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SKEWNESS AND KURTOSIS PROPERTIES OF INCOME DISTRIBUTION MODELS

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This paper explores the ability of some popular income distributions to model observed skewness and kurtosis. We present the generalized beta type 1 (GB1) and type 2 (GB2) distributions' skewness–kurtosis spaces and clarify and expand on previously known results on other distributions' skewness–kurtosis spaces. Data from the Luxembourg Income Study are used to estimate sample moments and explore the ability of the generalized gamma, Dagum, Singh–Maddala, beta of the first kind, beta of the second kind, GB1, and GB2 distributions to accommodate the skewness and kurtosis values. The GB2 has the flexibility to accurately describe the observed skewness and kurtosis.

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1. INTRODUCTION

Pareto's pioneering work in modeling the distribution of income was published more than a century ago. He observed that, in many cases, an approximately linear relationship existed between different income levels and the number of individuals receiving at least that level of income. While the Pareto distribution often provided an accurate model of the upper tail of the distribution, it did a poor job of describing the lower tail. Since inaccurate estimates of distributions can result in misleading policy implications, this led to the consideration of different distributions that more accurately modeled income. Gibrat's (1931) law of proportionate effect provided a theoretical foundation for the use of a two-parameter lognormal distribution, which was studied in more detail by Aitchison and Brown (1969). Battistin *et al.* (2009) used the lognormal to compare the distribution of

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income and consumption across households. Other two-parameter models include the gamma (Salem and Mount, 1974) and the Weibull (Bartels and Van Metelen, 1975). While these two-parameter models provide increased flexibility relative to single-parameter models, they do not allow for intersecting Lorenz curves, which frequently arise with income data.

Intersecting Lorenz curves can be obtained by adding a third parameter. Some common three-parameter models that have been used to model income and allow for intersecting Lorenz curves include the beta of the first kind (B1), the beta of the second kind (B2), the Dagum (DAGUM), and the Singh–Maddala (SM) distributions. Thurow (1970) used the B1 to explore explanatory factors associated with the distribution of income for whites and blacks in the United States. Chotikapanich *et al.* (2011) used the B2 to analyze global income inequality. Dagum's (1977) distribution was based on theoretical foundations and provided a significant improved fit in many applications. Singh and Maddala's (1976) distribution also provided an improved fit relative to the two-parameter models previously considered. The generalized gamma (GG) is another three-parameter model that permits intersecting Lorenz curves and yields improved fit relative to the lognormal and gamma distributions.

The generalized beta of the first and second kind (GB1 and GB2, respectively) are four-parameter models that include each of the previously described models as special or limiting cases. McDonald (1984) provided an early reference to the GB1 and GB2 and its special cases, along with applications. Distributional characteristics, such as moments and the Gini, Pietra, and Theil measures of inequality, can be expressed in terms of the distributional parameters. Other distributions, such as the double-Pareto-lognormal distributional distribution which have desirable properties and provide an excellent fit to empirical data (Kleiber and Kotz, 2003; Reed and Jorgensen, 2004; Reed and Wu, 2008), have been recently explored; the focus of this paper will be restricted to the GB1 and GB2 and its special cases.

There is a substantial literature describing the properties, estimation procedures, and applications of these distributions. Kleiber and Kotz's book provides an excellent summary of these issues and includes more than 500 references to the theoretical foundations and diverse applications of these and other distributions in economics and actuarial science (Kleiber and Kotz, 2003).

Maximum likelihood estimation is a common method of estimating the parameters of income distributions, although other methods have been used. Income data is often reported in a grouped format. Estimation with grouped data can be performed by maximizing a multinomial likelihood function or minimizing a chi-square goodness of fit statistic. Other estimators may be obtained by imposing restrictive assumptions (such as assuming that the observations appear at the midpoint of an income group) or by top coding—both of which ignore intra-group variability. These restrictions can impact estimator precision. Gastwirth (1972) studied the impact of grouping on estimating the Gini coefficient by deriving upper and lower bounds on the Gini coefficient. The lower bound assumes all incomes in an interval equal the average income, and the upper bound corresponds to distributing the income to maximize the spread within each group. McDonald and Ransom (1981) demonstrated that a failure to take account of sampling variation can lead to misleading results.

More recently, continuous income data have become increasingly available and have expanded possible estimation methods and analysis. These data include information drawn from the U.S. Census Bureau, Current Population Survey, and other sources, and they are readily available on the internet. The use of continuous observations yields more accurate estimation of such descriptive statistics as skewness, kurtosis, and Gini coefficients.

In this paper we explore the ability of the GB1 and GB2 distributions to model skewness and kurtosis. While many of the theoretical results are available in different sources, we summarize, clarify, and expand on previously known results, and derive new skewness–kurtosis spaces for the GB1 and GB2 distributions. In addition, we present previously unknown relationships between the skewness–kurtosis spaces for different distributions. We apply the results to the Luxembourg Income Study (LIS) for 13 countries, three definitions of income, and two time periods. The GB2 provides the flexibility to model the observed skewness and kurtosis levels in the cases considered.

The next section summarizes basic characteristics of a number of popular distributions of income (for models of positive income only). Their respective skewness–kurtosis spaces are described in the Appendix. Some new results, corrections to previously published results, and known results are given. Section 3 reports the observed skewness and kurtosis values for different countries, definitions of income, and time periods and compares them to the permissible values based on the distributions considered. Section 4 summarizes our findings.

2. The Models

Since many of the most commonly used models for the distribution of income are special cases of the generalized beta type 1 (GB1) and type 2 (GB2) distributions, we begin by defining them, their moments, and some of their special cases.

The GB1 and GB2 probability density functions (*pdf*s) are defined by:

$$GB1(y;a,b,p,q)=\frac{|a|\,y^{ap-1}\left(1-(y/b)^a\right)^{q-1}}{b^{ap}B(p,q)},\quad 0<y<b$$

$$GB2(y;a,b,p,q)=\frac{|a|\,y^{ap-1}}{b^{ap}B(p,q)\left(1+(y/b)^a\right)^{p+q}},\quad 0<y$$

with corresponding moments given by:

$$E_{GB1}(Y^h)=\frac{b^h\Gamma(p+q)\Gamma(p+h/a)}{\Gamma(p+q+h/a)\Gamma(p)}$$

$$E_{GB2}(Y^h)=\frac{b^h\Gamma(p+h/a)\Gamma(q-h/a)}{\Gamma(p)\Gamma(q)}.$$

The Pareto distribution can be viewed as a special case of the GB1:

$$\begin{aligned} Pareto(y; b, p) &= GB1(y; a=-1, b, p, q=1) \\ &= \frac{py^{-p-1}}{b^{-p}}, \quad b < y \end{aligned}$$

as can the beta of the first kind (B1), used by Thurow (1970):

$$\begin{aligned} B1(y; b, p, q) &= GB1(y; a=1, b, p, q) \\ &= \frac{y^{p-1}(b-y)^{q-1}}{b^p B(p,q)}, \quad 0 < y < b. \end{aligned}$$

The moments of the Pareto and B1 distributions can easily be obtained from expressions for the GB1 moments with appropriate substitutions.

The Singh–Maddala and Dagum distributions are obtained from the GB2 by substituting $p = 1$ and $q = 1$, respectively, into the GB2 *pdf* to obtain:

$$\begin{aligned} SM(y; a, b, q) &= GB2(y; a, b, p=1, q) \\ &= \frac{aqy^{a-1}}{b^a\left(1+(y/b)^a\right)^{q+1}}, \quad 0 < y \end{aligned}$$

$$\begin{aligned} DAGUM(y; a, b, p) &= GB2(y; a, b, p, q=1) \\ &= \frac{apy^{ap-1}}{b^{ap}\left(1+(y/b)^a\right)^{p+1}}, \quad 0 < y. \end{aligned}$$

The Dagum and Singh–Maddala distributions, respectively, are known as the Burr Type 3 and Burr Type 12 distributions in the statistics literature (Burr, 1942; Kleiber and Kotz, 2003).

The beta of the second kind (B2), used by Chotikapanich *et al.* (2011), is another three-parameter special case of the GB2:

$$\begin{aligned} B2(y; b, p, q) &= GB2(y; a=1, b, p, q) \\ &= \frac{y^{p-1}}{b^p B(p,q)(1+(y/b))^{p+q}}, \quad 0 < y. \end{aligned}$$

The moments of the SM, Dagum, and B2 distributions can easily be obtained from expressions for the GB2 moments with appropriate substitutions.

The generalized gamma (GG) was used by Kloek and van Dijk (1978), Taillie (1981), McDonald (1984), Atoda *et al.* (1988), and Bordley *et al.* (1996) to study the income distribution in a number of different countries. The GG *pdf* is obtained from the GB2 by taking the following limit:

$$\begin{aligned} GG(y; a, \beta, p) &= \lim_{q\to\infty} GB2\left(y; a, b = q^{1/a}\beta, p, q\right) \\ &= \frac{|a|\, y^{ap-1} e^{-(y/\beta)^a}}{\beta^{ap}\Gamma(p)}. \end{aligned}$$

The moments of the GG can be expressed as:

$$E_{GG}\left(Y^h\right)=\frac{\beta^h\Gamma(p+h/a)}{\Gamma(p)}.$$

The gamma (GAM), Weibull (W), lognormal (LN), and power function (PF) *pdf*s are the following special or limiting cases of the generalized gamma:

$$\begin{aligned}GAM(y;\beta,p)&=GG(y;a=1,\beta,p)\\&=\frac{y^{p-1}e^{-(y/\beta)}}{\beta^p\Gamma(p)}\end{aligned}$$

$$\begin{aligned}W(y;a,\beta)&=GG(y;a,\beta,p=1)\\&=\frac{ay^{a-1}e^{-(y/\beta)^a}}{\beta^a}\end{aligned}$$

$$\begin{aligned}LN\left(y;\mu,\sigma^2\right)&=\lim_{a\to0}GG\left(y;a,\beta=(a\sigma)^{2/a},p=\frac{a\mu+1}{a^2\sigma^2}\right)\\&=\frac{e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,y\sigma}\end{aligned}$$

$$\begin{aligned}PF(y;\beta,\theta)&=\lim_{a\to\infty}GG(y;\beta,p=\theta/a)\\&=\frac{\theta y^{\theta-1}}{\beta^\theta},\ 0<y<\beta.\end{aligned}$$

The moments of the gamma and Weibull distributions can easily be obtained from expressions for the GG moments with appropriate substitutions.

The moments for the LN and PF are given by:

$$E_{LN}\left(Y^h\right)=e^{h\mu+h^2\sigma^2/2}$$

$$E_{PF}\left(Y^h\right)=\beta^h\left(\frac{\theta}{\theta+h}\right).$$

Equations for the Pietra, Theil, and Gini measures of inequality expressed in terms of the distributional parameters have been derived by various authors and are summarized in Johnson *et al.* (1994), Kleiber and Kotz (2003), and McDonald and Ransom (2008) for the LN, GG, GB1, GB2, and special cases.

The purpose of this paper is to consider the ability of these distributions to model observed combinations of skewness and kurtosis arising in different income studies. We use the standardized skewness and kurtosis measures respectively defined by:

$$\gamma_1 = \frac{E(Y-\mu)^3}{\sigma^3} = \frac{E(Y^3) - 3E(Y^2)\mu + 2\mu^3}{\sigma^3}$$

$$\gamma_2 = \frac{E(Y-\mu)^4}{\sigma^4} = \frac{E(Y^4) - 4E(Y^3)\mu + 6E(Y^2)\mu^2 - 3\mu^4}{\sigma^4}$$

where μ and σ^2 in these equations denote the mean and variance of the random variable of interest. Standardized skewness and kurtosis are often denoted by $(\sqrt{\beta_1}, \beta_2)$ in the literature, but the notation (γ_1, γ_2) more clearly allows for positive and negative skewness.

Skewness and kurtosis can also be expressed in terms of the distributional parameters. For some *pdf*s the permissible skewness–kurtosis combinations yield relatively simple expressions of the distributional parameters. For example, parametric expressions for feasible skewness and kurtosis combinations for the gamma can be written in terms of the distributional parameter p as $\gamma_1 = 2/\sqrt{p}$ and $\gamma_2 = 3 + 6/p$, which can be rewritten as $\gamma_2 = 3 + (3/2)\gamma_1^2$. For other distributions, tractable expressions for permissible kurtosis in terms of skewness have not been obtained, but parametric expressions for skewness and kurtosis are available. The Pareto and lognormal are two examples of distributions with fairly simple parametric representations (see the Appendix) that trace out feasible skewness–kurtosis combinations in the (γ_1, γ_2) plane. Similarly, the Weibull corresponds to a line in the (γ_1, γ_2) plane.

For the three-parameter distributions, the feasible skewness–kurtosis combinations correspond to two-dimensional regions (also referred to as spaces) in the (γ_1, γ_2) plane, which are defined by upper (U) and lower (L) bounds. Rodriguez (1977) explores feasible skewness–kurtosis combinations for the SM distribution. Tadikamalla (1980) derives the upper and lower bounds defining the Dagum skewness–kurtosis space and demonstrates that it includes the SM space as a proper subset. Vargo *et al.* (2010) summarize the skewness–kurtosis space corresponding to a number of distributions (including the LN, B1, B2, GG, GAM, and SM) and provide expressions for bounding curves for some of the distributions. Pearson (1916) demonstrates that $\gamma_2 \geq (\gamma_1)^2 + 1$ for all distributions. This inequality gives the lower bound for any empirical or theoretical distribution, and we refer to it as AD_L. Klaassen *et al.* (2000) show that $\gamma_2 \geq (\gamma_1)^2 + 189/125$ defines a lower bound for unimodal distributions.

In the Appendix, we give upper and lower skewness–kurtosis bounds for the GB1 and GB2, summarize and expand on previously reported results, and provide explicit expressions for the bounding curves.

Figure 1 provides a graphical representation of the GB1, GB2, B1, B2, gamma, Pareto, lognormal, and normal skewness–kurtosis spaces. The normal corresponds to the point (0,3) in the (γ_1, γ_2) plane. The B1 and GB1 share the same skewness–kurtosis lower bound (represented in the figure as $B1_L$ and $GB1_L$, with bounds for other distributions labeled similarly); the lower bound for all distributions is AD_L. The gamma curve provides the upper bound for the B1 skewness–kurtosis space and the lower bound for the B2 space. The B2 allows for only positive skewness values, with the lower and upper bounds originating at (0,3). The

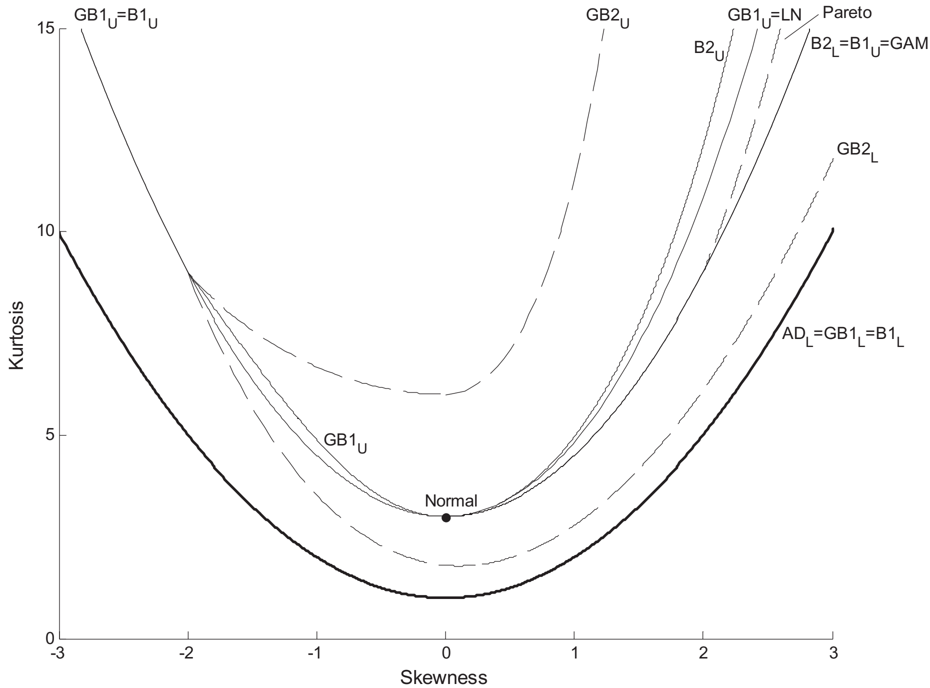

Figure 1. Skewness–Kurtosis Spaces for GB1, GB2, B1, B2, Gamma, Pareto, Lognormal, and Normal

Note: The "L" subscript represents the lower bound and the "U" subscript represents the upper bound for the respective distribution's skewness–kurtosis space.

GB2 skewness values are always greater than –2. For skewness values greater than –2, the GB2 lower bound is above the GB1 lower bound; however, the GB2 upper bound lies above the GB1 upper bound. While not obvious from the figure, the Pareto curve is contained in the B2 and GB2 spaces, but it lies above the upper bound for the more general GB1 distribution for skewness values exceeding 3.5. This is possible since the Pareto is a special case of the GB1 with $a = -1$, whereas the GB1, GB2, and their special cases correspond to $a \geq 0$ The Pareto also has a vertical asymptote at skewness of 7.07.

As many studies of income distributions employ special cases of the GB2 distribution, it is instructive to focus on the skewness–kurtosis spaces for the GB2 and its special cases, which are depicted in Figure 2. The GB2 upper bound lies above all of the upper bounds of its special and limiting cases. The Dagum and Singh–Maddala distributions share the same upper bound for positive skewness but differ slightly for negative skewness; the Dagum lower bound, however, lies below the Singh–Maddala lower bound (given by the Weibull skewness–kurtosis curve). The Dagum skewness–kurtosis space includes the SM space as a proper subset, which helps explain why the Dagum distribution often provides a better fit than the Singh–Maddala distribution. The upper bound for the GG corresponds to the LN curve for positive skewness. The GB2, Dagum/SM, and B2 upper

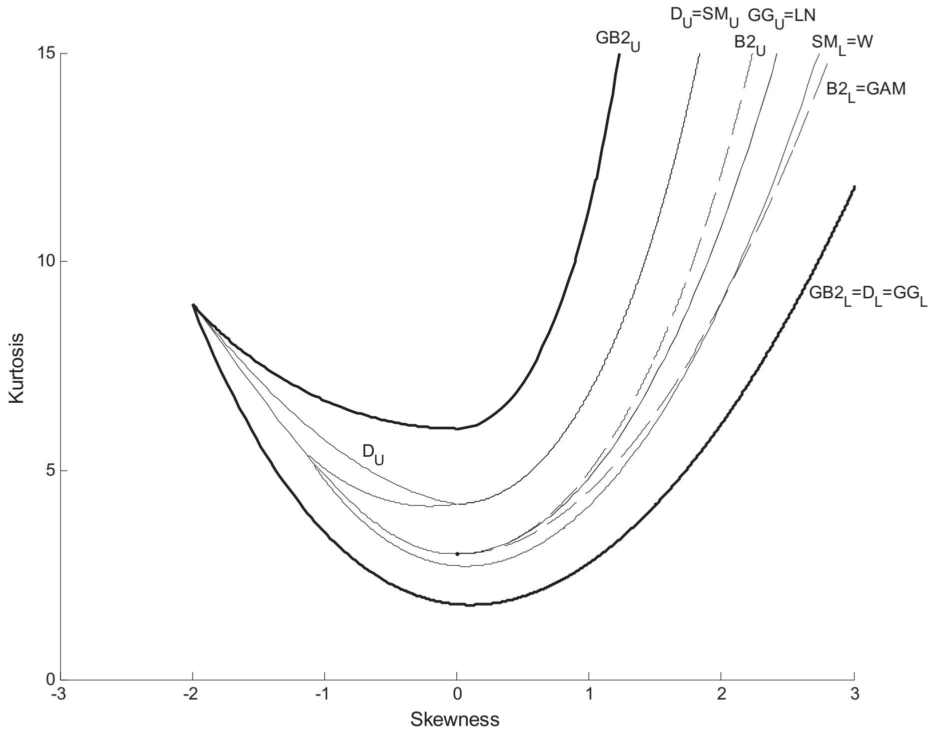

Figure 2. Skewness–Kurtosis Spaces for GB2, Dagum, Singh–Maddala, B2, Generalized Gamma, Lognormal, Weibull, and Gamma

Note: The "L" subscript represents the lower bound and the "U" subscript represents the upper bound for the respective distribution's skewness–kurtosis space.

bounds have vertical asymptotes at skewness values of 2.30, 4.28, and 5.66, respectively. The generalized gamma, Dagum, and GB2 all share the same lower bound, which lies above the all distribution lower bound. Not surprisingly, the GB2 skewness–kurtosis space includes the spaces for all of its limiting and special cases.

Table 1 reports upper and lower bounds (which define feasible skewness–kurtosis combinations) for the B1, B2, GG, Dagum, Singh–Maddala, GB1, and GB2. These bounds were used to construct Figures 1 and 2 and can assist researchers in selecting an appropriate distribution. Some of the distributions, such as the B2 and GG, have bounds that involve the skewness and kurtosis equations for other *pdf*s, such as the power function (PF), inverse gamma (IGAM), and log gamma (LGAM). Other distributions, such as the Dagum and SM, have bounds that are limiting or special cases of their own skewness and kurtosis equations. It is also worth noting that many bounds are segmented into two sections—one for the positive skewness part of the skewness–kurtosis plane and one for the negative skewness part. The Appendix includes skewness and kurtosis equations for all the distributions considered, a summary of the definitions and related properties of the additional *pdf*s mentioned above, and equations for the different skewness–kurtosis bounds.

TABLE 1

Bounds for Skewness–Kurtosis Spaces

pdf	Lower Bound	Upper Bound
B1	$(\gamma_1, \gamma_2)_{AD}$	$(\gamma_1, \gamma_2)_{GAM}$ and its mirror image
B2	$(\gamma_1, \gamma_2)_{GAM}$	$(\gamma_1, \gamma_2)_{IGAM}$
GG	$(\gamma_1, \gamma_2)_{PF}$	Negative skewness: $(\gamma_1, \gamma_2)_{LGAM}$ Positive skewness: $(\gamma_1, \gamma_2)_{LN}$
Dagum	$(\gamma_1, \gamma_2)_{PF}$	Negative skewness: $\lim_{a\to\infty}(\gamma_1, \gamma_2)_{DAGUM}$ Positive skewness: $(\gamma_1, \gamma_2)_{DAGUM}$ with $p = 1$
Singh–Maddala	$(\gamma_1, \gamma_2)_{Weibull}$	Negative skewness: $\lim_{a\to\infty}(\gamma_1, \gamma_2)_{SM}$ Positive skewness: $(\gamma_1, \gamma_2)_{SM}$ with $q = 1$
GB1	$(\gamma_1, \gamma_2)_{AD}$	Negative skewness: Mirror image of $(\gamma_1, \gamma_2)_{GAM}$ to the point (–2,9) and then $(\gamma_1, \gamma_2)_{LGAM}$ from (–2,9) to (0,3) Positive skewness: $(\gamma_1, \gamma_2)_{LN}$
GB2	$(\gamma_1, \gamma_2)_{PF}$	Negative skewness: $\lim_{p\to kq, a\to\infty}(\gamma_1, \gamma_2)_{GB2}$ Positive skewness: See the Appendix

Note: Equations for and details about the various bounds are found in the Appendix.

To illustrate the interpretation of the results in Table 1, consider the B1 distribution. The possible combinations of (γ_1, γ_2) that can be modeled by the B1 are defined by the region bounded below by the all distribution lower bound $(\gamma_2 = \gamma_1^2 + 1)$ and bounded above by the gamma skewness–kurtosis curve and its mirror image ($\gamma_2 = 3 + (3/2)\gamma_1^2$ for γ_1 real).

We now consider an application of these models to actual income data and investigate which *pdf*s are sufficiently flexible to accommodate observed skewness and kurtosis values.

3. Empirical Application: Luxembourg Income Study Data

Household income data were obtained from the Luxembourg Income Study (LIS) database for 13 countries.[1] The income measures we considered were earnings, gross income, and disposable income. Two time periods were used: Wave 5 of the survey (occurring in approximately 2000) and Wave 6 (occurring in approximately 2004). We looked at each country having data for each of the three income definitions for the time periods considered: Australia, Canada, Denmark, Finland, Germany, Israel, Norway, Poland, Sweden, Switzerland, Taiwan, the United Kingdom, and the United States. An advantage of using the LIS dataset is that the data from each country are uniformly formatted, especially with respect to the definition of income, thus facilitating inter-country comparisons. In all cases, income was measured in nominal local currency units. Because of government regulations and privacy laws, data on individual observations cannot be downloaded. Instead, we accessed the LIS microdata through their server to calculate the sample size, mean, variance, skewness, kurtosis, and Gini coefficient for each country, income definition, and year. Another advantage of using LIS data is that the income variables are continuous, not grouped, which makes the calculation of these measures more accurate.

[1]Luxembourg Income Study (LIS) Database, http://www.lisproject.org/techdoc.htm (multiple countries.

TABLE 2

Definitions of LIS Income Measures Used

Earnings (income before taxes and transfer payments)
- Gross cash wage and salary income
- Farm self-employment income
- Non-farm self-employment income

Gross income (income before taxes and after transfer payments)
- Earnings
- Cash property income (includes cash interest, dividends, rents, annuities, royalties, etc.)
- Private occupational and other pensions
- Public sector occupational pensions
- Sickness benefits
- Occupational injury and disease benefits
- Disability benefits
- Maternity and other family leave benefits
- Military/veterans/war benefits
- Other social insurance benefits
- State old-age and survivors benefits
- Child/family benefits
- Unemployment compensation benefits
- Social assistance cash benefits
- Near-cash benefits
- Alimony/child support
- Regular private transfers
- Other cash income

Disposable income (income after taxes and transfer payments)
- Gross income
- Minus:
  - Mandatory contributions for self-employed (includes social security, unemployment, etc.)
  - Mandatory employee contributions
  - Income taxes

Source: http://www.lisproject.org/techdoc/sumincvar.xls.

Table 2 summarizes the definitions of income used in this study. Earnings measures income before taxes and transfer payments. Gross income measures income and transfer payments before taxes are withheld. Disposable income measures income after adjusting for taxes and transfer payments. We followed the recommendation of the LIS group and used the weighted data, which can correct for non-sampling errors and sample bias. For additional details on the weighting procedures, see http://www.lisproject.org/techdoc.htm.

Table 3 reports the sample size and distributional characteristics of interest for households reporting positive income. Not surprisingly, the income data exhibit positive skewness. There is considerable variation in the estimated values for standardized skewness and kurtosis. One questionable observation in the Sweden 2005 data had a value for interest and dividends that was nearly 200 times as large as the next reported value, which greatly affected skewness and kurtosis; hence, the observation was dropped for all our analyses (see notes to Table 3). As measured by the Gini coefficient, transfer payments and taxes result in a more egalitarian distribution in 10 of the 13 countries considered, with only Australia, Taiwan, and the U.K. having similar Gini coefficients for earnings and disposable income. In all cases, taxes applied to gross income resulted in smaller Gini coefficients; however, the decrease in Switzerland and Poland was quite small.

TABLE 3
Moments and Gini Coefficients for LIS Annual Household Income Data

Country	Year	Definition	n	Mean	Std Dev	Skew	Kur	Gini
Australia	2001	Earnings	4,510	59,838	44,358	2.902	20.90	0.360
		Gross income	6,699	51,288	45,621	3.483	27.63	0.417
		Disposable income	6,697	41,786	31,475	2.993	24.74	0.370
	2003	Earnings	6,741	64,936	49,704	4.056	45.23	0.363
		Gross income	10,087	55,916	49,276	4.457	54.28	0.412
		Disposable income	10,086	44,870	33,037	3.209	32.66	0.365
Canada	2000	Earnings	22,298	56,295	57,237	6.555	95.91	0.437
		Gross income	28,936	56,859	56,367	7.688	139.02	0.413
		Disposable income	28,902	43,824	36,684	5.989	90.43	0.373
	2004	Earnings	21,594	61,690	63,207	6.741	117.54	0.451
		Gross income	27,776	64,030	61,225	7.109	132.43	0.412
		Disposable income	27,774	50,777	41,204	4.899	70.06	0.375
Denmark	2000	Earnings	59,549	364,336	267,158	2.130	22.28	0.386
		Gross income	81,916	367,090	284,729	7.934	241.12	0.360
		Disposable income	81,904	243,828	163,848	9.142	389.48	0.322
	2004	Earnings	59,824	402,943	308,210	2.978	38.06	0.392
		Gross income	83,220	407,209	310,706	6.289	144.85	0.358
		Disposable income	83,178	276,847	187,778	7.888	264.98	0.323
Finland	2000	Earnings	8,916	184,190	161,674	29.34	3537.9	0.406
		Gross income	10,420	199,933	238,812	83.84	14375	0.379
		Disposable income	10,415	145,051	145,006	59.21	7455.6	0.338
	2004	Earnings	9,358	35,526	27,538	1.723	11.70	0.410
		Gross income	11,226	39,312	45,717	22.00	845.86	0.389
		Disposable income	11,220	29,286	31,328	24.41	989.47	0.349
Germany	2000	Earnings	8,051	74,823	58,503	3.247	32.45	0.386
		Gross income	10,982	71,752	60,239	4.831	71.64	0.392
		Disposable income	10,982	52,682	39,517	6.886	169.17	0.345
	2004	Earnings	8,297	40,009	31,625	2.942	28.13	0.394
		Gross income	11,290	39,008	55,806	71.045	7580.9	0.393
		Disposable income	11,288	29,239	49,607	98.088	11873	0.346
Israel	2001	Earnings	4,382	149,884	152,131	4.815	67.21	0.457
		Gross income	5,768	145,455	153,356	6.164	95.35	0.447
		Disposable income	5,768	111,356	98,716	8.651	213.7	0.384
	2005	Earnings	4,762	147,007	139,566	3.939	43.15	0.447
		Gross income	6,259	148,009	156,337	7.028	110.60	0.441
		Disposable income	6,255	120,024	116,068	10.493	240.49	0.394
Norway	2000	Earnings	11,474	376,575	295,371	4.747	123.53	0.392
		Gross income	12,888	398,639	402,778	13.433	411.36	0.387
		Disposable income	12,870	300,334	312,958	19.976	787.06	0.355
	2004	Earnings	10,947	409,276	337,594	3.533	58.51	0.417
		Gross income	13,116	474,134	906,537	62.127	5679.68	0.396
		Disposable income	13,112	360,090	849,079	71.487	6964.48	0.370
Poland	1999	Earnings	24,201	19,542	23,610	48.927	4995.8	0.428
		Gross income	31,273	23,141	21,117	51.820	5890.7	0.325
		Disposable income	31,253	20,600	20,024	60.742	7320.1	0.323
	2004	Earnings	23,534	22,952	24,490	8.773	268.61	0.461
		Gross income	32,032	26,284	21,695	9.196	311.27	0.354
		Disposable income	32,027	24,414	20,412	10.281	384.77	0.352
Sweden	2000	Earnings	10,319	291,187	265,613	5.557	90.12	0.425
		Gross income	14,473	316,451	331,621	32.828	2417.71	0.379
		Disposable income	14,470	221,536	211,460	39.730	3285.33	0.347
	2005	Earnings	11,950	334,616	295,442	5.988	174.03	0.420
		Gross income	16,254	373,877	291,425	6.471	167.45	0.355
		Disposable income	16,251	269,439	181,934	4.749	87.15	0.327

TABLE 3 (*continued*)

Country	Year	Definition	n	Mean	Std Dev	Skew	Kur	Gini
Switzerland	2000	Earnings	3,015	93,973	70,423	11.226	363.34	0.328
		Gross income	3,641	96,491	76,939	11.326	285.40	0.320
		Disposable income	3,627	73,143	60,555	13.535	382.77	0.318
	2004	Earnings	2,596	97,945	60,182	1.777	11.40	0.320
		Gross income	3,267	98,638	61,917	4.091	57.03	0.305
		Disposable income	3,245	73,580	44,750	3.169	32.17	0.301
Taiwan	2000	Earnings	12,301	844,894	571,044	2.634	23.18	0.340
		Gross income	13,801	934,142	648,958	2.658	20.27	0.345
		Disposable income	13,800	893,449	609,208	2.562	19.60	0.341
	2005	Earnings	11,903	832,582	576,645	1.981	12.99	0.357
		Gross income	13,681	915,325	681,198	2.863	21.63	0.364
		Disposable income	13,679	872,331	640,298	2.878	22.41	0.359
U.K.	1999	Earnings	15,199	28,716	29,158	11.061	276.97	0.401
		Gross income	24,944	24,589	26,516	11.428	327.73	0.426
		Disposable income	24,830	19,596	20,515	14.157	471.86	0.398
	2004	Earnings	17,025	35,719	38,471	11.299	258.90	0.408
		Gross income	27,684	31,028	34,088	11.496	294.32	0.421
		Disposable income	27,574	24,800	27,417	12.965	332.26	0.398
U.S.	2000	Earnings	39,621	58,669	58,185	3.341	19.35	0.442
		Gross income	49,304	57,698	58,508	3.438	20.84	0.450
		Disposable income	49,294	44,785	39,883	3.511	23.91	0.404
	2004	Earnings	62,366	63,012	65,661	4.066	29.05	0.448
		Gross income	75,746	61,877	64,727	4.145	31.99	0.453
		Disposable income	75,736	49,534	46,568	4.371	38.97	0.414

Notes: All values are expressed in units of national currency in use at time of data collection. One questionable observation in the Sweden 2005 data has reported interest and dividends (which is included in gross and disposable income but not in earnings) of 1,526,993,544, whereas the next largest value is 8,448,972 and the mean value of interest and dividends, after dropping the outlier, is 8,794. Thus, the observation has been dropped. Keeping the observation results in a mean, standard deviation, skewness, kurtosis, and Gini coefficient for gross income of 374,602, 1,094,773, 1,305, 1,831,523, and 0.357, respectively. It results in a mean, standard deviation, skewness, kurtosis, and Gini coefficient for disposable income of 269,945, 758,053, 1,333, 1,883,894, and 0.328, respectively.

TABLE 4

Percentage of 78 Data Points Included in Each Skewness–Kurtosis Space

pdf	% of Data Included in Skewness–Kurtosis Space
GB2	100.00
Dagum	98.72
B2	84.62
SM	65.38
GB1	57.69
GG	57.69
B1	3.85

Table 4 reports the percentage of the 78 cases (13 countries, 3 income definitions, 2 time periods) included in each of the skewness–kurtosis spaces considered. While maximum likelihood estimators would not match sample and theoretical moments (as would method of moments estimators), these results shed light on the

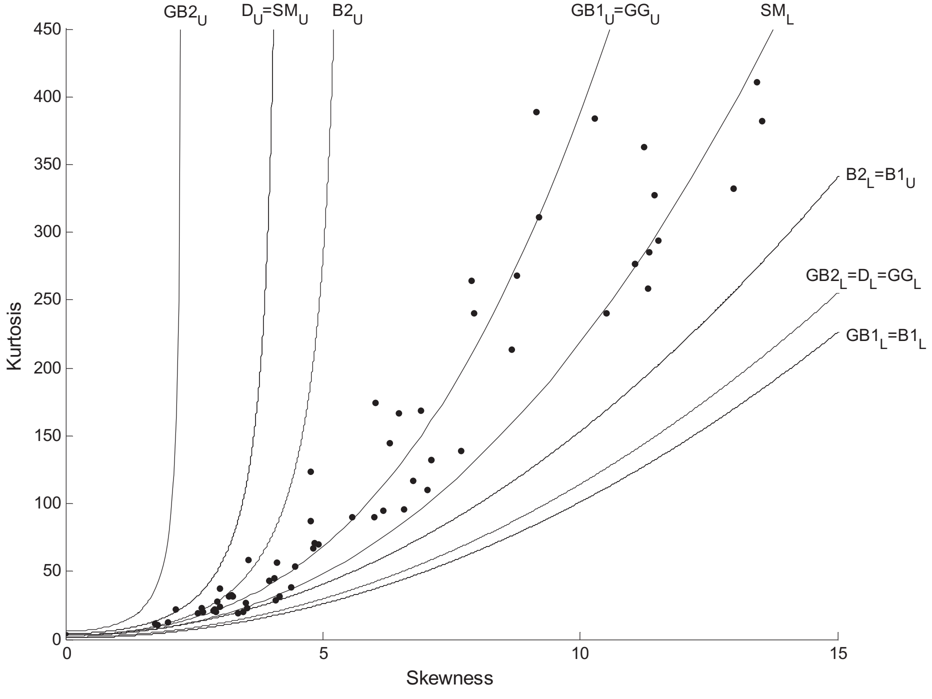

Figure 3. Skewness–Kurtosis Data Points and Skewness–Kurtosis Spaces

relative ability of the different *pdf*s to accommodate the observed distributional characteristics in the data considered. Only 3 of the 78 (3.85 percent) cases fall in the B1 skewness–kurtosis space, whereas the B2 space includes 66 of the 78 (84.62 percent) cases considered. The Dagum clearly outperforms the Singh–Maddala distribution, accounting for all but one of the observations. Although the GB1 lower bound lies below the GG lower bound, the two distributions perform equally well (none of the data points fall in the GB1's extended region).

Figure 3 illustrates 62 of the 78 observed skewness and kurtosis combinations along with the skewness–kurtosis spaces considered. The scale of the figure was selected to facilitate distinguishing the different bounds, with 16 of the larger skewness–kurtosis data points being omitted.

4. Summary and Conclusions

The ability of some popular income distributions to model distributional characteristics was investigated. The GB1 and GB2 skewness–kurtosis spaces were evaluated, and prior results on the spaces for the Pareto, lognormal, gamma, Weibull, generalized gamma, Dagum, Singh–Maddala, and beta distributions were given, expanded on, and compared. Of the models considered, the GB2 allowed for the highest kurtosis values for positive skewness, which appears to be important in modeling the distribution of income. The skewness–kurtosis values observed for 13 countries, three definitions of income, and two time periods were able to be modeled by the GB2. Of the three-parameter models, the Dagum performed the best and nearly as well as the more general GB2.

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Supporting Information

Additional Supporting Information may be found in the online version of this article:

Appendix: Skewness–Kurtosis Spaces for Select Distributions
Table A.1: λ_h's Used to Calculate Skewness and Kurtosis for Different *pdf*s