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THE IMPACT OF A MARGINAL SUBSIDY ON GINI INDICES

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This paper addresses the impact of a subsidy—an increase in someone's income—on generalized Gini inequality indices. We show that for any distribution of income there exists a "pivotal individual" such that an increment given to an individual poorer (resp. richer) than himself, decreases (resp. increases) inequality. We characterize the pivotal individual for relative and absolute Gini indices. We show that normative prescriptions about the preferred level of inequality aversion can also be formulated in terms of the pivotal, namely the richest individual that we find just to compensate.

JEL Codes: H2, I3

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1. INTRODUCTION

An index of inequality is called ethical if it implies and is implied by a welfare function, which usually captures both efficiency and equity considerations. A subsidy given to one individual has an ambiguous effect on this welfare function. An increase in someone's income, other things being equal, results in social welfare improvement. However, the subsidy may also reduce equality among individuals. As Weymark (1981) summarizes, "we do not assume that welfare is monotone (non-decreasing); the increase in inequality resulting from raising one person's income holding other incomes constant may not be balanced by the increase in total income."

This paper studies the impact of a subsidy given to one person on an inequality index. More precisely, we compute the marginal variation with respect to an increase in someone's income on the class of generalized Gini inequality indices (see Donaldson and Weymark, 1980; Weymark, 1981; Yitzhaki, 1983; Bossert, 1990). The main result of the paper stems from the intuition that a dollar given to a poor individual reduces inequality, but the same dollar increases inequality if it is given to a rich individual. Formally, we show that for any distribution of income there exists a "pivotal individual" such that an increment given to an individual poorer (resp. richer) than himself, decreases (resp. increases) the inequality index.

The subsidy considered in this paper can alternatively be seen as a collection of transfers. In the case of relative inequality, which is the main focus of this paper,

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we assume that the subsidy is collected through proportional taxation. In the case of absolute inequality, we suppose that collection is through a lump sum tax. Given that relative (resp. absolute) inequality is invariant to proportional (resp. lump sum) taxation, the first step can be eliminated in both cases and we focus our attention only onto the subsidy.

Section 2 shows the existence and characterizes the pivotal individual, while Section 3 explores the question regarding the identity of the pivotal individual as a normative choice.

2. The Pivotal Individual

2.1. *Notation and Definitions*

Consider a population of n individuals whose incomes y_i are drawn from some non-negative real interval D. We are concerned with the *distribution of income*, which we represent by the n-tuple $\mathbf{y} = (y_1, y_2, \ldots y_n)$ such that $\mathbf{y} \in Y_n(D) = \{\mathbf{y} \in D^n | 0 < y_1 \leq y_2 \leq \ldots \leq y_n\}$, where D is an interval of $\mathbb{R}$. By definition, individual $i = 1$ is the poorest and individual $i = n$ is the richest.[1] The mean of $\mathbf{y}$ is denoted by $\mu(\mathbf{y})$.

An *inequality index* $I : Y_n(D) \to \mathbb{R}$ is called *ethical* if it implies, and is implied by, a *social evaluation or welfare function* $W : Y_n(D) \to \mathbb{R}$ (see, e.g., Blackorby *et al*., 1999). For $\mathbf{x}$ and $\mathbf{y}$, two distributions of income with the same mean, we have that $I(\mathbf{x}) \leq I(\mathbf{y})$ if and only if $W(\mathbf{x}) \geq W(\mathbf{y})$.

Atkinson (1970), Kolm (1969), and Sen (1973) propose to link inequality indices and welfare functions in terms of an *equally distributed equivalent income* ξ. Given the n-vector of ones $\mathbf{1} = (1, \ldots, 1)$, ξ is defined by the relation $W(\xi\mathbf{1}) = W(\mathbf{y})$. The equivalent income ξ is that per capita income which, if distributed equally, is ethically indifferent to the given distribution. Assuming that W is continuous and increasing along the line of complete equality, ξ is unique for each $\mathbf{y} \in Y_n(D)$ and can be written as $\xi = \Xi(\mathbf{y})$. Thus, the function Ξ is a particular numerical representation of W.

A *relative inequality index* is homogenous of degree zero in income (i.e, satisfies $I(\lambda\mathbf{y}) = I(\mathbf{y})$ for λ strictly positive). The *Atkinson–Kolm–Sen index of relative inequality* is defined by

$$I(\mathbf{y}) = 1 - \frac{\Xi(\mathbf{y})}{\mu(\mathbf{y})}. \tag{1}$$

Blackorby and Donaldson (1978) have shown that the index defined by (1) is a relative index if and only if W is homothetic. As $\mu(\mathbf{y})$ is positively linearly homogenous, $I(\mathbf{y})$ is a relative index if $\Xi(\mathbf{y})$ is also positively linearly homogenous.

The *generalized Gini relative inequality indices* (Donaldson and Weymark, 1980; Weymark, 1981) are defined by (1) with

[1]Incomes ranked in non-decreasing (resp. non-increasing) order are called illfare-ranked (resp. welfare-ranked) by Donaldson and Weymark (1980).

$$\Xi(\mathbf{y}) = \sum_{i=1}^{n} h(i/n) y_i, \tag{2}$$

where h depends on the *relative rank* of the individual (i/n) and it satisfies

$$h(1/n) \geq h(2/n) \geq \ldots \geq h(n/n) > 0. \tag{3}$$

The requirement of $h(i/n)$ being non-increasing, which is given by (3), implies that $\Xi(\mathbf{y})$ attaches more significance to the income of the poorer individuals in the distribution. This restriction is necessary if the welfare function W is to be S-concave and the generalized Gini indices are to satisfy the Dalton *Transfer Principle* (Dalton, 1920). Inequality (3) plays an important role in our derivations. Additionally, by letting $\mathbf{y} = \xi\mathbf{1}$ in (2) we have that $\Sigma_i h(i/n) = 1$. Thus, the weights $h(i/n)$ are probabilities.[2]

2.2. *The Impact of a Marginal Subsidy*

Now we turn to our question: What is the impact on the inequality index, in the sense of (1) and (2), of a subsidy given to one individual? The more direct approach to the question is to consider that individual j receives an infinitesimal and rank-preserving subsidy ε. Thus, we would like to characterize the derivative of (1) with respect to the income y_j. However, inequality measures based upon the ranking of individuals' incomes are not differentiable everywhere. Consider the derivative of (1) at the point $y_i = y_{i+1}$ with respect to either y_i or y_{i+1}. In both cases, right-hand and left-hand derivatives are not the same, because the right-hand derivative with respect to y_i and the left-hand derivative with respect to y_{i+1} switch the ranking between these two individuals.

In order to avoid these concerns, we provide a more precise definition of the subsidy.

Definition (subsidy). *A distribution* $\mathbf{y}'$ *is obtained from* $\mathbf{y}$ *by a subsidy if and only if there exist* $\varepsilon > 0$ *and an individual* $j = 1, 2, \ldots n$ *such that* $y_i' = y_i$ *for all* $i \neq j$ *and* $y_j' = y_j + \varepsilon \leq y_{j+1}$.

Accordingly, a subsidy may be discrete or marginal, but sufficiently small to be rank preserving. By using such a definition,[3] there is no difficulty even if some individuals have the same income in $\mathbf{y}$. The change on inequality after the subsidy is given by

$$I(\mathbf{y}') - I(\mathbf{y}) = \frac{\varepsilon}{\mu(\mathbf{y}) + \varepsilon/n}\left(\frac{1}{n\mu(\mathbf{y})}\sum_{i=1}^{n} h(i/n) y_i - h(j/n)\right).$$

[2]For generalized Gini indices, probability functions $h(\cdot)$ depend on the population size n (see Donaldson and Weymark, 1980; Bossert, 1990). If the weights are independent of n, we have a subclass of generalized Ginis called *single-series Ginis*. In our context, n is fixed and we omit this dependence, although all our results can be seen as conditional to n.

[3]For an equivalent strategy, see Chateauneuf *et al.* (2002).

For a marginal increment, we define $\partial I(\mathbf{y})/\partial y_j = \lim_{\varepsilon\to 0^+}(I(\mathbf{y}') - I(\mathbf{y}))/\varepsilon$. We obtain

$$\frac{\partial I}{\partial y_j}(\mathbf{y}) = \frac{1}{\mu(\mathbf{y})}(h^*(\mathbf{y}) - h(j/n)), \tag{4}$$

with

$$h^*(\mathbf{y}) = \sum_{i=1}^{n} h(i/n)\lambda_i, \tag{5}$$

where in the last expression we normalize the income $\lambda_i = y_i/n\mu(\mathbf{y})$. In (4), the first term comes from the change in the mean $\mu(\mathbf{y})$ and the second term is the direct effect on y_j. From (4) and (3), we have $\partial I/\partial y_i \leq \partial I/\partial y_j$ for $i < j$. The sign of the change depends on the relationship between the weight associated to the receiver $h(j/n)$ and a *pivotal weight* $h^*(\mathbf{y})$ defined by (5). Notice that, by definition, for all $\mathbf{y} \in Y_n(D)$ we have $h^*(\mathbf{y}) \in (0,1]$ and thus it is comparable with any probability $h(j/n)$. As weights are ordered, the pivotal weight induces a *pivotal individual*.

Definition (pivotal individual). *For a given* $\mathbf{y} \in Y_n(D)$, *the* pivotal individual $j^*(\mathbf{y})$ *is defined by*

$$j^*(\mathbf{y}) = \max\left[j \mid h(j/n) \geq h^*(\mathbf{y}) = \sum\nolimits_i h(i/n)\lambda_i \right]. \tag{6}$$

In other words, the pivotal individual is the richest individual who can receive a subsidy without an increase in inequality. The next proposition demonstrates the existence of the pivotal individual (6) and it characterizes the behavior of the derivative (4).

Proposition 1. *For all* $\mathbf{y} \in Y_n(D)$ *and* $I(\mathbf{y})$ *given by (1) and (2), there exists a pivotal individual* $1 \leq j^*(\mathbf{y}) \leq n$ *given by (6), such that if a subsidy* ε *is given to an individual with rank* j, *then*

- *(a) inequality* $I(\mathbf{y})$ *non-increases if* $j \leq j^*(\mathbf{y})$. *The lower the rank of the receiver, the (weakly) stronger the decreasing effect.*
- *(b) inequality* $I(\mathbf{y})$ *increases if* $j > j^*(\mathbf{y})$. *The higher the rank of the receiver, the (weakly) stronger the increasing effect.*

Proof. *First we prove the existence of* j^* *(here we omit dependence on* $\mathbf{y}$*). Let* $h(1/n) = h(n/n)$ *so* $h(i/n) = 1/n$, *for all* i *and* $h^* = 1/n$ *in (5). In (6),* $j^* = n$. *Let* $h(1/n) > h(n/n)$. *We rewrite (5) as* $\Sigma_i(h(i/n) - h^*)\lambda_i = 0$. *Then we have* $\Sigma_i(h(i/n) - h(1/n))\lambda_i < 0$ *and* $\Sigma_i(h(i/n) - h(n/n))\lambda_i > 0$. *For continuity we have* $h(1/n) > h^* > h(n/n)$ *such that* $\Sigma_i(h(i/n) - h^*)\lambda_i = 0$, *which is (5). In (6),* $1 \leq j^* < n$. *Second, (a) and (b) are given by (4). For* $j > j^*$, $h(j/n) < h^*$ *by (6) and (a) follows from (4). For* $j \leq j^*$, $h(j/n) \geq h((j/n)^*)$ *by (3) and* $h((j/n)^*) \geq h^*$ *by (6). So* $h(j/n) \geq h^*$ *and (b) follows from (4).* ■

The pivotal individual divides the income distribution in two parts. A subsidy given to any individual in the distribution below (resp. above) that rank does not increase (resp. increases) inequality. In turn, the pivotal depends on both the distribution $\mathbf{y}$ and the inequality measure $I(\cdot)$.

Equation (6) characterizes the pivotal. From (1) and (2), we have a simple expression for the pivotal weight in terms of inequality

$$h^*(\mathbf{y}) = \frac{1}{n}(1 - I(\mathbf{y})). \tag{7}$$

As the pivotal weight decreases with inequality, the rank of the pivotal individual increases with $I(\mathbf{y})$. Alternatively, we can write (7) in terms of the equally distributed equivalent income $h^*(\mathbf{y}) = \Xi(\mathbf{y}/n\mu(\mathbf{y}))$.

As an example, we compute (7) for the widely used *Gini index* $G(\mathbf{y})$, defined by $h(i/n) = (2n(1 - i/n) + 1)/n^2$. In this case, the pivotal individual is defined by

$$(j/n)^*(\mathbf{y}) = \frac{G(\mathbf{y})+1}{2} + \frac{1}{2n}. \tag{8}$$

The interpretation of equation (8) is direct. Consider a society with a large n and $G(\mathbf{y}) = 0.40$. The rule (8) implies that any dollar given to the poorer 70 percent of the population decreases inequality in the sense of $G(\mathbf{y})$; any dollar given to the richer three deciles increases $G(\mathbf{y})$.

2.3. *Absolute Inequality*

While previous results focus on relative inequality, the pivotal individual also exists for absolute measures. An *absolute inequality index* is invariant to the addition of the same amount to each person's income (i.e., satisfies $A(\mathbf{y} + \lambda\mathbf{1}) = A(\mathbf{y})$ for all λ real). The absolute inequality index was introduced by Pollak (1971) and Kolm (1976a, 1976b). The so called *Kolm–Pollak index of absolute inequality* is defined by

$$A(\mathbf{y}) = \mu(\mathbf{y}) - \Xi(\mathbf{y}). \tag{9}$$

As in the previous section, we focus our work on the *generalized Gini absolute inequality indices* defined by (2) with the restriction (3).

Again, the effect of a subsidy ε given to one individual is in question here. As in the previous subsection, we define the derivative of absolute inequality using the limit $\varepsilon \to 0^+$ in order to obtain

$$\frac{\partial A}{\partial y_j}(\mathbf{y}) = \left(\frac{1}{n} - h(j/n)\right). \tag{10}$$

In this case, the pivotal weight is simply given by $(1/n)$. We define a pivotal individual for absolute inequality in similar terms such that (6). Consider j_A^* the richest individual who can receive a subsidy without an increase in absolute inequality (9), which is defined by $j_A^* = \max[j \mid h(j/n) \geq (1/n)]$. The analog to Proposition 1 for j_A^* and absolute inequality follows directly from (10).

From (7) we observe that $h^* \leq (1/n)$ except in the case of perfect equality. Accordingly, the pivotal individual has a lower rank for any absolute measure (9) compared to a relative one (1), i.e. $j_A^* \leq j^*(\mathbf{y})$ with equality in the case that $\mathbf{y} = \mu(\mathbf{y})\mathbf{1}$. This is in agreement with the claim that in many situations relative inequality decreases while absolute inequality increases.

We also observe that j_A^* does not depend on $\mathbf{y}$ although it changes for different sequences $h(i/n)$. The equivalent to (8) for the *absolute Gini index* is $(j/n)^* = 1/2 + 1/2n$. For a large n, the absolute Gini index increased for any transfer given to individuals richer than the median. Let us consider again the society with a large n and $G(\mathbf{y}) = 0.40$. Therefore, any transfer below the median decreases absolute and relative Gini indices; subsidies for individuals in deciles 6 and 7 increase (resp. decrease) the absolute (resp. relative) Gini index, and all transfers to the richer 30 percent increase both measures of Gini inequality.

3. The Pivotal Individual as a Normative Choice

The Previous section computes the pivotal individual for a given inequality index. Given a one to one relation between the index and the pivotal, the selection of the first necessarily implies the identity of the latter. But as inequality measures are a matter of ethical preferences, the same question can be posed in reverse. A practical normative choice is to identify the richer individual who receives a subsidy without an increase in inequality and to construct an inequality index consistent with this view. This section explores this issue.

A flexible approach in attaching inequality measures to value judgments is to specify a parameter ε to describe a family of indices. Atkinson (1970) called ε the *inequality aversion* or the relative sensitivity to transfers at different income levels. For his family of indices, ε ranges from zero, representing indifference to inequality, to infinity, representing the Rawlsian criterion or $min(y_i)$.

A parametric version of generalized Gini indices was presented by Donaldson and Weymark (1980).[4] For a parameter $\delta \geq 1$, the *single parameter Gini indices* $I_\delta(\mathbf{y})$ are defined by

$$h_\delta(i/n) = \left(\frac{n-i+1}{n}\right)^\delta - \left(\frac{n-i}{n}\right)^\delta. \tag{11}$$

Single parameter Gini indices are the subclass of generalized Gini indices which satisfies Dalton's (1920) *Population Principle*. For $\delta = 1$ and $\delta \to \infty$ the limits are the same such that $\varepsilon = 0$ and $\varepsilon \to \infty$ for the Atkinson index. For $\delta = 2$ we recover the standard *Gini index*.

An explicit expression for the pivotal individual is not possible for an arbitrary δ except for some simple cases. For $\delta = 1$, $h^*(\mathbf{y}) = 1/n$ and $j^*(\mathbf{y}) = n$. In this

[4]Parametric Ginis for income distributions defined in the continuum are discussed by Donaldson and Weymark (1983) and Yitzhaki (1983).

case, the welfare function ranks distributions solely according to total income and so a subsidy given to any individual increases welfare and does not decrease inequality. For $\delta = 2$ we have the standard Gini index discussed in (8). In the limit $\delta \to \infty$ the pivotal individual is again located in $j^*(\mathbf{y}) = n$. In this last case welfare solely depends on the minimum income, so again, a subsidy given to any individual increases welfare and does not decrease inequality.

For a general δ, a large n approximation is useful. As (11) is a difference, for large n and (i/n) bounded it can be approximated by the derivative $\delta n^{-1}(1 - i/n)^{\delta-1}$. We introduce this expression, evaluated in the pivotal rank $(j/n)^*$, in (7) and we obtain

$$(12) \qquad (j/n)^*(\mathbf{y}) = 1 - (1/\delta)(1 - I_\delta(\mathbf{y}))^{1/(\delta-1)}.$$

For $\delta = 2$, equation (12) gives $(G(\mathbf{y}) + 1)/2$ wich is the approximation of (8) for n large.

The general question in this section concerns the monotonicity between $(j/n)^*(\mathbf{y})$ and δ. A monotone relation assures that value judgments can be done directly over the identity of the pivotal individual $(j/n)^*(\mathbf{y})$. The following proposition shows this result for the expression (12).

Proposition 2. *Given $\mathbf{y} \in Y_n(D)$, for any $\delta > 1$, the rank of the pivotal individual $(j/n)^*(\mathbf{y})$ defined by (12) is strictly increasing in δ.*

Proof. *See Supplmentary Appendix.*

Proposition 3 claims that the stronger the inequality aversion of the decision maker (i.e. more convexity in $h_\delta(i/n)$), the higher the rank of the pivotal individual.

To summarize, the election of an inequality index from a family of parameterized measures is based on an ethical judgment regarding "inequality aversion." Alternatively, the question can be formulated in terms of the "pivotal individual": who is the richest individual that we find just and fair to compensate? In this sense, our results provide a simple and useful criterion for a normative prescription.

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Supporting Information

Additional Supporting Information may be found in the online version of this article:

Appendix: Proof of Proposition 2