ denotes the left-continuous inverse of $F_c(m)$, also called the quantile function. FDC induces a partial ordering: distribution $F_c(m)$ exhibits at least as much first-order discrimination as distribution $F_r(m)$ if and only if $\Gamma^1(t) \leq t$ for all $t \in [0, 1]$. In the top-right panel of Figure 1 we illustrate the FDC. From inspection of Figure 1, FDC dominance is equivalent to first-order stochastic dominance (FSD), because $\Gamma^1(t) \leq t$ for all $t \in [0, 1]$ is equivalent to $F_c(m)-F_r(m) \geq 0$ for all $m \in [0, z_c]$.

In a closely related contribution, Butler and McDonald (1987) proposed inter-distributional Lorenz curves (ILCs) defined in terms of normalized partial moments of the distributions, $\varphi_c(m;k)=\left[\int_0^m y^k f_c(y)dy\right]/E_c\left(y^k\right)$ and $\varphi_r(m;k)=\left[\int_0^m y^k f_r(y)dy\right]/E_r\left(y^k\right)$. They proposed two "natural" ILCs. The first ILC sets $k=0$ and then plots $\varphi_c(m;0)=F_c(m)$ and $\varphi_r(m;0)=F_r(m)$ at corresponding incomes. Here $\Gamma^1(t)=\varphi_r\left[\varphi_c^{-1}(t;0);0\right]$ is equivalent to the FDC above. Another ILC sets $k=1$ and plots $\varphi_c(m;1)=L_c[F_c(m)]$ and $\varphi_r(m;1)=L_r[F_r(m)]$, the Lorenz ordinates for a fixed m, at corresponding incomes in both populations. However, $\varphi_r\left[\varphi_c^{-1}(t;1);1\right]$ does not provide a refinement of FDC comparisons.

To obtain such a refinement, Le Breton *et al.* (2012) proposed integrating $\Gamma^1(t)$ as

$$\Phi^2(t)=\int_0^t \Gamma^1(u)du \text{ for all } t\in[0,1], \tag{1}$$

which they call a *second-order discrimination curve* (SDC). They show that orderings of discrimination patterns by this measure are equivalent to those generated by a discrimination index proposed by Gastwirth (1975). One feature of this refinement is a bit unusual. The first-order curve cumulates according to an income threshold, but the second-order curve cumulates according to a population proportion. The latter orderings resemble SSD, but they are not equivalent, except under the presence of the same uniform reference distribution in two discrimination patterns (Le Breton *et al.*, 2012, p. 1346).

2.2. *A New Approach*

We propose instead the second-order discrimination curve

$$\Gamma^2:\ [0,1]\rightarrow[0,1],\ \text{where } \Gamma^2(t)=T_r^{-1}(T_c(t)) \text{ for all } t\in[0,1], \tag{2}$$

$$T_c:\ [0,1]\rightarrow[0,\mu_c],\ \text{where } T_c(t)=\begin{cases}\dfrac{\int_0^{m(t)}(1-F_c(x))dx}{\int_0^{m(t)}f_c(x)dx} & t\in(0,1]\\ 0 & t=0\end{cases} \tag{3}$$

and where $m(t)=F_c^{-1}(t)$ and μ_c is the mean income of the comparison distribution.

Alternatively, we can write

$$T_c(t)=\begin{cases}Q_c(t)/t & t\in(0,1]\\ 0 & t=0\end{cases} \tag{4}$$

where $Q_c(t)=\int_0^t F_c^{-1}(p)dp=\int_0^{m(t)} xf_c(x)dx$ is the cumulative quantile function or the generalized Lorenz curve and $0\le p\le 1$ is a population proportion. Let $Q_r(t)$ and $T_r(t)$ be defined analogously for the reference distribution.

We can interpret $T_c(t)$ as the truncated mean income of the t poorest proportion of the comparison population or as the cumulative mean income up to the quantile $m(t)$ of the comparison population. Notice that $m(1){=}z$, $T_c(1){=}\mu_c$, and $T_r(1){=}\mu_r$. Notice further that both $T_c(t)$ and $T_r(t)$ are nondecreasing functions. This follows from equation (4), and $Q_c(t){=}\int_0^t F_c^{-1}(p)dp$, and from the fact that $F_c^{-1}(p)$ is a nondecreasing function. The latter implies that for any $p \leq t$, $Q_c(t) \leq \int_0^t F_c^{-1}(t)dp{=}F_c^{-1}(t)\int_0^t dp{=}F_c^{-1}(t)t$. Comparing the relevant areas in the top panels of Figure 1 confirms this result. Hence, we have $\frac{d}{dt}\left[\frac{Q_c(t)}{t}\right]= \frac{Q_c'(t)t-Q_c(t)}{t^2}=\frac{F_c^{-1}(t)t-Q_c(t)}{t^2} \geq 0,\ \forall\ 0<t<1$, and likewise for $T_r(t)$.

Given that $T_r(t)$ is nondecreasing, there is a well-defined left-continuous inverse, $T_r^{-1}(\mu)$, such that

(5)
$$T_r^{-1}:\ [0,\mu_r]\rightarrow[0,1],$$
$$\text{where }\ T_r^{-1}(\mu){=}inf\ \left\{t{=}F_r(s):\frac{\int_0^s xf_r(x)dx}{\int_0^s f_r(x)dx}\geq\mu\right\}\ \text{for all }\ \mu\in[0,\mu_r].$$

Define $T_c^{-1}(t)$ accordingly. It is the proportion of the (ascending-ordered) reference population with truncated mean income lower than or equal to μ. Therefore, $\Gamma^2(t)$ is the plot of $(T_r^{-1}(\mu),\ T_c^{-1}(\mu))$, or the plot of the inverse of the truncated mean income curves for the reference and the comparison distributions. The bottom-left panel of Figure 1 illustrates the inverse truncated mean income functions in equation (5), while the bottom-right panel displays the SDC in equation (2).

SDC induces a partial ordering: distribution $F_c(m)$ exhibits at least as much second-order discrimination as distribution $F_r(m)$ if and only if

(6)
$$\Gamma^2(t)\leq t\ \text{ for all }\ t\ \epsilon\ [0,1].^2$$

This is illustrated in bottom-right panel of Figure 1, where $\Gamma^2(t)$ is below the diagonal.

2.3. *Interpretation of Second-Order Discrimination and Related Indices*

Notice that condition (6) is equivalent to

(7)
$$\mathrm{T}_r^{-1}(\mu)\leq\mathrm{T}_c^{-1}(\mu)\ \text{ for all }\ \mu\ \epsilon\ [0,\mu_c],\ \text{ or}$$

[2]Eventually our SDC induces a further ordering on discrimination patterns: for any two distributions patterns $[F_r^1(m),F_c^1(m)]$ and $[F_r^2(m),F_c^2(m)]$, pattern $[F_r^1(m),F_c^1(m)]$ exhibits at least as much second-order discrimination as pattern $[F_r^2(m),F_c^2(m)]$ if and only if we have $\Gamma_1^2(t)\leq\Gamma_2^2(t)$ for all $t\ \epsilon\ [0,1]$, where $\Gamma_1^2(t)$ and $\Gamma_2^2(t)$ denote the SDCs for distribution patterns 1 and 2, respectively. A summary discrimination index consistent with this ordering is provided in Section 3 and Table A2.

$$\mathrm{T}_c(t) \leq \mathrm{T}_r(t) \text{ for all } t \in [0,1]. \tag{8}$$

Note that $\mu_r \geq \mu_c$ and $k_r \geq k_c$ are the standard necessary conditions for (6) to (8). We assume both conditions in Figure 1 to illustrate SSD and SDC. We interpret (6) through (8) as follows: the probability that a randomly selected individual in the reference population belongs to the subgroup with cumulated mean income μ is lower than or equal to the probability that a randomly selected individual in the comparison population belongs to the subgroup with the same cumulated mean income μ, for all μ. Note that the s value in expression (5) may not be the same for both populations, but the μ value (the cumulated mean income) is the same in both populations.

To see that this SDC ordering is consistent with the SSD ordering, notice that the conditions above are equivalent to SSD, because

$$T_c(t) \leq T_r(t) \;\; t \in (0,1], \tag{9}$$

is equivalent to

$$\frac{Q_c(t)}{t} \leq \frac{Q_r(t)}{t} \;\; t \in (0,1], \tag{10}$$

and thus to

$$Q_c(t) \leq Q_r(t) \;\; t \in (0,1], \tag{11}$$

and $T_c(0)=T_r(0)=Q_c(0)=Q_r(0)=0$ for $t=0$.

Given these results, we can define a class (Φ) of linear discrimination indices (ϕ) with positive weights, $w(t) > 0$:

$$\phi = \int_0^1 [Q_r(t) - Q_c(t)]w(t)dt. \tag{12}$$

Measures of this form are familiar from related literatures: the idea of linear, rank-dependent income inequality (Mehran, 1976), the contributions to risk and social welfare theory by Yaari (1987, 1988), and a class of aggregate income redistribution measures by Pfähler (1987, 1988). Using (12), we can state the proposition:

$$\phi \geq 0, \text{ for all } \phi \in \Phi \iff Q_r(t) - Q_c(t) \geq 0, \text{ for } t \in [0,1]. \tag{13}$$

The implication from right to left is obvious from the definition in (12). To prove the implication from left to right, suppose for a contradiction that $Q_r(t) - Q_c(t) < 0$ for some $t \in [0,1]$ and that $Q_r(t) - Q_c(t) > 0$ otherwise. Then there exist a number $k > 0$ and an interval $(a,b) \subset [0,1]$ on which

$$\int_a^b [Q_r(t) - Q_c(t)]dt < -k. \tag{14}$$

By choosing $w(t)=m > 0$ for $t \in (a,b)$, we have

(15) $$\int_a^b [Q_r(t)-Q_c(t)]mdt < -mk.$$

For all other values, $t \in [0,a] \cup [b,1]$, let $w(t)=h > 0$. We can always choose h to satisfy:

(16) $$\int_0^a [Q_r(t)-Q_c(t)]hdt+\int_b^1 [Q_r(t)-Q_c(t)]hdt=mk.$$

For instance, we can set

(17) $$h=\frac{mk}{\int_0^a [Q_r(t)-Q_c(t)]dt+\int_b^1 [Q_r(t)-Q_c(t)]dt}.$$

Then, combining (15) and (16), it cannot be true that

(18) $$\phi=\int_0^1 [Q_r(t)-Q_c(t)]w(t)dt \geq 0 \text{ for all } w(t) > 0.$$

Note further that the following three conditions are all equivalent:

(19) $$\phi \geq 0, \text{ for all } \phi \in \Phi$$

(20) $$Q_r(t)-Q_c(t) \geq 0, \text{ for all } t \in [0,1]$$

(21) $$T_r(t)-T_c(t) \geq 0, \text{ for all } t \in [0,1].$$

That is, proposition (13) and the proof that follows could be stated in terms of $T_r(t)$ and $T_c(t)$ instead of $Q_r(t)$ and $Q_c(t)$.

The equivalence of equations (19) through (21) enables us to refine rankings of discrimination patterns based on FDC or FSD criteria. For any fixed reference distribution, improvements in the comparison distribution in the sense of generalized Lorenz dominance or SSD (a mean-preserving contraction) reduce discrimination. Similarly, for any fixed comparison distribution, deterioration of the reference distribution in terms of generalized Lorenz dominance or SSD (a mean-preserving spread) reduces discrimination. In other words, narrowing of income (wage) gaps at lower incomes combined with an equal widening of income (wage) gaps at higher incomes results in a net reduction of second-order discrimination according to our measure.

Our refinement of first-order discrimination will seem reasonable to many researchers, because it is based on welfare dominance, but objections could be raised. For instance, it may not address all concerns about "glass-ceiling effects" [discrimination against persons with high incomes or wages (Blau and Kahn, 2016, p. 9)]. The numerical example in the Appendix provides illustrations of several points raised here.

For any two discrimination patterns $(F_r^1(m), F_c^1(m))$ and $(F_r^2(m), F_c^2(m))$ we can prove the proposition:

(22) $\phi_1 \geq \phi_2$, for all $\phi \in \Phi \iff Q_r^1(t) - Q_c^1(t) \geq Q_r^2(t) - Q_c^2(t)$, for $t \in [0, 1]$.

As with proposition (13), proposition (22) can be expressed in terms of $T_r(t)$ and $T_c(t)$ instead of $Q_r(t)$ and $Q_c(t)$. That is, we can compare the discrimination or inter-distributional inequality between two pairs of distributions by the areas between either the respective generalized Lorenz curves or the corresponding truncated mean income curves. Furthermore, we can compare two discrimination patterns when both the comparison and the reference distributions differ, which is not possible in LeBreton *et al.* (2012). We illustrate comparisons of this kind for senior versus non-senior whites, blacks, and Hispanics in the U.S. in Section 3.

2.4. *Links to Related Approaches*

Our formulation of second-order discrimination also has links to the distributional approach to earnings discrimination by Jenkins (1994), which is a different approach but has some mathematical similarities. From the class Φ of discrimination indices ϕ defined by (12), we can derive a general measure of discrimination in terms of the (weighted) area between the generalized Lorenz curves for the reference distribution $Q_r(t)$ and the comparison distribution $Q_c(t)$. This measure includes two important special cases. If we set the weights $w(t) = t(t-1)(1-t)^{(v-2)}$, for $v > 1$, we obtain the extended-Gini-based measure (Yitzhaki, 1983, Donaldson and Weymark, 1983). If we set $v=2$, then $w(t)=2$ and we obtain the standard-Gini-based measure, where ϕ equals twice the area between the generalized Lorenz curves for the comparison and reference distributions. Let $w(t) = -W'(t) > 0$. Then from (12),

$$\phi = \int_0^1 [Q_r(t) - Q_c(t)](-W'(t))dt \tag{23}$$

Using integration by parts, we can write:

$$\begin{aligned} \phi &= \int_0^1 [Q_r(t) - Q_c(t)](-W'(t))dt = (Q_r(1) - Q_c(1))(-W(1)) \\ &\quad - \int_0^1 \left[F_r^{-1}(t) - F_c^{-1}(t)\right](-W(t))dt, \end{aligned} \tag{24}$$

where $W(t)$ is nonnegative and decreasing in t ($W'(t) < 0$) to satisfy the original Yaari general form. Moreover, we can assume without loss of generality that $W(1)=0$, so our measure can be written in terms of income (wage) gaps of Jenkins (1994):

$$\phi = \int_0^1 \left[F_r^{-1}(t) - F_c^{-1}(t)\right] W(t)dt, \tag{25}$$

where $F_r^{-1}(t) - F_c^{-1}(t) = y_r(t) - y_c(t) = g(t)$ denotes the income (wage) gap at position $t = F_r(y) = F_c(y)$ of both distributions. For the extended-Gini case, the weights are $W(t) = v(1-t)^{(v-1)}$. For the standard Gini coefficient, the weights are $W(t)=2$ $(1-t)$. Thus, our analysis is consistent with a measure of rank-dependent positional wage gaps. In the appendix, we translate the analysis to the discrete context, as in Jenkins (1994), and provide a numerical illustration.[3]

Another measure proposed by Jenkins belongs to a closely related class. Let Y be the class of all indices φ:

$$\varphi = \int_0^1 \left[Q_{r,c}(t) - Q_c(t) \right] w(t) dt, \tag{26}$$

where $w(t)$ denotes positive weights and $Q_{r,c}(t)$ is the generalized concentration curve for the reference distribution with respect to the comparison distribution,

$$Q_{r,c}(t) = \int_0^t F_r^{-1}(F_c) dF_c. \tag{27}$$

Using integration by parts, as before, φ can be written as:

$$\varphi = \int_0^1 \left[F_r^{-1}(F_c) - F_c^{-1} \right] W(t) dt, \tag{28}$$

where $W(t)$ are positive and decreasing in t. For $W(t) = 2(1-t)$ we get the standard Gini version [the continuous counterpart of the measure C in equation (7) in Jenkins (1994, p. 87)] that corresponds to twice the area between the generalized concentration curve for the comparison distribution and the generalized Lorenz curve for the reference distribution. His C measure is therefore consistent with a measure of rank-dependent individual wage gaps and our proposal can be interpreted as a variation on it.

The most important feature that our proposal shares with Jenkins (1994) and a more recent contribution by del Rio *et al.* (2011) is that it takes into consideration the distribution of income (wage) gaps, not just their mean. Our aggregation of the wage gaps could be made using the same normative criteria that they use, but here the differences emerge. Jenkins (1994) differs from us by taking absolute values of the income (wage) gaps before aggregating, which discards the distinction between disadvantages and advantages of the comparison distribution relative to the reference distribution. Finding this approach unsatisfactory, del Rio *et al.* (2011) impose a focus axiom that aggregates only the *positive* income (wage) gaps, which restricts attention to the disadvantages of the comparison relative to the reference distribution. Our approach retains the distinction between the advantages and disadvantages of the comparison distribution relative to the reference distribution and takes both into consideration by examining the entire distribution; however, we consider only the anonymous, *positional* income (wage) gaps, not the *individual* income gaps, so we ignore re-rankings of *individuals* due to discrimination.

3. Application

To illustrate our approach to discrimination, we take up one of the most challenging problems confronting the U.S.: the relative well-being of the baby

[3]We owe the numerical illustration to an anonymous referee.

TABLE 1
U.S. Quantile Functions By Age And Race

	Seniors $F^{-1}(t)$			Non-Seniors $F^{-1}(t)$		
t	Whites (1)	Blacks (2)	Hispanics (3)	Whites (4)	Blacks (5)	Hispanics (6)
0.10	14,589.8	10,106.3	11,227.9	19,376.0	11,242.6	15,417.0
0.20	21,598.8	13,325.9	15,798.3	28,911.0	17,020.0	21,707.0
0.30	28,719.0	17,789.7	21,133.4	36,621.9	22,142.3	26,881.0
0.40	36,239.8	23,118.8	27,131.9	44,406.0	26,998.0	31,731.7
0.50	44,826.4	29,048.9	33,513.0	52,875.9	32,611.0	36,932.6
0.60	55,246.5	36,237.6	41,429.2	62,261.6	38,729.0	43,285.3
0.70	67,990.0	45,469.2	50,927.9	73,559.6	47,055.0	51,659.6
0.80	84,658.0	58,375.0	65,080.9	87,951.2	59,033.6	63,523.0
0.90	112,626.2	80,404.2	88,699.0	113,137.8	78,609.0	82,730.2
1.00	1,400,316.3	1,009,161.4	1,160,438.5	1,119,386.9	959,222.0	1,003,840.6

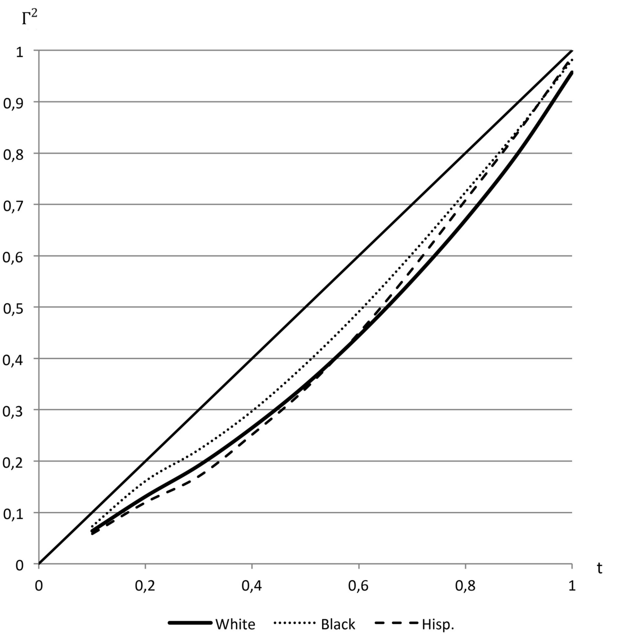
The quantile functions are based on household comprehensive incomes. We find crossings in the quantile functions, $F^{-1}(t)$, for the seniors and non-seniors in all ethnic groups [(1) vs. (4), (2) vs. (5), and (3) vs. (6)]. Quantile function crossings imply the absence of FSD, so the FDC crosses the diagonal in the unit square.

boomers as they enter into retirement (Kotlikoff and Burns, 2004), which will initiate a secular decline in the share of the population of working age and deplete the Social Security trust fund reserves. According to the calculations of the Office of the Chief Actuary for the Social Security Administration (2014), the combined Social Security and Disability Insurance reserves will be spent by 2033. At that point, present law would require that benefits be reduced to what incoming revenues can finance. This adjustment in retiree benefits, if implemented, can be likened to discrimination against seniors (greater inter-distributional income inequality between seniors and non-seniors). Here we illustrate how one could track the situation as it evolves.

The baby boomers were born between 1946 and 1964, so the oldest boomers recently began drawing Social Security benefits and the youngest boomers became 50 years old in 2014, the age at which persons qualify for AARP membership in the U.S. Using incomes for seniors (age 50 or older) and non-seniors (under age 50), we can construct quantile functions and check for first-order discrimination or inter-distributional inequality against seniors. To compare discrimination patterns, we create subsamples representing ethnic groups (whites, blacks, and Hispanics) and examine their FDCs and SDCs.

Our income data come from the Current Population Survey in 2006, 2009, and 2012, expressed in 2012 dollars. Given that the incomes of seniors depend heavily on government transfers (e.g. Social Security), we choose comprehensive incomes [cash income plus in-kind transfers (except government medical benefits, which are problematic to measure) minus taxes, but ignoring unrealized capital gains (also problematic to measure)] as the income concept. For this application we treat the household as both the income sharing unit and the unit of analysis, so our comparisons all involve household incomes.

Table 1 gives the quantile functions, $F^{-1}(t)$, for seniors and non-seniors in each ethnic group. Inspection of Table 1 reveals crossings in each group [column



The SDCs for all three groups lie below the diagonal, which implies second-order discrimination against seniors. There are crossings of the SDCs for whites and Hispanics and for Hispanics and blacks, but whites SDC dominate blacks, so discrimination is unambiguously greater for white seniors (relative to white non-seniors) than for blacks (relative to black non-seniors).

Figure 2. Second-Order Discrimination Curves for U.S. Whites, Blacks, and Hispanics: Seniors versus Non-Seniors

(1) vs. (4), (2) vs. (5), and (3) vs. (6)]. These crossings imply that there is no FDC dominance and that the FDCs for all the groups cross the diagonal in the unit square. To determine whether crossings occur at the second degree, we compare the truncated mean income functions in Table A1 of the Appendix. Here we find no crossings in any group; non-seniors dominate seniors in all groups. These findings imply second-order discrimination against seniors in all three groups. The same finding emerges from Table A2 in the Appendix, where we give the SDC ordinates, and in Figure 2, where the SDCs all lie below the diagonal. This example demonstrates the possibility of finding discrimination patterns even when the reference distribution (non-seniors) does not first-order dominate the comparison one (seniors).

Figure 2 also allows us to compare discrimination patterns across ethnic groups. Inspection of the three SDCs reveals two crossings: one involving whites and Hispanics and another involving Hispanics and blacks. In both cases, comparisons of discrimination patterns are inconclusive. In contrast, the SDC for whites lies everywhere below the SDC for blacks, so we infer that the discrimination (disadvantage) is greater for white seniors than for black seniors. At the same time it is important to remember that in Table 1, white seniors first-order dominate black and Hispanic seniors. Hence, *the greater disadvantage of white seniors is with respect to non-seniors in the same ethnic group*, not with respect to the seniors in other ethnic groups. We can aggregate across the income gaps to compare the degree of discrimination. Using a summary index based on twice the area between the SDC and the diagonal, we find the greatest discrimination for whites (0.2152), followed next by Hispanics in between (0.1996), and smallest for blacks (0.1421), as reported in the notes for Table A2.

4. Conclusion

Our proposed SDC is a natural refinement of the first ILC of Butler and McDonald (1987) and the FDC of Le Breton *et al*. (2012). We demonstrate equivalence between SDC orderings of two distributions truncated at any cumulative mean income and SSD orderings of the entire distributions. Using this equivalence, we can *quantify* the discrimination pattern for a pair of income (or wage) distributions as the area between the corresponding generalized Lorenz curves. In taking this approach, we incorporate the notion that a mean-preserving transfer in the comparison distribution from the top of the income (or wage) distribution to the bottom reduces overall discrimination. The opposite applies for the reference distribution.

Our formulation of the SDC advances discrimination comparisons in two other significant ways. First, it can be applied when the reference distribution does not first-order dominate the comparison one. Second, it allows rankings of discrimination patterns in situations where both the reference and comparison distributions differ. We illustrate both possibilities by comparing discrimination against U.S. seniors across ethnic groups. The price we must pay for these advances is that our approach, being based on SSD, can be applied only to variables that contain cardinal information, such as incomes and wages.

References

Blau, F. D. and L. M. Kahn, "The Gender Wage Gap: Extent, Trends, and Explanations," *National Bureau of Economic Research*, Working Paper 21913, January, 2016 http://www.nber.org/papers/w21913.

Butler, R. and J. McDonald, "Interdistributional Income Inequality," *Journal of Business and Economic Statistics*, 5, 13–8, 1987.

———, "Using Incomplete Moments to Measure Inequality," *Journal of Econometrics*, 42, 109–19, 1989.

Dagum, C., "Inequality Measures between Income Distributions with Applications," *Econometrica*, 48, 1791–803, 1980.

———, "Measuring the Economic Affluence between Populations of Income Receivers," *Journal of Business and Economic Statistics*, 5, 5–12, 1987.

del Río, C., C. Gradín, and O. Cantó, "The Measurement of Gender Wage Discrimination: The Distributional Approach Revisited," *Journal of Economic Inequality*, 9, 57–86, 2011.

Deutsch, J. and J. Silber, "Inequality Decomposition by Population Subgroup and the Analysis of Interdistributional Inequality," in Silber, J. (ed.), *Handbook of Income Inequality Measurement*, 363–97, Kluwer, Dordrecht, 1999.

Donaldson, D. and J. A. Weymark, "Ethically Flexible Gini Indices for Income Distributions in the Continuum," *Journal of Economic Theory*, 29, 353–58, 1983.

Ebert, U., "Measures of Distance between Income Distributions," *Journal of Economic Theory*, 32, 266–74, 1984.

Gastwirth, J. L., "Statistical Measures of Earnings Differentials," *The American Statistician*, 29, 32–5, 1975.

Gastwirth, J. L., T. K. Nayak, and J-L. Wang, "Statistical Properties of Measures of Between-Group Income Differentials," *Journal of Econometrics*, 42, 5–19, 1989.

Jenkins, S. P., "Earnings Discrimination Measurement: A Distributional Approach," *Journal of Econometrics*, 61, 81–102, 1994.

Kotlikoff, L. J. and S. Burns, *The Coming Generational Storm*, MIT Press, Cambridge, 2004.

Le Breton, M., A. Michelangeli, and E. Peluso, "A Stochastic Dominance Approach to the Measurement of Discrimination," *Journal of Economic Theory*, 147, 1342–50, 2012.

Mehran, F., "Linear Measures of Income Inequality," *Econometrica*, 44, 805–09, 1976.

Pfähler, W., "Redistributive Effect and Tax Progressivity: Evaluating a General Class of Aggregate Measures," *Public Finance/Finances Publiques*, 37, 1–31, 1987.

———, "Distributional Equity and Measurement of Tax Progressivity," in W. Eichhorn (ed.), *Measurement in Economics: Theory and Application to Economic Indices*, 533–48, Physica Verlag, Heidelberg, 1988.

Shorrocks, A. F., "On the Distance between Income Distributions," *Econometrica*, 50, 1337–39, 1982.

Social Security Administration, "Replacement Rates for Hypothetical Retired Workers," Actuarial Note Number 2014.9, July, 2014.

Yaari, M. E., "The Dual Theory of Choice under Risk," *Econometrica*, 55, 95–115, 1987.

———, "A Controversial Proposal Concerning Inequality Measurement," *Journal of Economic Theory*, 44, 381–97, 1988.

Yitzhaki, S., "On an Extension of the Gini Inequality Index," *International Economic Review*, 24, 617–28, 1983.

———, "Economic Distance and Overlapping Distributions," *Journal of Econometrics*, 61, 147–59, 1994.

Supporting Information

Additional supporting information may be found in the online version of this article at the publisher's web-site:

Table A1: U.S. Truncated Mean Income Functions by Age and Race

Table A2: Second-Order Discrimination Curves for U.S. Whites, Blacks, and Hispanics: Seniors versus Non-Seniors