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ON ENDOGENEITY BIAS: SOME GENERAL REMARKS AND CORRECTION OF THE *ROIW* PAPER BY BOURGUIGNON, FERREIRA, AND MENÉNDEZ (2007)

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This paper is a short note on the question of correcting for endogeneity bias in a regression. It also points out errors and omissions in the derivation of this bias in Bourguignon, Ferreira, and Menéndez (2007). We show that some assumptions needed for the derivation are not explicit and even under these assumptions, certain simplifications adopted are not valid. We support our points using simulation experiments.

JEL Codes: C1, D63, O15

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1. General Remarks

This paper is concerned with the problem of endogeneity bias, i.e. bias due to the correlation between the explanatory variables and the error term of a regression equation. It is no secret that the OLS estimator is biased in this case and the Instrumental Variables technique offers a valid (asymptotic) solution. The question we ask is whether there is a direct way to calculate the bias of the OLS estimator and correct for it. Bourguignon, Ferreira, and Menéndez (2007) (abbreviated as BFM (2007) hereafter) address this issue in a recent paper in the context of a study of the contribution of circumstance variables to the earnings inequality and derive an expression of the bias. We will first consider the question in general in this section, then discuss the derivation in BFM (2007) in Section 2, and finally present some simulation results in support of our arguments in Section 3.

We adopt the same notation as BFM (2007) in order to facilitate reference to the equations of the original paper.

Let us consider a regression equation as follows:[1]

$$y_i = X_i\gamma + u_i, \quad i = 1, \ldots, n, \tag{1}$$

with $u_i \sim i.i.d.(0, \sigma_u^2)$, X_i consisting of K variables, and where u_i is correlated with X_i. The OLS estimator of γ is given by:

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[1]Note that X_i denotes a **row** vector and **not** a column one as is usually the case. We have followed the same notation as BFM (2007).

(2) $$\hat{\gamma} = (\underline{X}'\underline{X})^{-1}\underline{X}'\underline{y} = \gamma + (\underline{X}'\underline{X})^{-1}\underline{X}'\underline{u},$$

denoting[2] the **column vector** of all the u_i's as $\underline{u}$, the **column vector** of all the y_i's as $\underline{y}$, and the **matrix** of all the X_i's as $\underline{X}$.

Thus the bias denoted as B can be written as:

(3) $$B = E(\hat{\gamma}) - \gamma = E((\underline{X}'\underline{X})^{-1}\underline{X}'\underline{u}).$$

There are two options for going further with the calculation of the above expectation.

- If $\underline{X}$ and $\underline{u}$ were independent, one could conclude that

$$E((\underline{X}'\underline{X})^{-1}\underline{X}'\underline{u}) = E((\underline{X}'\underline{X})^{-1}\underline{X}')E(\underline{u}) = 0.$$

- If $\underline{X}$ and $\underline{u}$ are (possibly) correlated, then one can go through the conditional expectation method:

(4) $$E((\underline{X}'\underline{X})^{-1}\underline{X}'\underline{u}) = E[E(\underline{X}'\underline{X})^{-1}\underline{X}'\underline{u}|\underline{X}] = E[(\underline{X}'\underline{X})^{-1}\underline{X}'E(\underline{u}|\underline{X})].$$

However, the expression of the conditional expectation $E(\underline{u}|\underline{X})$ cannot be derived in general. Normality assumption can provide an answer. Thus, if we assume that[3]

$$\begin{bmatrix} X_i' \\ u_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_x & \sigma_{xu} \\ \sigma_{xu}' & \sigma_u^2 \end{bmatrix}\right) \quad \forall i,$$

and independent for different i's, then we can write:

$$E(u_i | X_i) = \sigma_{xu}'\Sigma_x^{-1}X_i' = X_i\Sigma_x^{-1}\sigma_{xu}.$$

Hence

$$E(\underline{u}|\underline{X}) = \underline{X}\Sigma_x^{-1}\sigma_{xu},$$

leading to the following expression for the bias

(5) $$E[(\underline{X}'\underline{X})^{-1}\underline{X}'E(\underline{u}|\underline{X})] = E[(\underline{X}'\underline{X})^{-1}\underline{X}'\underline{X}\Sigma_x^{-1}\sigma_{xu}] = \Sigma_x^{-1}\sigma_{xu}.$$

The above equation gives us an analytical expression of the bias under normality.

Finally, we note that one can calculate **plim** $\hat{\gamma}$ (as $n \to \infty$) without the normality assumption, and under some standard limit assumptions concerning $\underline{X}$ and $\underline{u}$ we have:

(6) $$\text{plim}\ \hat{\gamma} = \gamma + \Sigma_x^{-1}\sigma_{xu}.$$

[2]Once again we have deviated from the usual notation of X for the matrix of all observations X_i, y for the vector of all observations y_i and u for the vector of all errors u_i, to align our notation with that of BFM (2007) as they use X and u to denote a single (generic) observation at some stage.

[3]We assume, without loss of generality, as done by BFM (2007), that X_i's are centered.

2. Correction of BFM (2007)

BFM (2007) analyze the contribution of circumstances to earnings inequality using a regression of earnings on some circumstance and effort variables. Since effort can also be influenced by circumstances, they specify additional equations for effort explained by relevant circumstance variables. In fact three effort variables are used and hence three effort equations are added. As the omitted variables of all these equations could possibly be correlated, the four equations form a simultaneous equation system. Further, the authors note that there is no need to estimate the structural form for computing the contribution of circumstances to inequality. It is sufficient to separately estimate the earnings equation and the reduced form. They also explain how both the direct effect (through the coefficients of circumstances on earnings) and the indirect effect (through the coefficients of circumstances on the effort variables which in turn affect earnings) can be obtained from these two estimations.

The authors strongly suspect that the error term (of the earnings equation) is likely to be correlated with the explanatory variables, and hence the OLS estimator will be biased. One solution is given by the instrumental variables technique but they argue that "an instrumental variable strategy is unlikely to succeed, since it is difficult to conceive of correlates of the circumstance variables that would not themselves have any direct influence on earnings." Thus they go on to calculate the possible bias of the OLS estimator. The bias is calculated as follows.

The authors write the regression equation in a general notation as:[4]

$$\ln w_i = X_i\gamma + u_i \tag{7}$$

where "u_i need not be orthogonal to the explanatory variables in X_i."

Denoting a single (say the i-th) observation as X and u (without the subscript i), the authors say that:

$$B = E(\hat{\gamma} - \gamma) = (X'X)^{-1}E(X'u) \tag{8}$$

and note that

$$E(X'u) = \underline{(\rho_{xu}\sigma_x)}\sigma_u,$$

where σ_u is the standard deviation of u, ρ_{xu} is the vector of correlations between the x variables and u, σ_x is the vector of the standard deviations of the x variables, and $\underline{(\rho_{xu}\sigma_x)}$ is a vector where each correlation is multiplied by the standard deviation of the corresponding x variable. Combining the last two formulae yields

$$B = (X'X)^{-1}\underline{(\rho_{xu}\sigma_x)}\sigma_u. \tag{9}$$

Next, the authors use

$$\hat{u} = u - X(\hat{\gamma} - \gamma)$$

[4]The only change in notation between our equation (1) and equation (7) is that y_i is replaced by $\ln w_i$, which is of no consequence for the derivation of the bias.

to write

$$u'u = \hat{u}'\hat{u} + (\hat{\gamma} - \gamma)' X'X (\hat{\gamma} - \gamma),$$

and

(10) $$\sigma_u^2 = E(u'u) = E(\hat{u}'\hat{u}) + E((\hat{\gamma} - \gamma)' X'X (\hat{\gamma} - \gamma)).$$

Then they replace $(\hat{\gamma} - \gamma)$ by its expectation B and get:

(11) $$\sigma_u^2 \simeq \hat{\sigma}_u^2 + B'X'XB.$$

Substituting the expression of the bias from (9), they finally get

(12) $$\sigma_u^2 = \hat{\sigma}_u^2 + (\underline{\rho_{xu}\sigma_x})'(X'X)^{-1}(\underline{\rho_{xu}\sigma_x})\sigma_u^2.$$

With their notation

(13) $$K = (\underline{\rho_{xu}\sigma_x})'(X'X)^{-1}(\underline{\rho_{xu}\sigma_x}),$$

one can write the bias as

(14) $$B = (X'X)^{-1}(\underline{\rho_{xu}\sigma_x})\hat{\sigma}_u^2 / (1-K).$$

There are several incorrect points in the above derivation. Some are notational and some more substantial. Let us discuss them one by one in detail.

1. The expectation operator is missing in the specification of the covariance matrix Σ in their paper. The correct specification should be that of the *theoretical* covariance matrix:

(15) $$\Sigma = E\begin{bmatrix} X'X & X'u \\ uX & u^2 \end{bmatrix}$$

omitting the subscript i (as done by the authors). Absence of the expectation operator in the above expression in their paper is not merely a *notational* error (or a typo) but is also a *computational* error as many subsequent expressions have mistakenly interchanged theoretical moments and empirical counterparts, thus leading to incorrect conclusions. Note that the last term is simply u^2 as u is a scalar here.

2. Given the correlation between X and u, and the non-linearity of the inverse operator, it is incorrect to write $E[(X'X)^{-1}X'u]$ as $(E(X'X))^{-1}E(X'u)$ which is implicit in result (8). Let us recall that the bias given in our result (5) does not arise from writing $E[(X'X)^{-1}X'u]$ as a product of $(E(X'X))^{-1}$ and $E(X'u)$ but from the conditional expectation derivation **under normality** as shown in the previous section. Thus there is a substantive element that is ignored in the result even if we correct the notational error.

   Under the normality assumption, we saw earlier that we get the product $\Sigma_x^{-1}\sigma_{xu}$. As these theoretical parameters are unknown, they have to be estimated. This can be done using their sample counterparts and then an *estimation* of the bias will be given by

$$\left(\frac{1}{N}X'X\right)^{-1}\underline{(\hat{\rho}_{xu}\hat{\sigma}_x)}\hat{\sigma}_u^2$$

where $\hat{\rho}_{xu}$, $\hat{\sigma}_x$, $\hat{\sigma}_u^2$ denote the *estimates* of the respective true parameters. The notation $(X'X)^{-1}\underline{(\rho_{xu}\sigma_x)}\sigma_u^2$ is not consistent as the first part of the expression contains an estimate and the second true values, plus $\frac{1}{N}$ is missing.[5]

Turning to the substance of the normality condition, it may not be justified in the empirical context in which many of the x's are categorical (or binary) variables (as we need *joint* normality of *both the explained and explanatory* variables). In the non-normality case we cannot derive the analytical expression of the exact bias.

3. Now we turn to their result reproduced in our (11). Looking at the passage from (10) to (11), we see that the authors have replaced $\hat{\gamma}-\gamma$ by its expectation B as "a convenient approximation" *inside* the Expectation operator. As a result of this approximation, the expectation operator only applies to $X'X$ in the whole expression. Now, $E(X'X)$ should have been replaced by $\left(\frac{1}{N}X'X\right)$, i.e. $N(X'X)^{-1}$ in our (12). The factor $\frac{1}{N}$ is once again missing in the expression (11) and consequently N in (12) (their equations (13) and (14)).

   Going beyond the $\frac{1}{N}$ factor, from a theoretical point of view, the "convenient approximation" is not justified as one cannot replace a random variable A by its expectation *inside* an expectation operator such as $E(ACA)$, especially when A is correlated with C. It is easily seen that $E((\hat{\gamma}-\gamma)'X'X(\hat{\gamma}-\gamma)) = E\left(u'X(X'X)^{-1}X'u\right)$ as $\hat{\gamma}-\gamma=(X'X)^{-1}X'u$. As such this expectation cannot be analytically calculated when X and u are correlated.

   Even continuing with the authors' simplification of replacing $\hat{\gamma}-\gamma$ by B (its expectation), one cannot conclude on its under-estimation as that depends on the nature of the covariance between $X'X$ and $(X'X)^{-1}X'u$ which is what is ignored in this "convenient approximation." In fact the authors go on to say that "This underestimation is likely to be small if the expected bias, B, is estimated with enough precision." This shows that they are discussing how far the *estimate* of B will be away from its true value and *not* whether it was legitimate to extract the bias outside the expectation operator. Taking the case when $E(X'u)=0$, i.e. $E(\hat{\gamma}-\gamma)=0$, the authors will get 0 for their approximation (as they will have $0 \cdot X'X \cdot 0$) whereas the true result is $E(u'X(X'X)^{-1}X'u) = E(trX(X'X)^{-1}X'uu') = trX(X'X)^{-1}X'E(uu') = \sigma^2 trP_X = K\sigma^2$ as is well known. Thus we do not get the right result for the expectation of the product even if we insert the exact

[5]We thank the referee for pointing out this additional omission.

bias inside the expectation. Therefore one can see that the "convenient approximation" is not valid irrespective of whether the estimate of the bias B is precise or not.

4. Finally, the first term of (11) is also incorrect as, recalling that u is a scalar, $E(\hat{u}'\hat{u}) = E(\hat{u}^2) \neq \hat{\sigma}_u^2$. In fact this expectation cannot be computed. Denoting the *column vector* of all $\hat{u}_i$'s as $\underline{\hat{u}}$, it is well known that $\underline{\hat{u}} = M_x \underline{u}$ where $M_x = I - \underline{X}(\underline{X}'\underline{X})^{-1}\underline{X}'$. Thus $E(\underline{\hat{u}}'\underline{\hat{u}}) = E(\underline{u}'M_x\underline{u}) = E(trM_x\underline{u}\underline{u}')$ One cannot go further in this exact moment calculation when X and u are correlated. The same problem remains when taking a single element of the vector of $\hat{u}$'s. Hence $E(\hat{u}^2) \neq \hat{\sigma}_u^2$, $\hat{\sigma}_u^2$ can only be an estimator of $E(u^2)$ and that too if $\hat{u}$ is a valid estimator of u, which it is not in this case as $\hat{\gamma}$ is biased.

Therefore *their* expression (13), reproduced in our equation (12), is not correct and hence their bias given by our equation (14) is not a correct approximation. In what follows we will see that correcting for the missing $\frac{1}{N}$ in front of $X'X$ helps but does not solve the problem completely especially when ρ_{xu} is unknown and has to be given *a priori* values.

3. Simulations

This section evaluates the bias given in BFM (2007) comparing it with the "true" theoretical and finite sample biases through a simulation exercise and finds that the BFM formula as given in the paper largely underestimates the "true" bias. If we correct it for the missing factor $\frac{1}{N}$ then the point estimate of the bias is close to the small sample and theoretical biases but it assumes that the true value of the correlation between X and u is known. However, the interval of the bias, which is what can be calculated in an empirical setting without knowing the true correlation, is always complex.

We present some simulation results to show the magnitude of the difference between the bias calculated using BFM (2007)'s formula (9), that obtained using our correct formula (either under normality or in limit) given by (5), and the corrected version of their formula inserting the factor $\frac{1}{N}$ which closely resembles an estimate of our correct formula.

Our first experiment is as follows. The regression equation is a very simple one given by

$$y_i = X_i\beta + u_i \tag{16}$$

where X is a scalar. We generate a certain number of observations of X and u from the bivariate normal distribution

$$\begin{bmatrix} X_i \\ u_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xu} \\ \sigma_{xu} & \sigma_u^2 \end{bmatrix}\right)$$

and calculate the corresponding values of y_i according to (16). Initially β is set at 0.7. The variance–covariance parameters of the above normal distribution are set at $\sigma_x^2 = 4$, $\sigma_u^2 = 1$, and $\sigma_{xu} = 1.5$. This gives a correlation coefficient between X and u, denoted as ρ_{xu}, equal to 0.75. The initial sample size is taken as 50 and 100 samples are generated. For each sample, $\hat{\beta}_{ols}$ is calculated and the following quantities are computed.

1. The mean of the 100 $\hat{\beta}_{ols}$ values.
2. The difference between the above mean and the true value of β, calling it the finite sample bias.
3. The expression given by (5), calling it the theoretical bias.
4. The expression given by (9), using the true value of ρ_{xu}, the estimated value of σ_x, and the value of σ_u^2 obtained using (12), calling it the BFM formula bias.[6]
5. The expression given by the author-corrected version of (9), using the true value of ρ_{xu}, the estimated value of σ_x, and the value of σ_u^2 obtained using the author-corrected version (12), calling it the BFM reported bias.
6. The BFM "confidence interval" for the bias using (a) $\rho_{xu} = -1$ and (b) $\rho_{xu} = +1$ instead of the true value of ρ_{xu} in the previous calculation.

After this base simulation, the true values of the parameters are varied to study the effect on the biases mentioned above. In particular different combinations of σ_{xu} and σ_u^2 are explored: (1.5, 1), (–1.5, 1), (1, 0.25), (0.5, 0.25), and (0.25, 0.25), giving the values of 0.75, –0.75, 1, 0.5, and 0.25 for ρ_{xu} respectively (σ_x^2 is maintained at 4 for all these cases). Some trials are repeated with the number of observations increased to 100, 1000, and 10,000, and/or the number of simulations to 200. The results are reported in Table 1.

Table 1 confirms that, for a known ρ_{xu}, the theoretical bias, the small sample bias, and the corrected BFM bias (with the $\frac{1}{N}$ factor) are all of the same order of magnitude. The formula given in BFM (2007) gives a very low incorrect bias. In a real setting where the true value of ρ_{xu} is unknown, BFM (2007) suggest exploring various values for ρ_{xu} ranging from –1 to 1 to obtain "confidence intervals." Taking these limits, we get the confidence intervals to be entirely complex in all models with one x. This is not surprising and is due to the fact that when $\rho_{xu} = 1$ or -1, the (corrected) K value is given by $\left(\frac{1}{N}X'X\right)^{-1}\hat{\sigma}_x^2$, which is $\frac{N}{N-1}$ when there is only one x (assuming σ_x^2 is estimated by $\frac{1}{N-1}(X'X)$). Thus K will always be >1, implying $1 - K < 0$ and hence the two limits of the interval will always be complex.[7]

As a first step we decided to gradually narrow down the range of values for ρ_{xu} to see the effect on the bias range. Results are in Table 2. The interval for the bias becomes real for values of $|\rho_{xu}|$ less than 1 but is rather wide for all ranges of ρ_{xu} and

[6]We were subsequently informed that the bias reported in BFM (2007) was actually calculated using the correct version (see next point). Hence we call this one the BFM formula bias as this is what one would obtain if one were to calculate the bias according to the formula given in their paper.

[7]In case σ_x^2 is estimated by $\frac{1}{N}(X'X)$ then $K = 1$ and we will be dividing by 0 and the interval will become $]-\infty, \infty[$.

TABLE 1

Simulation Results: Different Models with One x

Trials	I	II	III	IV	V	VI	VII	VIII	IX	X
β	0.7	0.7	0.7	0.7	0.3	0.7	0.7	0.7	0.7	0.7
σ_x^2	4	4	4	4	4	4	4	4	4	4
σ_{xu}	1.5	1.5	1.5	–1.5	1.5	1	0.25	0.5	–1.5	1.5
σ_u^2	1	1	1	1	1	0.25	0.25	0.25	1	1
ρ_{xu}	0.75	0.75	0.75	–0.75	0.75	1	0.25	0.5	–0.75	0.75
$\hat{\sigma}_u^2$	0.4521	0.4436	0.4474	0.4301	0.4478	$7.721 \cdot 10^{-32}$	0.2374	0.1864	0.4395	0.4370
$\hat{\sigma}_x^2$	4.0292	4.1796	3.9553	3.9945	4.0157	4.1045	4.0398	4.0129	3.9950	4.0028
Mean $\hat{\beta}_{ols}$	1.0752	1.0781	1.0761	0.3275	0.6735	0.9500	0.7610	0.8218	0.3250	1.0747
Finite sample bias	0.3752	0.3781	0.3761	–0.3725	0.3735	0.2500	0.0610	0.1218	–0.3750	0.3747
Theoretical bias	0.3750	0.3750	0.3750	–0.3750	0.3750	0.2500	0.0625	0.1250	–0.3750	0.3750
BFM formula bias	0.0052	0.0050	0.0026	–0.0025	0.0025	$1.2325 \cdot 10^{-18}$	$6.1537 \cdot 10^{-4}$	0.0011	$-2.4910 \cdot 10^{-4}$	$-2.4784 \cdot 10^{-4}$
BFM reported bias	0.3941	0.3842	0.3881	–0.3784	0.3854	$1.2201 \cdot 10^{-15}i$	0.0636	0.1264	–0.3767	0.3747
BFM conf. int.*	0 – 2.4007i	0 – 2.3403i	0 – 3.3834i	0 – 0.32989i	$0 - 1.2201 \cdot 10^{-15}i$	0 – 2.4484i	0 – 2.1741i	0 – 10.4947i	0 – 33.0429i	$-3.3076 \cdot 10^{-4}$
BFM conf. int.**	0 + 2.4007i	0 + 2.3403i	0 + 3.3834i	0 + 0.32989i	$0 + 1.2201 \cdot 10^{-15}i$	0 + 2.4484i	0 + 2.1741i	0 + 10.4947i	0 + 33.0429i	$-3.3076 \cdot 10^{-4}$
No. of observ.	50	50	100	100	100	100	100	100	1,000	10,000
No. of simulations	100	200	200	200	200	200	200	200	200	200

*Lower bound with $\rho_{xu} = -1$.
**Upper bound with $\rho_{xu} = +1$.

TABLE 2

Simulation Results: Exploring Different Intervals for ρ_{xu}

Model X of Table 1 with one x						
Theoretical bias: 0.3750, true ρ_{xu} = 0.7, n = 10,000, nsim = 200						
Trials ρ	–1, +1	–0.9, +0.9	–0.8, +0.8	–0.7, +0.7	–0.6, +0.6	–0.3, +0.3
Finite sample bias	0.3747	0.3747	0.3747	0.3747	0.3747	0.3747
BFM reported bias	0.3747	0.3747	0.3747	0.3747	0.3747	0.3747
BFM conf. int.*	0 – 33.0429i	–0.6833	–0.4404	–0.3246	–0.2482	–0.1040
BFM conf. int.**	0 + 33.0429i	+0.6833	+0.4404	+0.3246	+0.2482	+0.1040
Model VII of Table 1 with one x						
Theoretical bias: 0.0625, true ρ_{xu} = 0.25, n = 10,000, nsim = 200						
Trials ρ	–1, +1	–0.9, +0.9	–0.8, +0.8	–0.7, +0.7	–0.5, +0.5	–0.3, +0.3
Finite sample bias	0.0625	0.0624	0.0625	0.0626	0.0625	0.0625
BFM reported bias	0.0624	0.0625	0.0625	0.0624	0.0626	0.0625
BFM conf. int.*	0 – 24.1702i	–0.4998	–0.3228	–0.2369	–0.1399	–0.0761
BFM conf. int.**	0 + 24.1702i	+0.4998	+0.3228	+0.2369	+0.1399	+0.0761

*Lower bound.
**Upper bound.

TABLE 3

Simulation Results: Generating Random Values for ρ_{xu}

Model X of Table 1 with one x						
Theoretical bias: 0.3750, true ρ_{xu} = 0.7, n = 10,000, nsim = 200						
Trials ρ	1	2	3	4	5	6
BFM lower bound	–2.3695	–1.0002	–1.4910	–1.5456	–2.4868	–3.7873
Corresponding ρ_{xu}	–0.9903	–0.9495	–0.9762	–0.9778	–0.9912	–0.9962
BFM upper bound	1.5513	1.6388	0.9206	0.17.1995	1.4123	2.6721
Corresponding ρ_{xu}	0.9780	0.9802	0.9412	0.9998	0.9736	0.9924
Model VII of Table 1 with one x						
Theoretical bias: 0.0625, true ρ_{xu} = 0.25, n = 10,000, nsim = 200						
Trials ρ	1	2	3	4	5	6
BFM lower bound	–1.9250	–0.7316	–1.0909	–0.9150	–1.3856	–0.9991
Corresponding ρ_{xu}	–0.9922	–0.9495	–0.9762	–0.9667	–0.9850	–0.9991
BFM upper bound	1.6526	1.1995	0.6735	3.7962	1.2019	1.3657
Corresponding ρ_{xu}	0.9894	0.9802	0.9412	0.9979	0.9803	0.9846

includes zero. As a further exercise, we also generated 100 random values of ρ_{xu} from a uniform distribution [–1, 1] and calculated the BFM reported bias for each of them. The resulting range for the bias values, excluding the ρ_{xu} values that give a complex interval, is given in Table 3. For a proper interpretation of these results, we report different trials that correspond to different sets of 100 random values for ρ_{xu}. A comparison of the bias ranges obtained with that calculated with the true value of ρ_{xu} indicates that the interval is rather large even when it is not complex, and contains zero.

TABLE 4

Simulation Results: Different Models with Two x's

Trials	I	II	III	IV	V
β	$\begin{pmatrix}0.7\\1.4\end{pmatrix}$	$\begin{pmatrix}0.7\\1.4\end{pmatrix}$	$\begin{pmatrix}0.7\\1.4\end{pmatrix}$	$\begin{pmatrix}0.7\\1.4\end{pmatrix}$	$\begin{pmatrix}0.7\\1.4\end{pmatrix}$
Σ	$\begin{pmatrix}4 & 1 & 1.5\\1 & 4 & 1.5\\1.5 & 1.5 & 1\end{pmatrix}$	$\begin{pmatrix}4 & 1 & 1.5\\1 & 4 & 1.5\\1.5 & 1.5 & 1\end{pmatrix}$	$\begin{pmatrix}4 & 1 & 1.5\\1 & 4 & 1.5\\1.5 & 1.5 & 1\end{pmatrix}$	$\begin{pmatrix}4 & 1 & 1\\1 & 4 & -1\\1 & -1 & 1\end{pmatrix}$	$\begin{pmatrix}4 & 1 & 1\\1 & 4 & -1\\1 & -1 & 1\end{pmatrix}$
Mean $\hat{\beta}_{ols}$	$\begin{pmatrix}1.0002\\1.7002\end{pmatrix}$	$\begin{pmatrix}0.9999\\1.6999\end{pmatrix}$	$\begin{pmatrix}0.9997\\1.7000\end{pmatrix}$	$\begin{pmatrix}1.0323\\1.0695\end{pmatrix}$	$\begin{pmatrix}1.0330\\1.6072\end{pmatrix}$
Finite sample bias	$\begin{pmatrix}0.3002\\0.3002\end{pmatrix}$	$\begin{pmatrix}0.2999\\0.2999\end{pmatrix}$	$\begin{pmatrix}0.2997\\0.3000\end{pmatrix}$	$\begin{pmatrix}0.3323\\-0.3305\end{pmatrix}$	$\begin{pmatrix}0.3330\\-0.3328\end{pmatrix}$
Theoretical bias	$\begin{pmatrix}0.3\\0.3\end{pmatrix}$	$\begin{pmatrix}0.3\\0.3\end{pmatrix}$	$\begin{pmatrix}0.3\\0.3\end{pmatrix}$	$\begin{pmatrix}0.2\\0.2\end{pmatrix}$	$\begin{pmatrix}0.2\\0.2\end{pmatrix}$
BFM reported bias	$\begin{pmatrix}0.3083\\0.3086\end{pmatrix}$	$\begin{pmatrix}0.3105\\0.3097\end{pmatrix}$	$\begin{pmatrix}0.3081\\0.3079\end{pmatrix}$	$\begin{pmatrix}0.3378\\-0.3388\end{pmatrix}$	$\begin{pmatrix}0.3349\\-0.3341\end{pmatrix}$
ρ(lower), ρ(upper)	–1,+1	–0.9, 0.9	–0.8, 0.8	–1, +1	–0.8, 0.8
BFM conf. int.*	$\begin{pmatrix}-0.0382i\\-0.0382i\end{pmatrix}$	$\begin{pmatrix}-0.1649i\\-0.1641i\end{pmatrix}$	$\begin{pmatrix}-0.3161-0.5282i\\-0.3131-0.5280i\end{pmatrix}$	$\begin{pmatrix}-0.2977i\\-0.2986i\end{pmatrix}$	$\begin{pmatrix}-0.4649-1.1388i\\-0.4602-1.1349i\end{pmatrix}$
BFM conf. int.**	$\begin{pmatrix}0.0382i\\0.0382i\end{pmatrix}$	$\begin{pmatrix}0.1649i\\0.1641i\end{pmatrix}$	$\begin{pmatrix}0.3161+0.5282i\\0.3131+0.5280i\end{pmatrix}$	$\begin{pmatrix}0.2977i\\0.2986i\end{pmatrix}$	$\begin{pmatrix}0.4649+1.1388i\\0.4602+1.1349i\end{pmatrix}$
No. of observ.	1000	1000	1000	1000	1000
No. of simulations	100	100	100	100	100

*For ρ(lower).
**For ρ(lower).

TABLE 5

Simulation Results: Exploring Different Intervals for ρ_{xu} in the Two x's Case

$$\Sigma = \begin{pmatrix} 4 & 1 & 1.5 \\ 1 & 4 & 1.5 \\ 1.5 & 1.5 & 1 \end{pmatrix}; \text{ theoretical bias: } \begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix}, \text{ n = 1000, nsim = 100}$$

Trials ρ	–1, +1	–0.8, +0.8	–0.7, +0.7	–0.3, +0.3
BFM conf. int.*	$\begin{pmatrix} -0.0382i \\ -0.0382i \end{pmatrix}$	$\begin{pmatrix} -0.1577-0.6141i \\ -0.1549-0.6126i \end{pmatrix}$	$\begin{pmatrix} -0.1899 \\ -0.1905 \end{pmatrix}$	$\begin{pmatrix} -0.0407 \\ -0.0407 \end{pmatrix}$
BFM conf. int.**	$\begin{pmatrix} +0.0382i \\ +0.0382i \end{pmatrix}$	$\begin{pmatrix} 0.1577+0.6141i \\ 0.1549+0.6126i \end{pmatrix}$	$\begin{pmatrix} 0.1899 \\ 0.1905 \end{pmatrix}$	$\begin{pmatrix} 0.0407 \\ 0.0407 \end{pmatrix}$

$$\Sigma = \begin{pmatrix} 4 & 1 & 1.5 \\ 1 & 4 & 1.5 \\ 1.5 & 1.5 & 1 \end{pmatrix}; \text{ theoretical bias: } \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}, \text{ n = 1000, nsim = 100}$$

Trials ρ	–1, +1	–0.8, +0.8	–0.7, +0.7	–0.3, +0.3
BFM conf. int.*	$\begin{pmatrix} -0.4919i \\ -0.4909i \end{pmatrix}$	$\begin{pmatrix} -0.9140-1.7877i \\ -0.9118-1.7280i \end{pmatrix}$	$\begin{pmatrix} -0.5808 \\ -0.5803 \end{pmatrix}$	$\begin{pmatrix} -0.1235 \\ -0.1234 \end{pmatrix}$
BFM conf. int.**	$\begin{pmatrix} 0.4919i \\ 0.4909i \end{pmatrix}$	$\begin{pmatrix} 0.9140+1.7877i \\ 0.9118+1.7280i \end{pmatrix}$	$\begin{pmatrix} 0.5808 \\ 0.5803 \end{pmatrix}$	$\begin{pmatrix} 0.1235 \\ 0.1234 \end{pmatrix}$

*Lower bound.
**Upper bound.

Next, we decided to consider two x's in the equation to see if the bias range result obtained in the single x case, where the sign of $(1 - K)$ can be theoretically derived, is special or extends to a general case (where we cannot get a theoretical sign for $(1 - K)$). We conducted another series of simulations with two x variables with various parameter values and different ranges for the correlation parameters between the two x's and u. Results are presented in Tables 4 and 5.

Without commenting on all the results in detail, it can once again be seen that the bias calculated according to the formula as given in BFM (2007) is far from the actual bias and heavily underestimated (in absolute terms) whereas the $\frac{1}{N}$-corrected version gives values that are close to the theoretical bias. But this is just one part of the conclusions and not the main one in our opinion as these BFM biases are calculated using the true value(s) of ρ_{xu} which is(are) never known.

In a setting where the true value(s) of ρ_{xu} is(are) unknown, the "confidence intervals" suggested by BFM (2007) using the limits –1 and 1 for ρ_{xu} are entirely complex in all situations. In the two x's case, the interval remains complex even for certain values of $|\rho_{xu}|$ less than 1, up to the range –0.8, 0.8. Narrowing the range further produces a real interval but still spans a wide range of values for the bias. The "estimated" bias can be either several times greater the actual bias or many times lower than than the latter depending on the ρ value chosen. Some intervals can even exclude the true bias. Thus the "confidence intervals" do not provide any useful information on the order of magnitude of the actual bias. This conclusion is not specific to any particular setting but holds across different settings.

Thus we can conclude that it is not possible in a general setting to calculate or approximate the bias of the OLS estimator when the explanatory variables are correlated with the disturbance term, except when we know the true values of the covariance parameters (or of the estimated parameters themselves). The methodology of calculation of the bias and "confidence intervals" presented by BFM (2007) is not correct and the results do not provide the correct range of bias of their OLS estimates. This also casts doubts on their empirical conclusions based on these estimates.

Reference

Bourguignon, F., F. H. G. Ferreira, and M. Menéndez, "Inequality of Opportunity in Brazil," *Review of Income and Wealth*, 53, 585–618, 2007.