the review of income and wealth

Review of Income and Wealth
Series 63, Number 4, December 2017
DOI: 10.1111/roiw.12260

THE THEIL INDICES IN PARAMETRIC FAMILIES OF INCOME DISTRIBUTIONS—A SHORT REVIEW

BY JOSÉ MARÍA SARABIA*, VANESA JORDÁ AND LORENA REMUZGO

Department of Economics, University of Cantabria

The Theil indices (Theil, 1967) are widely used measures for studying the degree of concentration and inequality in size income distributions. Their property of decomposability makes these indices especially useful in applied economic analysis. This paper is a synthetic review of the Theil indices for the most important and popular parametric income distributions. Extensions to higher dimensions are sketched.

JEL Codes: C13, C46, D63

Keywords: Champernowne distribution, generalized gamma distribution, generalized beta distributions, κ-generalized income distribution

1. INTRODUCTION

The study of economic inequality has received a huge amount of attention in the literature and is still a field of intense ongoing research. The debate about the uneven distributional effects of globalization or the consequences of the current financial crisis on income disparities have reinforced the interest in this topic.

Several inequality measures have been proposed in the literature. Among them, the most commonly used is the Gini index, the interpretation of which is very intuitive in terms of the area between the egalitarian line and the Lorenz curve. Notwithstanding its popularity, this indicator presents some rigidities with respect to other inequality measures. In particular, it does not allow us to vary the sensitivity of the index to redistributional movements at specific income ranges, which can point out different evolutions of disparities when no Lorenz dominance can be found. A generalization of this index was proposed by Yitzhaki (1983), which includes a parameter to allow for different degrees of "aversion to inequality". In this line, the generalized entropy (GE) inequality measures also include a sensitivity parameter which, in this case, varies the importance given to the differences at the upper tail. This index becomes more sensitive to changes at the top of the distribution as the parameter increases. The two limiting cases of this family of inequality measures are obtained when the parameter is set to 1 and 0 (hereinafter T_1 and T_0), corresponding to the Theil entropy index and the mean log deviation (MLD), respectively. These two measures have been widely used to

*Correspondence to: José María Sarabia, Department of Economics, University of Cantabria, 39005-Santander, Spain (sarabiaj@unican.es).

investigate the evolution of income inequality (see, e.g., Bourguignon and Morrisson, 2002; Chotikapanich *et al.*, 2007; Jorda *et al.*, 2014; Warner *et al.*, 2014). The Theil entropy index treats equally differences in all parts of the distribution, while the MLD is more sensitive to changes at the bottom tail. An appealing feature of the GE measures is the property of additive decomposability, which makes these indices especially useful for measuring socioeconomic inequalities.

Closed expressions of the GE measures can be obtained straightforwardly using the theoretical moments of the parametric distributions, but the formulas for the Theil indices are not so simple. In this paper, we present a convenient expression for the Theil indices (Theil, 1967) which allows us to obtain these inequality measures for the most relevant parametric families of income distributions and to derive closed expressions for multivariate versions of these indices. We review these formulas for the most common income distributions and obtain some expressions that were not available in the existing literature. In particular, we have obtained these measures for the Champernowne (1952) distribution, the generalized gamma distribution and the generalized beta distributions of the first and second kind (McDonald, 1984), the κ-generalized income distribution (Clementi *et al.*, 2007), and several Pareto and gamma-type distributions. The moments of these families are also included to facilitate the computation of the GE measures.

The contents of this paper are the following. In Section 2, we present general expressions for the Theil entropy index and the MLD, which are especially convenient to obtain these indicators in a closed form for a battery of parametric distributions. Section 3 presents the Theil indices for many classical distributions, including the Champernowne distribution and the three especial cases of the generalized beta family. The Pareto hierarchy is also studied and recently developed functional forms, such as the κ-generalized distribution, are also included. Section 4 describes the decomposition of the GE measures. Extensions to higher dimensions are presented in Section 5. Finally, Section 6 concludes the paper.

2. Computation of the Theil Indices

Let X be an absolutely continuous non-negative random variable with probability density function (pdf) $f(x)$ and support (l, u), where $0 \leq l < u < \infty$. Assume that the mathematical expectation of X is finite and is given by $E(X)=\mu$.

The class of GE measures is given by

$$\text{GE}(\theta)=\frac{1}{\theta(\theta-1)}\left[E\left(\frac{X}{\mu}\right)^{\theta}-1\right], \quad \theta \neq 0,1, \tag{1}$$

with $E(X^{\theta}) < \infty$. The limiting cases of the GE measures for $\theta=0$ and $\theta=1$ can be obtained as special cases of the integral

$$\int_l^u x^r (\log x)^m f(x) \mathrm{d}x = E[X^r (\log X)^m], \tag{2}$$

with $r=0$, $m=1$, and $r=m=1$ respectively. In order to compute the previous integral (2), we have the following simple lemma.

Lemma 1. *Let X be an absolutely continuous random variable such that $E(X^r)<\infty$, for some positive integer $r\in\{1,2,\ldots,\infty\}$. If,*

$$g(r)=E(X^r),$$

we have

$$\frac{d^m}{dr^m}g(r)=E[X^r(\log X)^m],\ m=1,2,\ldots$$

Proof. The proof is direct and will be omitted. ■

If we define

$$u(r,m)=E[X^r(\log X)^m],$$

we have

$$u(0,1)=E(\log X)$$

and

$$u(1,1)=E(X\log X).$$

Now, the relation of the previous quantities with the Theil indices is direct. If we consider the two Theil indices

$$T_0(X)=-E\left(\log\frac{X}{\mu}\right)$$

and

$$T_1(X)=E\left(\frac{X}{\mu}\log\frac{X}{\mu}\right),$$

we have

$$T_0(X)=-u(0,1)+\log\mu, \tag{3}$$

$$T_1(X)=\frac{1}{\mu}u(1,1)-\log\mu. \tag{4}$$

These expressions for the Theil indices are especially convenient to obtain these measures in a closed form for different parametric distributions. Equation (2) entails as particular cases the formulas used by Jenkins (2009) to obtain the two limiting GE measures: GE(0) for $r = 0$ and $m = 1$ and GE(1) for $r = 1$, $m = 1$. Lemma 1 not only applies to the unidimensional case, but also to derive higher-order moments, which are used to obtain the multivariate versions of these indices in Section 5.

3. Parametric Income Models

In this section, we report the Theil indices of the main parametric income distributions.

Table 2 includes the Theil indices for a number of classical income distributions, which are defined in terms of their pdf or the cumulative distribution function (cdf) as presented in Table 1, where the moments are also included.

The T_1 index of the classical Pareto distribution can be found in Kleiber and Kotz (2003). Next, we include the Pareto hierarchy of distributions introduced by Arnold (2015). These indices have been obtained from Table 1 and equations (3 and 4). For the lognormal distribution, both coefficients coincide (Kleiber and Kotz, 2003). For the Champernowne distribution (Champernowne, 1953), the formulas for these coefficients are new. Finally, we have included the Theil indices for the three McDonald (1984) income distributions: the generalized gamma (GG) distribution and the generalized beta distributions of the first and second kind (GB1 and GB2). The Theil indices for the GB2 distribution were obtained by Jenkins (2009). For the GG distribution, the T_1 index was obtained by McDonald and Ransom (2008) for $a = 1$ (the classical gamma distribution). These results were also reported by Salem and Mount (1974) and McDonald (1984). To our knowledge, the formulas for the GB1 distribution are new. For the case of the classical beta distribution ($a = 1$), the formula for T_1 was considered by (McDonald, 1981) and Pham-Gia and Turkkan (1992).

Table 4 reports the Theil indices for other income distributions. The corresponding pdfs or cdfs and moments are given in Table 3.

First, we include the indices for the Pareto-lognormal (Colombi, 1990) and for the double Pareto-lognormal distribution, developed by Reed (2003) and studied further by Reed and Jorgensen (2004). These indices were obtained by Hajargasht and Griffiths (2013). The single and double Pareto distributions are emerging distributions for income that are derived from a flexibilization of the assumptions of the economic model that yields the lognormal distribution. This family has good theoretical properties and some evidence has been found in favor of its performance to fit income data (see Hajargasht and Griffiths, 2013).

The Stoppa (1990) distribution is an extension of the classical Pareto distribution, which is obtained as a power transformation of the cumulative distribution function of the Pareto distribution. The log-gamma distribution was proposed by Taguchi *et al.* (1993) and is also considered as a generalized Pareto distribution, since the classical Pareto distribution corresponds to the choice $p = 1$. The inverse gamma distribution is used in Bayesian analysis as a prior distribution in several problems. Note that equation (5.69) in Kleiber and Kotz (2003) related to the moments of the inverse gamma distribution is not correct, and hence we report the right expression in Table 3. If $p = 1/2$, we obtain the Levy

TABLE 1

The Probability Density Function (pdf) or Cumulative Distribution Function (cdf) and Moments of Some Classical Income Distributions.

Distribution	Pdf $f(x)$ or Cdf $F(x)$	rth Moment $E(X^r)$
Classical Pareto	$f(x;\alpha,\sigma)=\frac{\alpha\sigma^\alpha}{x^{\alpha+1}},\ x\geq\sigma>0$	$\frac{\alpha\sigma^r}{\alpha-r},\ \alpha>r$
Pareto II	$f(x;\alpha,\sigma)=\frac{\alpha}{\sigma(1+x/\sigma)^{\alpha+1}},\ x\geq 0$	$\frac{\sigma^r\Gamma(\alpha-r)\Gamma(r+1)}{\Gamma(\alpha)},\ \alpha>r$
Pareto III	$F(x)=1-\frac{1}{[1+(x/\sigma)^{1/a}]},\ x\geq 0$	$\sigma^r\Gamma(1-ra)\Gamma(1+ra),\ -1/a<r<1/a$
Pareto IV	$F(x)=1-\frac{1}{[1+(x/\sigma)^{1/a}]^\alpha},\ x\geq 0$	$\frac{\sigma^r\Gamma(\alpha-ra)\Gamma(1+ra)}{\Gamma(\alpha)},\ -1/a<r<\alpha/a$
Lognormal	$f(x;\mu,\sigma)=\frac{1}{\sigma x\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right\},\ x\geq 0$	$\exp\left(\mu r+\frac{1}{2}r^2\sigma^2\right)$
Gamma	$f(x;b,p)=\frac{x^{p-1}e^{-x/b}}{b^p\Gamma(p)},\ x\geq 0$	$\frac{b^r\Gamma(p+r)}{\Gamma(p)},\ p+r>0$
Champernowne	$f(x;\alpha,\theta,x_0)=\frac{\alpha\sin(\theta)}{2\theta x\{\cosh[\alpha\log(x/x_0)]+\cos(\theta)\}},\ x\geq 0$	$x_0^r\frac{\pi\sin(r\theta/\alpha)}{\theta\sin(r\pi/\alpha)},\ -\alpha<r<\alpha$
GG	$f(x;a,b,p)=\frac{ax^{ap-1}e^{-(x/b)^a}}{b^{ap}\Gamma(p)},\ x\geq 0$	$\frac{b^r\Gamma(p+\frac{r}{a})}{\Gamma(p)}$
GB1	$f(x;a,b,p,q)=\frac{ax^{ap-1}[1-(x/b)^a]^{q-1}}{b^{ap}B(p,q)},\ 0\leq x\leq b$	$\frac{b^r\Gamma(p+r/a)\Gamma(p+q)}{\Gamma(p+q+r/a)\Gamma(p)}.$
GB2	$f(x;a,b,p,q)=\frac{ax^{ap-1}}{b^{ap}B(p,q)[1+(x/b)^a]^{p+q}},\ x\geq 0$	$\frac{b^r\Gamma(p+r/a)\Gamma(q-r/a)}{\Gamma(p)\Gamma(q)},\ -ap<r<aq$

Notes: $\Gamma(z)$ is the usual gamma function and $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the usual beta function.

TABLE 2
THE THEIL INDICES OF SOME CLASSICAL INCOME DISTRIBUTIONS.

Distribution	$T_0(X)$	$T_1(X)$
Classical Pareto	$-\frac{1}{\alpha}+\log\frac{\alpha}{\alpha-1},\ \alpha>1$	$\frac{1}{\alpha-1}-\log\frac{\alpha}{\alpha-1},\ \alpha>1$
Pareto II	$\gamma+\psi(\alpha)-\log(\alpha-1),\ \alpha>1$	$1-\gamma-\psi(\alpha-1)+\log(\alpha-1),\ \alpha>1$
Pareto III	$\log[\Gamma(1-a)\Gamma(1+a)],\ a<1$	$a[\psi(1+a)-\psi(1-a)]-\log[\Gamma(1-a)\Gamma(1+a)],\ a<1$
Pareto IV	$a[\gamma+\psi(\alpha)]+\log\left[\frac{\Gamma(1+a)\Gamma(\alpha-a)}{\Gamma(\alpha)}\right],\ \alpha>a$	$a[\psi(1+a)-\psi(\alpha-a)]-\log\left[\frac{\Gamma(1+a)\Gamma(\alpha-a)}{\Gamma(\alpha)}\right],\ \alpha>a$
Lognormal	$\frac{\sigma^2}{2}$	$\frac{\sigma^2}{2}$
Gamma	$-\psi(p)+\log p$	$\psi(p+1)-\log p$
Champernowne	$\log\left(\frac{\pi}{\theta}\frac{\sin(\theta/\alpha)}{\sin(\pi/\alpha)}\right)$	See the Appendix
GG	$-\frac{\psi(p)}{a}+\log\frac{\Gamma(p+\frac{1}{a})}{\Gamma(p)}$	$\frac{\psi(p+\frac{1}{a})}{a}-\log\frac{\Gamma(p+\frac{1}{a})}{\Gamma(p)}$
GB1	$-\frac{\psi(p)-\psi(p+q)}{a}+\log\frac{\Gamma(p+q)\Gamma(p+\frac{1}{a})}{\Gamma(p)\Gamma(p+q+\frac{1}{a})}$	$\frac{\psi(p+\frac{1}{a})-\psi(p+q+\frac{1}{a})}{a}-\log\frac{\Gamma(p+q)\Gamma(p+\frac{1}{a})}{\Gamma(p)\Gamma(p+q+\frac{1}{a})}$
GB2	$-\frac{\psi(p)-\psi(q)}{a}+\log\frac{\Gamma(p+\frac{1}{a})\Gamma(q-\frac{1}{a})}{\Gamma(p)\Gamma(q)}$	$\frac{\psi(p+\frac{1}{a})-\psi(q-\frac{1}{a})}{a}-\log\frac{\Gamma(p+\frac{1}{a})\Gamma(q-\frac{1}{a})}{\Gamma(p)\Gamma(q)},\ q>1/a$

Notes: $\gamma=0.5772156649$ is Euler's constant; $\psi(z)=\frac{d\log\Gamma(z)}{dz}=\frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function.

TABLE 3

THE PROBABILITY DENSITY FUNCTION (PDF) OR CUMULATIVE DISTRIBUTION FUNCTION (CDF) AND MOMENTS FOR OTHER PARAMETRIC INCOME DISTRIBUTIONS.

Distribution	Pdf $f(x)$ or Cdf $F(x)$	rth Moment $E(X^r)$
Pareto-lognormal	$f(x)=\frac{\alpha}{x}\phi\left(\frac{\log x-\mu}{\sigma}\right)R(x_1),\ x\geq 0$	$\frac{\alpha}{\alpha-r}\exp\left(r\mu+\frac{r^2\sigma^2}{2}\right),\ \alpha>r$
Double Pareto-lognormal	$f(x)=\frac{\alpha\beta}{(\alpha+\beta)x}\phi\left(\frac{\log x-\mu}{\sigma}\right)\{R(x_1)+R(x_2)\},\ x\geq 0$	$\frac{\alpha\beta}{(\alpha-r)(\beta+r)}\exp\left(r\mu+\frac{r^2\sigma^2}{2}\right),\ \alpha>r$
Stoppa	$F(x)=\left[1-\left(\frac{x}{\sigma}\right)^{-\alpha}\right]^{\theta},\ x\geq\sigma>0$	$\sigma^r\theta B\left(1-\frac{r}{\alpha},\theta\right),\ r<\alpha$
Log-gamma	$f(x;\beta,p)=\frac{\beta^p x^{-(\beta+1)}\{\log(x)\}^{p-1}}{\Gamma(p)},\ x\geq 1$	$\left(\frac{\beta}{\beta-r}\right)^p,\ \beta>r$
Inverse gamma	$f(x)=\frac{\lambda^p}{\Gamma(p)}x^{-p-1}e^{-\lambda/x},\ x\geq 0$	$\lambda^r\frac{\Gamma(p-r)}{\Gamma(p)},\ p>r$
Log-Gompertz	$F(x;a,b)=\exp\{-(x/b)^{-a}\},\ x\geq 0$	$b^r\Gamma\left(1-\frac{r}{a}\right),\ r<a$
κ distribution	$f(x;\alpha,\beta,\kappa)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\frac{\exp_\kappa[-(x/\beta)^\alpha]}{\sqrt{1+\kappa^2(x/\beta)^{2\alpha}}}$	$\frac{\beta^r}{(2\kappa)^{r/\alpha}}\frac{\Gamma(1+\frac{r}{\alpha})}{1+\frac{r}{\alpha}\kappa}\frac{\Gamma(\frac{1}{2\kappa}-\frac{r}{2\alpha})}{\Gamma(\frac{1}{2\kappa}+\frac{r}{2\alpha})},\ -\alpha/\kappa<r<\alpha/\kappa$

Notes: $\phi(z)$ is the pdf of the standard normal distribution; $R(z)=[1-\Phi(z)]/\phi(z)$ is the Mills ratio, where $\Phi(z)$ is the cdf of the standard normal distribution; $x_1=\alpha\sigma-(\log x-\mu)/\sigma$, $x_2=\beta\sigma-(\log x-\mu)/\sigma$; and $\exp_\kappa(x)=(\sqrt{1+\kappa^2x^2}+\kappa x)^{1/\kappa}$ is the deformed exponential function.

TABLE 4
Theil Indices in Other Parametric Income Distributions

Distribution	$T_0(X)$	$T_1(X)$
Pareto-lognormal	$-\frac{1}{\alpha}+\frac{\sigma^2}{2}-\log\left(\frac{\alpha-1}{\alpha}\right),\ \alpha>1$	$\frac{1}{\alpha-1}+\frac{\sigma^2}{2}+\log\left(\frac{\alpha-1}{\alpha}\right),\ \alpha>1$
Double Pareto-lognormal	See the Appendix	See the Appendix
Stoppa	$-\frac{\gamma+\psi(1+\theta)}{\alpha}+\log\left\{\theta B\left(1-\frac{1}{\alpha},\theta\right)\right\},\ \alpha>1$	$\frac{\psi(1-\frac{1}{\alpha}+\theta)-\psi(1-\frac{1}{\alpha})}{\alpha}-\log\left\{\theta B\left(1-\frac{1}{\alpha},\theta\right)\right\},\quad \alpha>1$
Log-gamma	$-\frac{p}{\beta}+p\log\frac{\beta}{\beta-1},\ \beta>1$	$\frac{p}{\beta-1}-p\log\frac{\beta}{\beta-1},\ \beta>1$
Inverse gamma	$\psi(p)-\log(p-1),\ p>1$	$\psi(p-1)+\log(p-1),\ p>1$
Log-Gompertz	$-\frac{\gamma}{a}+\log\Gamma\left(1-\frac{1}{a}\right),\ a>1$	$-\frac{\psi(1-\frac{1}{a})}{a}-\log\Gamma\left(1-\frac{1}{a}\right),\ a>1$
κ distribution	See the Appendix	See the Appendix

distribution, which is a special case of the stable distributions and it arises as the first-passage ties in Brownian motion. Formulas for the log-Gompertz distribution are also included.

Finally, we present the Theil indices for the κ-generalized income distribution (see Clementi *et al*., 2007, 2008, 2012). This distribution has been proposed as a descriptive model for the distribution and dispersion of income within a population based on the deformed exponential and logarithmic functions recently introduced by Kaniadakis (2001). In a reasonably large number of cases, the statistical performance of this model is not considered inferior to those of the Singh–Maddala, the Dagum, and the GB2 distributions, whereas its ability to match the entropy of the data dominates these families. These formulas appear in Clementi *et al*. (2010).

4. Decomposition of the Generalized Entropy Measures

In this section, we review the decomposition properties of the Theil index. We consider two kinds of decompositions: by factors and by population subgroups.

We begin with the decomposition by factors. Let $X_1, X_2, \ldots, X_m$ be independent and positive random variables with finite expectations and consider the new random variable $X=\prod_{i=1}^{m} X_i$. Then, it is direct to show that

$$T_k(X)=\sum_{i=1}^{m} T_k(X_i), \ \ k=0,1. \tag{5}$$

Example 1. *Let X be a Pareto-lognormal distribution with pdf given in the first row of Table* 3*. By construction, $X=X_1 \cdot X_2$, where X_1 is a classical Pareto distribution with parameters α and σ, and X_2 a classical lognormal distribution with parameters μ and σ^2, both of which are independent. Then, using (5) for $T_0(X)$, we have*

$$T_0(X)=T_0(X_1)+T_0(X_2)=-\frac{1}{\alpha}+\log\frac{\alpha}{\alpha-1}+\frac{\sigma^2}{2},$$

which is the formula for the Pareto-lognormal in Table 4*. A similar reasoning holds for the computation of $T_1(X)$ using (5).*

Example 2. *Let X be a GB2 distribution with shape parameters (a, p, q) and unit scale. It is well known that X can be written as $X=\frac{X_1}{X_2}$, where X_1 and X_2 are independent generalized gamma distributions, with unit scale parameters and shape parameters (a, p) and (a, q) respectively (see Kleiber and Kotz, 2003, ch. 6). The random variable X_2^{-1} is distributed as a inverted generalized gamma distribution with Theil parameter $T_0(X_2^{-1})=\frac{\psi(q)}{a}-\log\frac{\Gamma(q-1/a)}{\Gamma(q)}$, if $q>1/a$. Then,*

$$T_0(X)=T_0(X_1)+T_0(X_2^{-1})=-\frac{\psi(p)}{a}+\log\frac{\Gamma(p+1/a)}{\Gamma(p)}+\frac{\psi(q)}{a}-\log\frac{\Gamma(q-1/a)}{\Gamma(q)},$$

and we obtain the formula in Table 2. In a similar way, we obtain the $T_1(X)$ index for the GB2 distribution.

Empirical analyses of income distributions frequently require the decomposition of inequality by population subgroups, to account for different income patterns by sex, region, or social status, for instance. The generalized entropy measures are of especial interest, being the sole indices of relative inequality which are additively decomposable into a weighted sum of inequality within each group and another component that captures differences in mean income across groups (between-group inequality) (Shorrocks, 1980).

If we divide the population of a country into K mutually exclusive groups, the income distribution can be defined as a mixture of the distributions of those groups weighted by their population shares:

$$f(x)=\sum_{k=1}^{K}\lambda_k f_k(x;\Theta),$$

where λ_k stands for the population weights of the groups. The global cdf would be the integral of the pdf given by the previous equation:

$$F(x)=\sum_{k=1}^{K}\lambda_k F_k(x;\Theta).$$

The mean of this distribution is a weighted sum of the mean income of each group, given by

$$\mu=\sum_{k=1}^{K}\lambda_k\mu_k,$$

where μ_k is the first moment of the income distribution, which can be obtained from Tables 1 and 3.

Overall GE measures can be expressed in terms of the analytical expression of the decomposition, given by

$$GE(\theta)=\sum_{k=1}^{K}\lambda_k^{1-\theta}s_k^{\theta}GE(\theta)_k+\frac{1}{\theta(\theta-1)}\left(\sum_{k=1}^{K}\lambda_k\left(\frac{\mu_k}{\mu}\right)^{\theta}-1\right), \tag{6}$$

where λ_k and $GE(\theta)_k$ are the population share and the GE measure of the group k. s_k stands for the proportion of mean income of the group k in the national mean: $s_k=\frac{\lambda_k\mu_k}{\mu}=\frac{\lambda_k\mu_k}{\sum_{k=1}^{K}\lambda_k\mu_k}$.

The first component in equation (6) measures the within-group inequality, given by the weighted sum of disparities in each group. The weights $\omega_k = \lambda_k^{1-\theta} s_k^{\theta}$ depend on the proportion of the population in each group and the share of their per capita income in the overall mean. Interestingly, these weights add up to one only for the two limit cases corresponding to the Theil indices, which can be expressed as

$$T_1 = \sum_{k=1}^{K} s_k T_{1k} + \sum_{k=1}^{K} s_k \log\left(\frac{\mu_k}{\mu}\right),$$

$$T_0 = \sum_{k=1}^{K} \lambda_k T_{0k} + \sum_{k=1}^{K} \lambda_k \log\left(\frac{\mu}{\mu_k}\right),$$

where T_1 and T_0 are, respectively, the Theil and the MDL indices of group k, which can be taken from Tables 2 and 4.

Interestingly, only the Theil entropy measure is consistent with the concept "income weighted decomposability," which defines the decomposition coefficients in terms of the income shares (Bourguignon, 1979). Alternative candidates for the weights would be population shares, thus resulting in the so-called "population weighted decomposability" which is only satisfied by the MDL. It should be noted that the within-group component is a weighted average of inequalities within groups only for these two measures. This is an interesting feature from an accounting point of view (Lambert, 2001). Theil (1969) pointed out a relevant advantage of these measures. The term $1 - \sum_{k=1}^{K} \omega_k$ is proportional to the between-group component, and hence the decomposition coefficients are independent of differences between groups only for the two especial cases T_0 and T_1. The second component reflects the level of inequality that would exist if all individuals in the group had the same income. This is equivalent to removing any differences within the group by replacing each income by the mean of the group.

5. Extensions to Higher Dimensions

In this section, we sketch possible extensions of these indices to higher dimensions. Let $\boldsymbol{X} = (X_1, \ldots, X_m)^\top$ be an m-dimensional vector with non-negative components. We define the m-dimensional Theil indices as

$$T_0^{(m)}(\boldsymbol{X}) = -E\left[\log\left(\frac{X_1 \cdots X_m}{\mu_{12\ldots m}}\right)\right] \tag{7}$$

and

$$T_1^{(m)}(\boldsymbol{X}) = -E\left[\frac{X_1 \cdots X_m}{\mu_{12\ldots m}} \log\left(\frac{X_1 \cdots X_m}{\mu_{12\ldots m}}\right)\right], \tag{8}$$

where

$$\mu_{12\ldots m}=E(X_1\cdots X_m)<\infty.$$

For the bivariate case $m=2$, expression (7) can be written as

$$T_0^{(2)}(\boldsymbol{X})=-E(\log X_1)-E(\log X_2)+\log\mu_{12}, \tag{9}$$

and (8) can be written as

$$T_1^{(2)}(\boldsymbol{X})=\mu_{12}^{-1}[E(X_1X_2\log X_1)+E(X_1X_2\log X_2)-\mu_{12}\log\mu_{12}], \tag{10}$$

where $\mu_{12}=E(X_1X_2)$. For computing the terms in (10), we define

$$u(r_1,r_2)=E(X_1^{r_1}X_2^{r_2}).$$

Then

$$v_1(r_1,r_2)=\frac{\partial u(r_1,r_2)}{\partial r_1}=E(X_1^{r_1}X_2^{r_2}\log X_1), \tag{11}$$

and

$$v_2(r_1,r_2)=\frac{\partial u(r_1,r_2)}{\partial r_2}=E(X_1^{r_1}X_2^{r_2}\log X_2). \tag{12}$$

In consequence,

$$v_1(1,1)=E(X_1X_2\log X_1)$$

and

$$v_2(1,1)=E(X_1X_2\log X_2).$$

If $\boldsymbol{X}=(X_1,X_2)^\top$ is a bivariate lognormal distribution with joint pdf

$$f_{\boldsymbol{X}}(x_1,x_2)=\frac{1}{2\pi x_1x_2\sqrt{|\Sigma|}}\exp\left\{-\frac{1}{2}(\log\boldsymbol{x}-\boldsymbol{\mu})^\top\Sigma^{-1}(\log\boldsymbol{x}-\boldsymbol{\mu})\right\},$$

where $\boldsymbol{x}=(x_1,x_2)^\top$, $\boldsymbol{\mu}=(\mu_1,\mu_2)^\top$. We have

$$E(X_1^{r_1}X_2^{r_2})=\exp\left(\boldsymbol{r}^\top\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{r}^\top\Sigma\boldsymbol{r}\right),$$

in which $\boldsymbol{r}=(r_1,r_2)^\top$. Using (9)–(10) and (11)–(12), we obtain

$$T_0^{(2)}(\boldsymbol{X})=T_1^{(2)}(\boldsymbol{X})=\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\sigma_{12},$$

where both bivariate indices are the same, similar to the univariate case.

6. Conclusions

In this paper, we have reviewed some of the existing formulas of the Theil indices for several parametric models of income distributions. The GE family of inequality measures has been widely used to study concentration and inequality. Its property of decomposability makes these indices particularly useful for studying socioeconomic inequalities. This family includes a sensitivity parameter that varies the weight given to income differences at the upper tail. Despite its importance, the expressions for these inequality measures are only available for a few families of distributions. The derivation of closed expressions of the GE measures is relatively straightforward using the theoretical moments of the parametric distributions. However, the Theil indices are limiting cases of this family and they cannot be obtained so directly.

We have presented a convenient general expression for the Theil indices which allows us to obtain these inequality measures for many of the most important parametric income distributions and to derive a multivariate extension of these measures. This paper has included closed expressions of both Theil indices for the Champernowne (1952) distribution, the generalized gamma distribution and the generalized beta distributions of the first and second kinds (McDonald, 1984), the κ-generalized income distribution (Clementi *et al.*, 2007), and several Pareto and gamma-type distributions. We have described the decomposition of the GE measures and the proposed extension of the Theil indices to higher dimensions.

Acknowledgments

The authors thank the Ministerio de Economía y Competitividad (project ECO2013-48326-C2-2-P) and the Ministerio de Educación, Cultura y Deporte (FPU13/02155) for partial support of this work. We are grateful for the constructive suggestions provided by the editor and the reviewers, which improved the paper.

References

Arnold, B. C., *Pareto Distributions*, 2nd edn, Chapman & Hall, London/CRC Press, Boca Raton, FL, 2015.

Bourguignon, F., "Decomposable Income Inequality Measures," *Econometrica*, 901–20, 1979.

Bourguignon, F. and C. Morrisson, "Inequality among World Citizens: 1820–1992," *American Economic Review*, 92, 727–44, 2002.

Champernowne, D. G., "The Graduation of Income Distributions," *Econometrica*, 20, 591–615, 1952.

———, "A Model of Income Distribution," *The Economic Journal*, 318–51, 1953.

Chotikapanich, D., W. E. Griffiths, and D. P. Rao, "Estimating and Combining National Income Distributions Using Limited Data," *Journal of Business & Economic Statistics*, 25, 97–109, 2007.

Clementi, F., T. Di Matteo, M. Gallegati, and G. Kaniadakis, "The κ-Generalized Distribution: A New Descriptive Model for the Size Distribution of Incomes," *Physica A*, 387, 3201–8, 2008.

Clementi, F., M. Gallegati, and G. Kaniadakis, "κ-Generalized Statistics in Personal Income Distribution," *European Physical Journal B*, 57, 187–93, 2007.

———, "A Model of Personal Income Distribution with Application to Italian Data," *Empirical Economics*, 39, 559–91, 2010.

———, "A Generalized Statistical Model for the Size Distribution of Wealth," *Journal of Statistical Mechanics: Theory and Experiment*, 12, P12006, 2012.

Colombi, R., "A New Model of Income Distribution: The Pareto-Lognormal Distribution," in C. Dagum and M. Zenga (eds), *Income and Wealth Distribution, Inequality and Poverty*, Springer, New York, 18–32, 1990.
Hajargasht, G. and W. E. Griffiths, "Pareto-Lognormal Distributions: Inequality, Poverty, and Estimation from Grouped Income Data," *Economic Modelling*, 33, 593–604, 2013.
Jenkins, S. P., "Distributionally-Sensitive Indices and the GB2 Distribution," *Review of Income and Wealth*, 55, 392–8, 2009.
Jorda, V., J. M. Sarabia, and F. Prieto, "On the Estimation of the Global Income Distribution Using a Parsimonious Approach," in J. Bishop and J. G. Rodríguez (eds), *Economic Well-Being and Inequality*, Research on Economic Inequality, vol. 22, Emerald, Bingley, U.K., 115–45, 2014.
Kaniadakis, G., "Non-linear Kinetics Underlying Generalized Statistics," *Physica A*, 296, 405–25, 2001.
Kleiber, C. and S. Kotz, *Statistical Size Distributions in Economics and Actuarial Sciences*, John Wiley & Sons, Inc., Hoboken, NJ, 2003.
Lambert P. J., *The Distribution and Redistribution of Income*, Manchester University Press, Manchester, 2001.
McDonald, J. B., "Some Issues Associated with the Measurement of Income Inequality," in C. Taillie, G. P. Patil, and B. A. Baldessari (eds), *Statistical Distributions in Scientific Work*, Applications in Physical, Social, and Life Sciences, vol. 6, Reidel, Dordrecht, 161–79, 1981.
———, "Some Generalized Functions for the Size Distribution of Income," *Econometrica*, 52, 647–63, 1984.
McDonald, J. B. and M. Ransom, "The Generalized Beta Distribution as a Model for the Distribution of Income," in D. Chotikapanich (ed.), *Modeling Income Distributions and Lorenz Curves*, Springer, New York, 147–66, 2008.
Pham-Gia, T. and N. Turkkan, "Determination of the Beta Distribution from its Lorenz Curve," *Mathematical and Computer Modelling*, 16, 73–84, 1992.
Reed, W. J., "The Pareto Law Of Incomes - An Explanation and an Extension," *Physica A: Statistical Mechanics and its Applications*, 319, 469–86, 2003.
Reed, W. J. and M. Jorgensen, "The Double Pareto-Lognormal Distribution—A New Parametric Model for Size Distributions," *Communications in Statistics—Theory and Methods*, 33(8), 1733–53, 2004.
Salem, A. B. and T. D. Mount, "A Convenient Descriptive Model of Income Distribution: The Gamma Density," *Econometrica*, 42, 1115–27, 1974.
Shorrocks, A. F., "The Class of Additively Decomposable Inequality Measures," *Econometrica* 48, 613–25, 1980.
Stoppa, G., "A New Model for Income Size Distributions," in C. Dagum and M. Zenga (eds), *Income and Wealth Distribution, Inequality and Poverty*, Springer, New York, 33–41, 1990.
Taguchi, T., H. Sakurai, and S. Nakajima, "A Concentration Analysis of Income Distribution Model and Consumption Pattern: Introduction of Logarithmic Gamma Distribution and Statistical Analysis of Engel Elasticity," *Statistica*, 53, 31–57, 1993.
Theil, H., *Economics and Information Theory*, North-Holland, Amsterdam, 1967.
———, "A Multinomial Extension of the Linear Logit Model," *International Economic Review*, 10, 251–9, 1969.
Warner, D., D. S. Prasada Rao, W. E. Griffiths and D. Chotikapanich, "Global Inequality; Levels and Trends, 1993–2005: How Sensitive are These to the Choice of PPPs and Real Income Measures?" *Review of Income and Wealth*, 60, 281–304, 2014.
Yitzhaki, S., "On an Extension of the Gini Inequality Index," *International Economic Review*, 24, 617–28, 1983.

Supporting Information

Additional Supporting Information may be found in the online version of this paper at the publisher's web-site:

Appendix: Supporting Data