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THE GENERALIZED UNIT VALUE INDEX FAMILY

BY LUDWIG VON AUER*

Universität Trier, Germany

This study introduces a group of Generalized Unit Value indices that evaluate price level changes. The approach is to transform the original price and quantity data into numbers that all relate to a common unit which provides the same intrinsic worth to the consumers. When, in the transformed data, an equivalence of worth is present, then even incommensurables can be aggregated by the standard unit value method. The group of Generalized Unit Value indices includes some well-known (Laspeyres, Paasche, Banerjee), barely known (Lehr, Davies), and previously quite unknown price indices. Using a Generalized Unit Value price index as a deflator yields a particularly appealing and useful quantity index.

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1. INTRODUCTION

Recently, the Unit Value (UV) index has been the subject of several studies, for example de Haan (2002, 2004, 2007), Dalén (2001), and Silver (2010, 2011). These studies are concerned with computing the overall price change of products that differ but at the same time provide the same function, e.g. washing machines. These authors suggest the application of an adapted version of the UV index that they call the Quality Adjusted UV index. The modification comes in the form of quality adjustment factors that take into account the quality differences that exist between the products.

As one possibility, hedonic regression techniques are suggested for the estimation of these quality adjustment factors. Hedonic regression and some other quality adjustment methods, however, require the availability of external data concerning the inherent qualitative characteristics of the products involved. If this information is unavailable, then these inherent product differences must be handled in a manner that relies solely upon the observable price and quantity data from the marketplace.

Many years ago, Lehr (1885, pp. 37–39) and Davies (1924, pp. 182–86) developed methods for accomplishing this task. The approach is to transform the

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*Correspondence to: Ludwig von Auer, Fachbereich IV – Finanzwissenschaft, Universitätsring 15, D-54286 Trier, Germany (vonauer@uni-trier.de).

original price and quantity data into numbers that all relate to a common unit which provides the same intrinsic worth to the consumers. When, in the transformed data, an equivalence of worth is present, then even incommensurables can be aggregated by the standard unit value method. Lehr (1885, pp. 37–38) and Davies (1924, pp. 182–83) argued that their approach could be also used to aggregate heterogeneous products. Their noteworthy insights are the point of departure for this present research.

First, this study elucidates the arguments of Lehr and Davies and extends them to a general framework referred to as the Generalized Unit Value (GUV) index family. All family members share a unifying feature; their calculation requires no supplementary sources of information, such as measurements of the innate qualitative characteristics that are responsible for the differences in the products. They are based solely upon the observed price and quantity data from the marketplace.

Second, this study classifies some well-known, hardly known, and previously quite unknown price indices into the GUV index family. It demonstrates that the Laspeyres and Paasche indices, as well as those proposed by Davies (1924, p. 185) and Lehr (1885, p. 39), are members of this GUV index family.

Third, if price differences of similar products accurately reflect their quality differences, then the GUV indices are extremely useful for aggregating the prices of these products.

This paper is organized as follows. Some background material concerning price measurement utilizing the UV index is contained in Section 2. The applicability of this form of price index is demonstrated for the case of identical products. An amended version is presented for use with those products that are defined to be similar. The similar products considered have product differences that are observable and measurable. Section 3 considers the case of heterogeneous products where the auxiliary information concerning product differences is unavailable. The family of GUV indices is introduced in this section as well. Concluding remarks are contained in Section 4.

2. Preliminaries

In the Consumer Price Index Manual (a joint publication of the ILO, IMF, OECD, UNECE, Eurostat, and The World Bank), Boldsen and Hill (2004, p. 164) recommend the UV index for identical products. An axiomatic justification for the use of the UV index for identical products can be found in Auer (2008, pp. 9–11), and Silver (2011, p. 557) proposes an analogous argument. An elaborate formal examination of the unit value index is provided by Balk (1998, p. 8), who explores the link between economic theory and the UV index. He emphasizes that product homogeneity is essential for the appropriateness of the UV index. Ivancic and Fox (2013) demonstrate how an empirical examination of product homogeneity can be conducted.

2.1. *The Unit Value Index*

Consider N identical products that are sold in the marketplace in both the base time period, $t = 0$, and a comparison period, $t = 1$. Furthermore, assume that the prevailing market conditions permitted these products to sell for different prices. Let p_i^t $(i = 1, \ldots, N)$ denote the unit price of product i in time period t. Similarly, let x_i^t denote the number of units transacted. Consequently, the value aggregates, $V^t = \sum p_i^t x_i^t$, are the total expenditure on the goods traded. The unit value (Segnitz, 1870, p. 184), P_{UV}^t, in time period t is:

$$P_{UV}^t = \frac{\sum p_i^t x_i^t}{\sum x_i^t} = \frac{V^t}{\sum x_i^t}.$$

The UV index (Drobisch, 1871a, p. 39; 1871b, p. 149), P_{UV}, is a ratio of unit values and it is used to measure the average price level change between the base and comparison time periods:

$$(1) \qquad P_{UV} = \frac{P_{UV}^1}{P_{UV}^0} = \frac{V^1}{V^0} \frac{\sum x_i^0}{\sum x_i^1}.$$

The product quantity summations, $\sum x_i^t$, yield accurate results because the product-identifying units being summed are identical. If the products considered are classified as almost identical, then the UV index continues to be the appropriate choice. This situation occurs when the products differ only with respect to the location and/or moment of purchase within a given time period.

2.2. *The Amended Unit Value Index*

When products are similar, but not identical, an amended version of the UV index is required. Similar products are defined as having innate differences that are observable and measurable. Such product differences occur frequently and stem from such things as quality levels, operating features, or simply the size of the packaging. These products have dissimilar product-identifying units and, consequently, they are unsuitable for the quantity summations in the UV index (1). The situation is correctable, however, by the inclusion of N product transformation rates, z_i $(i = 1, \ldots, N)$. They are defined to be an appropriate number of common units per product-identifying unit. Accordingly, the transformed prices, p_i^t / z_i, become monetary units per common unit of product i and the transformed quantities, $x_i^t z_i$, are the number of common units transacted in the form of product i. By definition, these common units are identical and, for that reason, reliable results are now obtained from the quantity summations, $\sum x_i^t z_i$. The value aggregates, V^t, remain unaffected by this transformation.

The functioning of these transformation rates, z_i, can best be illustrated by an example. Two similar products are presented in Table 1. They are gift boxes that contain the same assorted chocolates and differ only with respect to their net weight. Product B contains 300 grams while the smaller box, product S, contains only 200 grams. Assume that if producers were called upon to produce 600 grams of chocolates, they would be indifferent between producing two of the larger 300-gram boxes or three of the smaller 200-gram boxes. Moreover, consumers are indifferent in their consuming preferences. Finally, both types of packaging are not equally accessible to all consumers.

TABLE 1
EXAMPLE; SIMILAR PRODUCTS

	$t = 0$		$t = 1$	
	Price	Quantity	Price	Quantity
Product B	12	2	12	4
Product S	6	4	9	4

TABLE 2
PRICES AND QUANTITIES RELATING TO COMMON UNITS

	$t = 0$		$t = 1$	
	Price	Quantity	Price	Quantity
Product B	8	3	8	6
Product S	6	4	9	4

The product-identifying units, the big and small boxes, are not identical; therefore, the price and quantity data of product B and/or product S must be transformed. A convenient set of transformation rates are $z_B = 1.5$ and $z_S = 1.0$. These rates transform the product-identifying units into a common identical unit, the 200-gram portion. The transformed prices and quantities are presented in Table 2. For example, the transformed quantity, $x_B^1 z_B = 4 \cdot 1.5 = 6$, is the total quantity of 200-gram portions transacted during the comparison period in the form of 300-gram boxes. Each of these six common units is sold at the price $p_B^1 z_B = 12/1.5 = 8$.

Applying the unit value formula to the transformed data yields the amended unit value in time period t:

$$P_{AUV}^t = \frac{\sum (p_i^t / z_i)(x_i^t z_i)}{\sum x_i^t z_i} = \frac{V^t}{\sum x_i^t z_i}.$$

Consequently, the Amended Unit Value (AUV) index is:

$$(2) \qquad P_{AUV} = \frac{P_{AUV}^1}{P_{AUV}^0} = \frac{V^1}{V^0} \frac{\sum x_i^0 z_i}{\sum x_i^1 z_i}.$$

This index measures the change in the unit value of a common unit.

The numerical example yields $P_{AUV} = 1.225$. This indicates a 22.5 percent increase in the price level of assorted chocolates. This index is invariant to multiples of the transformation rates. For example, multiplying the transformation rates by 200 simply reduces the common unit into the 1-gram portion. Nevertheless, the numerical value of the AUV index (2) remains unaltered. In the case of identical products, $z_i = z$, the AUV index (2) simplifies to the UV index (1). If the transacted product quantities remain constant over time, then the AUV index (2) simplifies to the ratio of value aggregates, V^1/V^0.

2.3. *Implications for Practical Work*

The amended unit value index is not original to this study. It stems from the work of Dalén (2001, p. 11) and de Haan (2002, pp. 81–82). Furthermore, additional elucidations, together with some empirical applications, can be found in de Haan (2004, pp. 6–7). A related proposal is provided by Silver (2010, p. S220; 2011, p. 561). In all of these publications the price indices derived were concerned with the problem of aggregating the price changes of similar products into some average price change. By similar products these authors envisioned those products that serve the same purpose in consumption, yet have distinct quality differences. Accordingly, the formulas were labeled the Quality Adjusted UV index.

The authors point out that if the data required for the quality adjustment factors are not directly observable, then some form of estimation will be required. Hedonic regressions or similar techniques are recommended for the job. These estimation techniques, however, require the availability of auxiliary information concerning product quality. Can anything be accomplished if this information is unavailable? Moreover, when products serve completely different purposes, i.e., they are heterogeneous, is it still possible to use some amended version of the UV index for price measurement purposes?

3. The Generalized Unit Value Index

If two products are identical, then a unit of either product will provide the same amount of intrinsic worth to the consumer. It is not the equivalence in the tangible make-up of these products (e.g., their chemical, material, or technological characteristics) but this identical unit worth that permits the quantity summations in the UV index (1) to produce reliable results. In the context of similar products, the use of the quality adjustment factors is an attempt to emulate the case of the identical products. The quality adjustment factors convert the price and quantity data into numbers that all relate to units that provide the same amount of intrinsic worth to the consumer. This common unit is identical for all products and, therefore, the quantity summations can take place.

Accordingly, the essential prerequisite for reliable price measurements does not depend upon the products in question having the same tangible make-up. What is essential, however, is the presence of an identical worth unit. When an equivalence of worth is present, then a sufficient condition exists for appropriate price measurement and even incommensurables can be added in a price level calculation. Once the diverse product-identifying units have been transformed into a suitable number of intrinsic-worth units, then a meaningful quantity summation is possible. This is the first of two essential messages in the aforementioned studies by Lehr (1885, pp. 37–38) and Davies (1924, pp. 183–84).

3.1. *Background*

In the example presented in Section 2.2, two similar products with different product-identifying units were transformed into a common unit. This common unit was the 200-gram portion and it provided an identical worth to consumers. After transforming the data, the AUV index (2) provided a proper price measurement. A precondition for using the AUV index (2), however, is that all of the necessary information for determining the values of the transformation rates, z_i, is readily available. Unfortunately, in practice this is rarely the case.

When the distinguishing characteristics defining the product differences are unavailable, de Haan (2002, p. 82) recommended using a time period in which the products were being sold in the marketplace and they were preferably in a state of equilibrium. In this case, the observed prices could be used to assess the implicit worth of the products.

Lehr (1885, pp. 37–39), publishing in the German language, discussed the implications of this proposition many years earlier. Unaware of this research, however, Davies (1924, pp. 183–85), some 40 years later, independently took a similar position. Going far beyond the proposal of de Haan, who was concerned only with similar products, these authors claimed and justified that the transformation rates calculated from available price data could be applied to the case of heterogeneous products as well. This is the second essential message contained in the studies authored by Lehr and Davies.

Regrettably, these studies did not receive the attention they deserved. Perhaps, these thoughts were considered to be too unorthodox at the time. It would be an unwarranted mistake, however, to dismiss this inspiration prematurely. Whether the approach is reasonable or not should be judged on the basis of the results obtained. If the resulting price indices correspond to some existing and highly respected price index formulas and if they produce reasonable results, then sufficient justification for the underlying approach should have been demonstrated.

3.2. *The Definition*

The N assessed-value transformation rates, $\hat{z}_i$ $(i = 1, \ldots, N)$, are a certain number of intrinsic-worth units per product-identifying unit. The numerical magnitudes of these rates are determined by the appraisal of the intrinsic worth of the products that was made. Some straightforward appraisal methods will be introduced in Section 3.3.

Replacing the above-defined transformation rates, z_i, in the AUV index (2) by these (assessed-value) transformation rates, $\hat{z}_i$, yields the basic formula for the Generalized Unit Value (GUV) index:

$$(3) \qquad P_{GUV} = \frac{P_{GUV}^1}{P_{GUV}^0} = \frac{\left[\sum (p_i^1/\hat{z}_i) x_i^1 \hat{z}_i\right] / \left(\sum x_i^1 \hat{z}_i\right)}{\left[\sum (p_i^0/\hat{z}_i) x_i^0 \hat{z}_i\right] / \left(\sum x_i^0 \hat{z}_i\right)} = \frac{V^1}{V^0} \frac{\sum x_i^0 \hat{z}_i}{\sum x_i^1 \hat{z}_i}.$$

This index measures the change in the unit value of an intrinsic-worth unit. Multiplying all of the transformation rates, $\hat{z}_i$, by an arbitrary constant does not alter the value of the GUV index (3). In other words, the value of the index is not dependent upon the absolute values of these rates, $\hat{z}_i$, but rather their ratios, $\hat{z}_{ij} = \hat{z}_i / \hat{z}_j$ $(i, j = 1, \ldots, N)$. This reflects the fact that the intrinsic worth unit can be arbitrarily scaled up and down.

The basic GUV index formula (3) produces many different price indices depending upon how the transformation rates, $\hat{z}_i$, are computed. In order to qualify as a legitimate member of the GUV index family, however, the selected definition of the transformation rates must conform to some common characteristics that can be stated in the form of formal axioms. Inspiration for the construction of these axioms can be obtained from traditional axiomatic index theory. Auer (2011, p. 12) proposes several axioms that relate to the transformation rates, $\hat{z}_i$. Since the value of the GUV index (3) depends on the transformation ratios, $\hat{z}_{ij} = \hat{z}_i / \hat{z}_j$, and not on the absolute values of the transformation rates, $\hat{z}_i$, it is preferable to define axioms that relate to the transformation ratios, $\hat{z}_{ij}$.

Hereinafter, as a matter of convenience, bold print will signify a column vector, e.g., $\mathbf{p}^0 = (p_1^0, p_2^0, \ldots, p_N^0)'$. The subscript "-*ij*" indicates a column vector that contains all elements, except for the i^{th} and the j^{th} ones.

The computation of a GUV index must be feasible in situations in which no other information than price and quantity data are available. Therefore, the transformation ratios, $\hat{z}_{ij}$, should depend exclusively on the available price and quantity vectors: $\hat{z}_{ij} = \hat{z}_{ij}(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1)$. Furthermore, the function for computing the transformation ratios, $\hat{z}_{ij}(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1)$, should be the same for all (ordered) pairs of products *i* and *j*. The prices and quantities of these two products should have a specific position in the formula, whereas permutations of all other products *k* (with $k \neq i, j$) should not affect the transformation ratio $\hat{z}_{ij}$. These very basic postulates can be combined in the following axiom:

Z1 *The* ***Base axiom*** *postulates that the function used for computing the transformation ratios,* $\hat{z}_{ij}$ $(i, j = 1, \ldots, N)$*, must utilize only observed price and quantity data, must be the same for all (ordered) pairs of products, and must be independent of uniform permutations of the vectors* $\mathbf{p}^0_{-ij}$, $\mathbf{x}^0_{-ij}$, $\mathbf{p}^1_{-ij}$, *and* $\mathbf{x}^1_{-ij}$:

$$\begin{aligned}\hat{z}_{ij} &= z\left(p_i^0, x_i^0, p_i^1, x_i^1, p_j^0, x_j^0, p_j^1, x_j^1, \mathbf{p}^0_{-ij}, \mathbf{x}^0_{-ij}, \mathbf{p}^1_{-ij}, \mathbf{x}^1_{-ij}\right) \\ &= z(p_i^0, x_i^0, p_i^1, x_i^1, p_j^0, x_j^0, p_j^1, x_j^1, \mathbf{p}^{0\prime}_{-ij}, \mathbf{x}^{0\prime}_{-ij}, \mathbf{p}^{1\prime}_{-ij}, \mathbf{x}^{1\prime}_{-ij}),\end{aligned}$$

where the vectors $\mathbf{p}^{0\prime}_{-ij}$, $\mathbf{x}^{0\prime}_{-ij}$, $\mathbf{p}^{1\prime}_{-ij}$, $\mathbf{x}^{1\prime}_{-ij}$ *are uniform permutations of the vectors* $\mathbf{p}^0_{-ij}$, $\mathbf{x}^0_{-ij}$, $\mathbf{p}^1_{-ij}$, *and* $\mathbf{x}^1_{-ij}$.

Z2 *The* ***Transitivity axiom*** *postulates that for all triples of products* $(i, j, k = 1, \ldots, N)$*, the following relationship must hold:*

$$\hat{z}_{ij} = \hat{z}_{ik}\hat{z}_{kj}.$$

Z3 *The* ***Weak Monotonicity axiom*** *postulates that the values of the transformation ratios,* $\hat{z}_{ij}$ $(i, j = 1, \ldots, N)$*, are weakly monotonically increasing with the observed market prices* p_i^0 *and* p_i^1,

$$\frac{\partial \hat{z}_{ij}}{\partial p_i^0} \geq 0 \quad \textit{and} \quad \frac{\partial \hat{z}_{ij}}{\partial p_i^1} \geq 0,$$

and weakly monotonically decreasing with the observed market prices p_j^0 *and* p_j^1,

$$\frac{\partial \hat{z}_{ij}}{\partial p_j^0} \leq 0 \quad \textit{and} \quad \frac{\partial \hat{z}_{ij}}{\partial p_j^1} \leq 0.$$

Z4 *The* ***Price Dimensionality axiom*** *postulates that the transformation ratios, $\hat{z}_{ij}$, should not be affected by a change in the currency:*

$$\hat{z}_{ij}(\eta \mathbf{p}^0, \mathbf{x}^0, \eta \mathbf{p}^1, \mathbf{x}^1) = \hat{z}_{ij}(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1), \quad \textit{with } \eta > 0.$$

Z5 *The* ***Commensurability axiom*** *postulates that a change in the units of measure of some product i and/or some product j should change the value of the transformation ratio, $\hat{z}_{ij}$, by the same proportion:*

$$\begin{aligned} &\hat{z}_{ij}(p_i^0\lambda, x_i^0/\lambda, p_i^1\lambda, x_i^1/\lambda, p_j^0\phi, x_j^0/\phi, p_j^1\phi, x_j^1/\phi, \mathbf{p}_{-ij}^0, \mathbf{x}_{-ij}^0, \mathbf{p}_{-ij}^1, \mathbf{x}_{-ij}^1) \\ &\quad = (\lambda/\phi)\,\hat{z}_{ij}(p_i^0, x_i^0, p_i^1, x_i^1, p_j^0, x_j^0, p_j^1, x_j^1, \mathbf{p}_{-ij}^0, \mathbf{x}_{-ij}^0, \mathbf{p}_{-ij}^1, \mathbf{x}_{-ij}^1), \end{aligned}$$

with λ, $\phi > 0$.

Utilizing axioms Z1 through Z5, the GUV index family can be defined as follows:

Definition 1. *The basic GUV index formula (3) defines a family of price indices that differ from one another by the selected definition of the (assessed-value) transformation rates, $\hat{z}_i$ ($i = 1, \ldots, N$). Moreover, the selected definition must conform to axioms Z1 through Z5.*

3.3. *Some Members of the GUV Index Family*

The GUV indices differ from one another depending upon the precise manner in which the transformation rates, $\hat{z}_i$, are computed. Their values depend upon an assessment of intrinsic product worth. A straightforward appraisal of this worth is found by setting the number of intrinsic-worth units equal to the number of monetary units required to purchase the product. Using the observed prices in the base time period, p_i^0, yields the transformation rates:

$$\text{(GUV-1)} \qquad \hat{z}_i = p_i^0.$$

Substituting the expression (GUV-1) into the basic GUV index formula (3) yields quite a surprising result, the Paasche (1874, p. 172) index, P_P:

$$\text{(4)} \qquad P_{GUV} = \frac{V^1}{V^0}\frac{\sum x_i^0 p_i^0}{\sum x_i^1 p_i^0} = \frac{\sum p_i^1 x_i^1}{\sum p_i^0 x_i^1} = P_P.$$

For expository purposes, reconsider the example presented in Table 1 with two very different products. Product B is a package of 12 AA batteries and product S is a box of table salt (net weight: 1.2 kilogram). Utilizing expression (GUV-1) yields $\hat{z}_B = 12$. This number says that the original unit of measurement (a package of 12 batteries) is equivalent to 12 intrinsic-worth units. Therefore, an intrinsic-worth unit is commensurate to one battery. Analogously, (GUV-1) yields $\hat{z}_S = 6$ which says that the original unit of measurement (1.2 kilogram of salt) is equivalent to six intrinsic-worth units. Therefore, an intrinsic-worth unit is commensurate not only to one battery but also to a 200-gram portion of salt. It is this equivalence that in the basic GUV index formula (3) allows for a meaningful summation over the transformed quantities, $\sum x_i^t \hat{z}_i$. It should be noted that, as an implication of utilizing definition (GUV-1), the base period price of an intrinsic-worth unit is 1, because this is the price of both, one battery and 200 gram of salt.

Expression (GUV-1) uses the observed prices in the base time period, p_i^0, to numerically measure intrinsic worth. Using instead the observed prices in the comparison period, p_i^1, the transformation rates are:

(GUV-2) $$\hat{z}_i = p_i^1.$$

Substituting the rates defined by (GUV-2) into the basic GUV index formula (3) and simplifying, the Laspeyres (1871, p. 306) index, P_L, is obtained:

(5) $$P_{GUV} = \frac{V^1}{V^0}\frac{\sum x_i^0 p_i^1}{\sum x_i^1 p_i^1} = \frac{\sum p_i^1 x_i^0}{\sum p_i^0 x_i^0} = P_L.$$

The Paasche and Laspeyres indices are bona fide members of the GUV index family. These two indices are usually described as tracking the change in the cost of some fixed group of products or as a weighted average of the individual price changes that occur in the products involved. A very different interpretation of these two indices, however, is provided by the specification of the basic GUV index formula (3). There, they measure the change in the unit value of an intrinsic-worth unit.

The Laspeyres and Paasche indices produce different numerical results because their measurement of intrinsic worth is based upon different time periods. Utilizing the arithmetic, geometric, or harmonic mean, the observed prices from both time periods can be used to form the required appraisals of product worth:

(GUV-3) $$\hat{z}_i = \left(p_i^0 + p_i^1\right)/2$$

(GUV-4) $$\hat{z}_i = \sqrt{p_i^0 p_i^1}$$

(GUV-5) $$\hat{z}_i = 2\cdot\left[1/p_i^0 + 1/p_i^1\right]^{-1}.$$

Substituting the transformation rates defined by (GUV-3) into the basic GUV index formula (3) and simplifying, yields the Banerjee (1977, p. 25) index, P_B:

(6) $$P_{GUV} = \frac{V^1}{V^0}\frac{\sum x_i^0\left(p_i^0 + p_i^1\right)/2}{\sum x_i^1\left(p_i^0 + p_i^1\right)/2} = \frac{V^1}{V^0}\frac{\left(V^0 + V^{10}\right)}{\left(V^1 + V^{01}\right)} = P_B,$$

where $V^{st} = \sum p_i^s x_i^t$. Inserting the rates defined by (GUV-4) into the basic GUV index formula (3) and simplifying, yields the Davies (1924, p. 185) index, P_D:

(7)
$$P_{GUV} = \frac{V^1}{V^0} \frac{\sum x_i^0 \sqrt{p_i^0 p_i^1}}{\sum x_i^1 \sqrt{p_i^0 p_i^1}} = P_D.$$

Therefore, the Banerjee and the Davies indices are also members of the GUV index family.

A common feature among the transformation rate definitions (GUV-1) to (GUV-5) is the fact that they are all exclusively based upon un-weighted market prices. Other GUV indices are possible that attach weights to these prices. For example, the expenditure shares,

$$w_i = \frac{p_i^0 x_i^0}{p_i^0 x_i^0 + p_i^1 x_i^1},$$

could be used to weight the observed prices from the two time periods in a geometric mean:

(GUV-6)
$$\hat{z}_i = \left(p_i^0\right)^{w_i} \left(p_i^1\right)^{1-w_i}.$$

Applying a harmonic mean yields:

(GUV-7)
$$\hat{z}_i = \frac{1}{w_i / p_i^0 + (1-w_i)/p_i^1} = \frac{p_i^0 x_i^0 + p_i^1 x_i^1}{x_i^0 + x_i^1}.$$

Inserting the rates defined by (GUV-7) into the basic GUV index formula (3), yields the Lehr (1885, p. 39) index, P_{Le}:

$$P_{GUV} = \frac{V^1}{V^0} \frac{\sum \left(p_i^0 x_i^0 + p_i^1 x_i^1\right) x_i^0 / \left(x_i^0 + x_i^1\right)}{\sum \left(p_i^0 x_i^0 + p_i^1 x_i^1\right) x_i^1 / \left(x_i^0 + x_i^1\right)} = P_{Le}.$$

Consequently, the Lehr index is also a bona fide member of the GUV index family. For an alternative interpretation of the Lehr index as an implicit price index, see Balk (2008, p. 8).

The axioms Z1 through Z5 are basic requirements that the formula for computing the tranformation ratios, $\hat{z}_{ij}$, must satisfy. Expressions (GUV-1) through (GUV-7) satisfy all of the axioms Z1 through Z5 (proofs are in the Appendix). Therefore, the price indices corresponding to expressions (GUV-1) through (GUV-7) are all GUV indices in the sense of Definition 1.

3.4. *Further Z-Axioms*

The axioms Z1 through Z5 are probably not contentious. Additional axioms can help to differentiate further between better or worse definitions of the transformation ratios, $\hat{z}_{ij}$, and, thus, between the more or less reasonable GUV indices. A selection of some possible additional axioms is listed below. Not all of the definitions (GUV-1) through (GUV-7) satisfy all of these axioms (proofs are in the Appendix).

Z6 *The **Strict Monotonicity axiom** postulates that the values of the transformation ratios,* $\hat{z}_{ij}$ *(i, j = 1, . . . ,N), are strictly monotonically increasing with the observed market prices* p_i^0 *and* p_i^1*,*

$$\frac{\partial \hat{z}_{ij}}{\partial p_i^0} > 0 \quad and \quad \frac{\partial \hat{z}_{ij}}{\partial p_i^1} > 0,$$

and strictly monotonically decreasing with the observed market prices p_j^0 *and* p_j^1*,*

$$\frac{\partial \hat{z}_{ij}}{\partial p_j^0} < 0 \quad and \quad \frac{\partial \hat{z}_{ij}}{\partial p_j^1} < 0.$$

Obviously, any definition of the transformation ratios, $\hat{z}_{ij}$, that satisfies axiom Z6 will also satisfy axiom Z3 as well. Therefore, axiom Z6 can be viewed as a tightening of the condition posited by axiom Z3. Only the rates defined by definitions (GUV-1) and (GUV-2) violate axiom Z6.

Z7 *The **Proportionality axiom** postulates that:*

$$\text{(8)} \qquad if \quad \frac{p_i^0}{p_j^0} = \frac{p_i^1}{p_j^1} = \theta, \quad then\ \hat{z}_{ij} = \theta, \quad with\ i,\ j = 1, \ldots, N.$$

Z8 *The **Weak Mean Value axiom** postulates that:*

$$\text{(9)} \qquad \min\left(\frac{p_i^0}{p_j^0}, \frac{p_i^1}{p_j^1}\right) \le \hat{z}_{ij} \le \max\left(\frac{p_i^0}{p_j^0}, \frac{p_i^1}{p_j^1}\right), \quad with\ i,\ j = 1, \ldots, N.$$

For $p_i^0 / p_j^0 = p_i^1 / p_j^1 = \theta$, expression (9) degenerates to expression (8). Therefore, axiom Z8 represents a tightening of axiom Z7. Only definitions (GUV-6) and (GUV-7) violate axioms Z7, and consequently Z8.

Z9 *The **Independence axiom** postulates that the values of the transformation ratios,* $\hat{z}_{ij}$*, are independent from all products k ≠ i, j (i, j, k = 1, . . . , N).*

This condition is satisfied by definitions (GUV-1) through (GUV-7). Additional axioms could easily be added to this list.

Axioms Z1 through Z9 are postulates on the formulas for calculating the transformation ratios, $\hat{z}_{ij}$. The GUV-1 through GUV-7 indices satisfy most of these axioms. However, do these GUV indices offer a reliable alternative to the Fisher index,

$$\text{(10)} \qquad P_F = \sqrt{P_L P_P} = \frac{V^1}{V^0} \frac{\sqrt{V^0 V^{10}}}{\sqrt{V^1 V^{01}}},$$

and the Walsh index,

$$(11) \qquad P_W = \frac{\sum p_i^1 \sqrt{x_i^0 x_i^1}}{\sum p_i^0 \sqrt{x_i^0 x_i^1}} ?$$

Many price statisticians consider these two price indices as particularly reliable. Again, axiomatic index theory can be used to conduct a comparison. The relevant axioms for this comparison are not axioms Z1 through Z9, but those defined with respect to the complete price index formulas, P.

A thorough axiomatic comparison of the GUV-1 through GUV-7 indices, the Fisher index, and the Walsh index can be found in Auer (2011, pp. 18–30). The relevance of the axiomatic approach and the individual axioms continues to be a subject of controversy (see, e.g., Auer, 2001, 2008). It is probably fair to infer, however, that a sufficient number of the GUV indices possess an axiomatic profile that is good enough to accept the proposition that the GUV methodology is a rational approach for the generation of reliable price index formulas.

3.5. *Further Members of the GUV Index Family*

As an alternative to the direct computation of the transformation rates, $\hat{z}_i$, one could start by computing a set of transitive transformation ratios, $\hat{z}_{ij}$ ($i, j = 1, \ldots, N$). In a second step, these ratios could be used to calculate the transformation rates, $\hat{z}_i$ ($i = 1, \ldots, N$), required for the GUV index formula (3).

For example, one could calculate the transformation ratios, $\hat{z}_{ij}$, by taking an average of the price ratios p_i^0 / p_j^0 and p_i^1 / p_j^1. Using the geometric mean yields formula (GUV-4). Using the arithmetic mean gives

$$(12) \qquad \hat{z}_{ij} = \frac{1}{2}\left(\frac{p_i^0}{p_j^0} + \frac{p_i^1}{p_j^1} \right).$$

However, the transformation ratios, $\hat{z}_{ij}$, resulting from formula (12) are not transitive. To obtain transitive ratios, $\hat{z}_{ij}^*$, procedures familiar from interregional price indices can be utilized. The most prominent is the GEKS method attributed to Gini (1924, 1931), Éltetö and Köves (1964), and Szulc (1964). Using the formula

$$(13) \qquad \hat{z}_{ij}^* = \prod_{k=1}^{N} \left(\hat{z}_{ik} \hat{z}_{kj} \right)^{1/N},$$

the GEKS method converts the intransitive transformation ratios, $\hat{z}_{ij}$, into transitive ones, $\hat{z}_{ij}^*$ ($i, j = 1, \ldots, N$). Obviously, this approach violates axiom Z9.

Transitivity of the transformation ratios, $\hat{z}_{ij}^*$, ensures that numbers $\hat{z}_1, \hat{z}_2, \ldots,$ $\hat{z}_N$ exist that are determined up to a factor of proportionality, such that for all i, $j = 1, \ldots, N$:

$$\hat{z}_{ij}^{*} = \frac{\hat{z}_i}{\hat{z}_j}.$$

Inserting the numbers $\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_N$, in the GUV index formula (3), the index number can be computed. These multi-step GUV index formulas are not considered further in the paper.

3.6. *Implications for Practical Work*

As pointed out before, many price statisticians consider the Fisher index (10) and the Walsh index (11) as particularly accurate. In applied work, will the GUV-3 through GUV-7 indices closely approximate those of Fisher and Walsh? The Walsh index always approximates the Fisher index very closely. The only difference between the Fisher index and the GUV-3 index, that is, the Banarjee index, is the manner in which the value aggregates V^t and V^{st} are combined. The Fisher index (10) utilizes a geometric mean, while the Banerjee index (6) employs the arithmetic mean. Therefore, their index numbers are usually very close to each other.

The GUV-3 through GUV-5 indices are all based on the same basic GUV index formula (3). They differ from one another only with respect to the manner in which the prices p_i^0 and p_i^1 are averaged to obtain the transformation rates $\hat{z}_i$. Therefore, not only the GUV-3 index (Banerjee index) approximates the Fisher index very closely, but also the GUV-4 index (Davies index) and the GUV-5 index.

The GUV-6 index and the GUV-7 index (Lehr index) are merely weighted variants of the GUV-3 and GUV-5 indices. Since the expenditure on some product usually fluctuates only modestly between two adjacent time periods, the weights, w_i, do not deviate much from 0.5. As a consequence, the index numbers of the GUV-6 and GUV-7 indices also closely approximate those of their unweighted counterparts GUV-3 and GUV-5, and therefore, the Fisher index.

The numerical example listed in Table 1 confirms these considerations. The price level change between the base and comparison time period using the various GUV indices defined above is presented in Table 3. As can be seen from the Laspeyres and Paasche index numbers, in this numerical example, the potential for

TABLE 3
Example; GUV Index Family and Traditional Indices

GUV Index	Name	Index Number
GUV-1	Paasche: P_P	1.167
GUV-2	Laspeyres: P_L	1.250
GUV-3	Banerjee: P_B	1.212
GUV-4	Davies: P_D	1.207
GUV-5	–	1.203
GUV-6	–	1.219
GUV-7	Lehr: P_{Le}	1.212
	Fisher: P_F	1.208
	Walsh: P_W	1.207

deviations between the index numbers is considerably larger than in any real dataset. Nevertheless, as Table 3 shows, the index numbers produced by the Fisher and Walsh indices are very similar to those produced by the GUV-3 through GUV-7 indices. It can be concluded that in practical applications, the index numbers produced by the GUV-3 through GUV-7 price indices will be barely distinguishable from those produced by the Fisher and Walsh indices.

When deriving price indices, most price statisticians seek to decompose the value aggregate ratio, V^1/V^0, into an overall price change, P, and an overall quantity change, Q:

$$\frac{V^1}{V^0} = PQ. \tag{14}$$

Substituting the basic GUV index formula (3) for P and solving for Q yields:

$$Q = \frac{\sum x_i^1 \hat{z}_i}{\sum x_i^0 \hat{z}_i}. \tag{15}$$

This is a very appealing quantity index because it is simply the ratio of the sum of transformed quantities in the comparison period divided by those in the base time period.

For practical purposes, this quantity index can also be expressed in the following additive form:

$$Q - 1 = \sum \left(\frac{x_i^1}{x_i^0} - 1 \right) \frac{x_i^0 \hat{z}_i}{\sum x_j^0 \hat{z}_j}. \tag{16}$$

The percentage change in the overall quantity, Q–1, is a weighted average of the percentage changes of the individual products, $x_i^1 / x_i^0 - 1$, where each product's weight is given by its base period share of the sum of transformed quantities. Based on formula (16), practitioners can easily determine the contribution of each individual product or of a group of products to the percentage change in overall quantity.

In the numerical example presented in Table 1, the GUV-4 index, that is, the Davies price index (7), yields $z_B = 12$, $z_S = 7.3485$, and $P_D = 1.207$. The quantity index can be computed from formulas (14) or (15): $Q = 1.450$. Formula (16) reveals that the 45.0 percent increase in the overall quantity can be attributed entirely to the quantity change in product B:

$$\left(\frac{x_B^1}{x_B^0} - 1 \right) \frac{x_B^0 \hat{z}_B}{x_B^0 \hat{z}_B + x_S^0 \hat{z}_S} = \left(\frac{4}{2} - 1 \right) \frac{2 \cdot 12}{2 \cdot 12 + 4 \cdot 7.3485} = 0.45$$

$$\left(\frac{x_S^1}{x_S^0} - 1 \right) \frac{x_S^0 \hat{z}_S}{x_B^0 \hat{z}_B + x_S^0 \hat{z}_S} = \left(\frac{2}{2} - 1 \right) \frac{4 \cdot 7.3485}{2 \cdot 12 + 4 \cdot 7.3485} = 0.$$

4. Concluding Remarks

The inflation rate is normally computed as a weighted average of individual price changes. Alternatively, this rate could be evaluated by comparing average price levels. This methodology is a standard approach in international or interregional price comparisons. In past research on intertemporal price measurement, however, the comparison of price levels has received limited attention. This study has attempted to remedy this situation by introducing a group of Generalized Unit Value (GUV) indices. It has demonstrated that the GUV index family includes the well-known Paasche, Laspeyres, and Banerjee, as well as the hardly known Davies and Lehr indices.

The GUV indices are also useful for practical purposes. If the prices of similar products accurately reflect their quality differences, then the GUV indices could be used to aggregate the prices of these products. This is particularly attractive, where hedonic analysis and other sophisticated quality adjustment methods are either impossible to conduct or are deemed excessively extravagant in terms of the resources they require.

This study has asserted that the GUV indices produce reliable results in the case of heterogeneous products, too. The use of the GUV index variants GUV-3 to GUV-7 represent viable alternatives to some of the most highly respected traditional price indices, for example, the Fisher index. This is particularly useful for statistical agencies that decompose the value aggregate ratio into an overall price change (the "deflator") and an overall quantity change. When using a GUV price index as deflator, the overall quantity change is simply the ratio of the sum of transformed quantities in the comparison period, divided by those in the base time period. From this formula, it is easy to calculate the contribution of each individual product to the overall quantity change.

Some promising areas for future research exist. A systematic investigation into how these GUV indices relate to economic theory could prove to be a fruitful endeavor. The stochastic approach to index theory usually assumes that all observed price ratios are realizations of some random variable with an expected value equal to the "common inflation". The GUV approach suggests the pursuit of a stochastic analysis that is based upon a less contentious assumption. It assumes that for each pair of products the price ratio observed during the base period and the price ratio observed during the comparison period represent realizations of a random variable with an expected value equal to the the product's transformation ratios. Based upon this assumption, one could compare the statistical properties of the estimators of the transformation ratios used by the various GUV indices. The GUV indices could also be applied in other measurement situations not specifically referred to in this study. An obvious area could involve interregional price comparisons.

Appendix

This appendix contains proofs of the axiomatic results listed in Table A1.

TABLE A1
Axiomatic Properties of the $\hat{z}_i$-Formulas

	GUV-1 P_L	GUV-2 P_P	GUV-3 P_B	GUV-4 P_D	GUV-5	GUV-6	GUV-7 P_{Le}
Z1 Base	▲	▲	▲	▲	▲	▲	▲
Z2 Transitivity	▲	▲	▲	▲	▲	▲	▲
Z3 Weak Monotonicity	▲	▲	▲	▲	▲	▲	▲
Z4 Price Dimensionality	▲	▲	▲	▲	▲	▲	▲
Z5 Commensurability	▲	▲	▲	▲	▲	▲	▲
Z6 Strict Monotonicity	▽	▽	▲	▲	▲	▲	▲
Z7 Proportionality	▲	▲	▲	▲	▲	▽	▽
Z8 Weak Mean Value	▲	▲	▲	▲	▲	▽	▽
Z9 Independence	▲	▲	▲	▲	▲	▲	▲

Note: A filled triangle indicates "axiom satisfied" and an empty triangle indicates "axiom violated".

Most of the proofs are trivial, because the formulas (GUV-1) through (GUV-7) imply that the transformation rate of each product, $\hat{z}_i$, is independent of the price and quantity data of all other products j ($j \neq i$).

Z1 Base: The transformation ratios associated with the GUV-1 through GUV-7 indices are all defined by $\hat{z}_{ij} = \hat{z}_i / \hat{z}_j$, where $\hat{z}_i$ and $\hat{z}_j$ are computed from the respective formulas (GUV-1) through (GUV-7). Therefore, they satisfy this axiom.

Z2 Transitivity: The transformation ratios associated with the GUV-1 through GUV-7 indices are all defined by $\hat{z}_{ij} = \hat{z}_i / \hat{z}_j$. Therefore, they satisfy this axiom.

Z3 Weak Monotonicity: Formulas (GUV-1) through (GUV-7) have the following derivatives:

$$\frac{\partial \hat{z}_i}{\partial p_i^0} \geq 0, \quad \frac{\partial \hat{z}_i}{\partial p_i^1} \geq 0, \quad \frac{\partial \hat{z}_j}{\partial p_i^0} = \frac{\partial \hat{z}_j}{\partial p_i^1} = 0.$$

Therefore, the transformation ratios, $\hat{z}_{ij} = \hat{z}_i / \hat{z}_j$, satisfy this axiom.

Z4 Price Dimensionality: In all formulas (GUV-1) through (GUV-7), multiplying the prices of products i and j by the same scalar η yields the new transformation rates $\hat{z}_i' = \eta \hat{z}_i$ and $\hat{z}_j' = \eta \hat{z}_j$, where $\hat{z}_i$ and $\hat{z}_j$ are the original transformation rates. Therefore, the new transformation ratio, $\hat{z}_{ij}' = \hat{z}_i' / \hat{z}_j'$, is identical to the original one, $\hat{z}_{ij} = \hat{z}_i / \hat{z}_j$.

Z5 Commensurability: In all formulas (GUV-1) through (GUV-7), changing the unit of measurement of products i and j by the factors λ and ϕ, yields the new transformation rates $\hat{z}_i' = \lambda \hat{z}_i$ and $\hat{z}_j' = \phi \hat{z}_j$, where $\hat{z}_i$ and $\hat{z}_j$ are the original ones. Therefore, the new transformation ratio is

$$\hat{z}'_{ij} = \frac{\hat{z}'_i}{\hat{z}'_j} = \frac{\lambda}{\phi}\frac{\hat{z}_i}{\hat{z}_j} = \frac{\lambda}{\phi}\hat{z}_{ij}.$$

Z6 Strict Monotonicity: Formulas (GUV-3) through (GUV-7) have the following derivatives:

$$\frac{\partial \hat{z}_i}{\partial p_i^0} > 0, \quad \frac{\partial \hat{z}_i}{\partial p_i^1} > 0, \quad \frac{\partial \hat{z}_j}{\partial p_i^0} = \frac{\partial \hat{z}_j}{\partial p_i^1} = 0.$$

Therefore, the respective transformation ratios, $\hat{z}_{ij} = \hat{z}_i / \hat{z}_j$, satisfy this axiom. Formulas (GUV-1) and (GUV-2) violate the axiom, because

$$\frac{\partial \hat{z}_i}{\partial p_i^1} = 0 \quad \text{and} \quad \frac{\partial \hat{z}_j}{\partial p_i^0} = \frac{\partial \hat{z}_j}{\partial p_i^1} = 0 \quad \text{(GUV-1)}$$

$$\frac{\partial \hat{z}_i}{\partial p_i^0} = 0 \quad \text{and} \quad \frac{\partial \hat{z}_j}{\partial p_i^0} = \frac{\partial \hat{z}_j}{\partial p_i^1} = 0 \quad \text{(GUV-2)}.$$

Z7 Proportionality: Suppose that $p_i^0 = \theta p_j^0$ and $p_i^1 = \theta p_j^1$. In all formulas (GUV-1) through (GUV-5), the resulting transformation rates are $\hat{z}_i = \theta \hat{z}_j$. Therefore, $\hat{z}_{ij} = \theta$.

(GUV-6) yields

$$\begin{aligned} \hat{z}_i &= \left(p_i^0\right)^{w_i}\left(p_i^1\right)^{1-w_i} \\ &= \left(\theta p_j^0\right)^{w_i}\left(\theta p_j^1\right)^{1-w_i} \\ &= \theta\left(p_j^0\right)^{w_i}\left(p_j^1\right)^{1-w_i} \end{aligned}$$

and

$$\hat{z}_j = \left(p_j^0\right)^{w_j}\left(p_j^1\right)^{1-w_j}$$

and therefore

$$\hat{z}_{ij} = \frac{\hat{z}_i}{\hat{z}_j} = \theta \frac{\left(p_j^0\right)^{w_i}\left(p_j^1\right)^{1-w_i}}{\left(p_j^0\right)^{w_j}\left(p_j^1\right)^{1-w_j}}$$

which is not equal to θ, unless $w_i = w_j$.

(GUV-7) yields

$$\begin{aligned} \hat{z}_{ij} = \frac{\hat{z}_i}{\hat{z}_j} &= \frac{\left(p_i^0 x_i^0 + p_i^1 x_i^1\right)\left(x_j^0 + x_j^1\right)}{\left(x_i^0 + x_i^1\right)\left(p_j^0 x_j^0 + p_j^1 x_j^1\right)} \\ &= \theta \frac{\left(p_j^0 x_i^0 + p_j^1 x_i^1\right)\left(x_j^0 + x_j^1\right)}{\left(x_i^0 + x_i^1\right)\left(p_j^0 x_j^0 + p_j^1 x_j^1\right)} \\ &= \theta \frac{p_j^0 x_i^0 x_j^0 + p_j^1 x_j^0 x_i^1 + p_j^0 x_i^0 x_j^1 + p_j^1 x_i^1 x_j^1}{p_j^0 x_i^0 x_j^0 + p_j^1 x_i^0 x_j^1 + p_j^0 x_j^0 x_i^1 + p_j^1 x_i^1 x_j^1} \end{aligned}$$

which is not equal to θ, unless

$$p_j^1 x_j^0 x_i^1 + p_j^0 x_i^0 x_j^1 = p_j^1 x_i^0 x_j^1 + p_j^0 x_j^0 x_i^1$$

$$p_j^1 \left(x_j^0 x_i^1 - x_i^0 x_j^1 \right) = p_j^0 \left(x_j^0 x_i^1 - x_i^0 x_j^1 \right)$$

$$p_j^1 = p_j^0$$

Z8 Weak Mean Value: (adopted from Eichhorn and Voeller, 1990, p. 332). Define $\theta_{\min} := \min\left(p_i^0 / p_j^0, p_i^1 / p_j^1\right)$ and $\theta_{\max} := \max\left(p_i^0 / p_j^0, p_i^1 / p_j^1\right)$. Therefore, $p_i^0 \geq p_j^0 \theta_{\min}$, $p_i^1 \geq p_j^1 \theta_{\min}$, $p_i^0 \leq p_j^0 \theta_{\max}$, and $p_i^1 \leq p_j^1 \theta_{\max}$. Axiom Z7 (Proportionality) implies that

$$\theta_{\min} = \hat{z}_{ij}\left(p_j^0 \theta_{\min}, x_i^0, p_j^1 \theta_{\min}, x_i^1, p_j^0, x_j^0, p_j^1, x_j^1, \mathbf{p}_{-ij}^0, \mathbf{x}_{-ij}^0, \mathbf{p}_{-ij}^1, \mathbf{x}_{-ij}^1\right)$$

$$\theta_{\max} = \hat{z}_{ij}\left(p_j^0 \theta_{\max}, x_i^0, p_j^1 \theta_{\max}, x_i^1, p_j^0, x_j^0, p_j^1, x_j^1, \mathbf{p}_{-ij}^0, \mathbf{x}_{-ij}^0, \mathbf{p}_{-ij}^1, \mathbf{x}_{-ij}^1\right).$$

Axiom Z3 (Weak Monotonicity) implies that

$$\hat{z}_{ij}\left(p_j^0 \theta_{\min}, x_i^0, p_j^1 \theta_{\min}, x_i^1, p_j^0, x_j^0, p_j^1, x_j^1, \mathbf{p}_{-ij}^0, \mathbf{x}_{-ij}^0, \mathbf{p}_{-ij}^1, \mathbf{x}_{-ij}^1\right) \leq \hat{z}_{ij}\left(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}\right)$$

$$\hat{z}_{ij}\left(p_j^0 \theta_{\max}, x_i^0, p_j^1 \theta_{\max}, x_i^1, p_j^0, x_j^0, p_j^1, x_j^1, \mathbf{p}_{-ij}^0, \mathbf{x}_{-ij}^0, \mathbf{p}_{-ij}^1, \mathbf{x}_{-ij}^1\right) \geq \hat{z}_{ij}\left(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}\right).$$

Therefore,

$$\theta_{\min} \leq \hat{z}_{ij}\left(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}\right) \leq \theta_{\max}.$$

This implies that formulas which satisfy axioms Z7 and Z3, also satisfy axiom Z8. The formulas (GUV-1) through (GUV-5) satisfy axioms Z7 and Z3. Axiom Z8 represents a tightening of axiom Z7. Therefore, formulas (GUV-6) and (GUV-7) violate axiom Z8.

Z9 Independence: In all formulas (GUV-1) through (GUV-7), the value of the transformation rate of each product i ($i = 1, \ldots, N$), $\hat{z}_i$, is independent of all other products. Therefore, the transformation ratios, $\hat{z}_{ij} = \hat{z}_i / \hat{z}_j$, are independent of all products $k \neq i, j$ ($i, j, k = 1, \ldots, N$).

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