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HOW TO MEASURE LIVING STANDARDS AND PRODUCTIVITY

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I set out a general algorithm for calculating true cost-of-living indices when demand is not homothetic and when the number of products may be large. The non-homothetic case is the important one empirically (Engel's Law). The algorithm can be applied in both time series and cross section. It can also be used to estimate true producer price indices and Total Factor Productivity in the presence of input-biased economies of scale and technical change. The basic idea is to calculate a chain index of prices but with actual budget (cost) shares replaced by compensated shares, i.e. what the shares would have been if consumers (firms) faced actual prices but their utility (output) were held constant at some reference level. The compensated shares can be derived econometrically from the same data as are required for the construction of conventional index numbers. The algorithm is illustrated by applying it to estimating true PPPs for 141 countries and 100 products within household consumption, using data from the World Bank's latest International Comparison Program.

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1. Introduction

This paper sets out an algorithm for measuring the true cost of living in the important case where demand is non-homothetic. The algorithm can be applied in both time series and cross section, e.g. cross-country studies of living standards. Essentially the same algorithm can be applied to the parallel problem of measuring the price of producers' inputs, which in turn is a step on the road to measuring technical change. The algorithm is practical since it requires no more data than is needed to calculate conventional index numbers. And in principle it can be implemented at the same level of product detail at which conventional index numbers are constructed by national statistical agencies.

Economic theory tells us how to measure the true cost of living: estimate the expenditure function econometrically and then calculate the Konüs price index. The Konüs price index for period t relative to some other period r is defined as the

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ratio of the (minimum) cost of achieving a given utility level at the prices of period t to the cost of achieving the same utility level at the prices of period r (Konüs, 1939); the utility level can be that of t, r, or any other period. If we know the expenditure function then we can calculate the Konüs price index, for any chosen utility level. Similarly, economic theory tells us how to measure the true index of the cost of a producer's inputs: estimate the producer's cost function and calculate the analogue of the Konüs price index. If we know the cost function, then we also know the degree of economies of scale, the size of any input biases in economies of scale, the growth rate of technical change, and the size of any input biases in technical change.

However, though much work has been done on estimating systems of consumer demand or producers' cost functions, the results of these studies are not typically employed by other economists in empirical work. For example, when macro economists study inflation empirically, they do not usually employ their micro colleagues' estimates of expenditure functions. Rather they use consumer price indices constructed by national statistical agencies. The reason is clear. The economic approach cannot be applied at a level useful for other empirical economists because of data limitations.

1.1. *The Data Problem*

The economic approach cannot be employed because the number of parameters to be estimated is large and the number of observations is comparatively small. In other words the problem is a purely practical one which might in theory be solved just by waiting long enough (possibly for hundreds of years). This causes a dilemma for the empirical economist who is unwilling to wait. Either the economic approach must be abandoned and index numbers employed instead, or the data must be aggregated and the economic approach applied at a higher level. The first way, I shall argue later, is perfectly all right if demand (for consumer goods or producer inputs) is homothetic. But if it is not, then index numbers will not measure what they are supposed to measure. The second approach is more relevant to testing economic theory rather than using it. In practice, empirical economists tend to use the index numbers (for output, inputs, and prices) supplied to them by statistical agencies, without asking too many questions about the assumptions on which they are based.[1]

The data problem can be illustrated by taking the Quadratic Almost Ideal Demand System (QAIDS) for N products of Banks *et al.* (1997) as an example. In the expenditure function of this system there are $\frac{1}{2}(N-1)(N+2)$ independent parameters relating to the consumer's response to prices, and $2(N-1)$ independent parameters relating to the consumer's response to income, for a total (excluding a scale parameter) of $\frac{1}{2}(N-1)(N+6)$ independent parameters. The QAIDS is a

[1]See, for example, the remarks of Tobin (1987) on the contributions of Irving Fisher to index number theory: "These index number issues do not seem as important to present-day economists as they did to Fisher. Knowing that they are intrinsically unsolvable, we finesse them and use uncritically the indexes that government statisticians provide." Of course, I do not agree that these "index number issues" are "intrinsically unsolvable," otherwise I would not have written this paper.

system of $N - 1$ independent equations for the expenditure shares. Roughly speaking, each of these equations contains on average $\frac{1}{2}(N+2)$ independent coefficients relating to prices and two coefficients relating to income. To have any chance of estimating these coefficients econometrically, we must have more observations than coefficients; i.e., if we have T aggregate time series observations, then we require $T > \frac{1}{2}(N+6)$.

This is where the empirical study of demand and the practice of index number construction part company. National statistical agencies construct their indices of the cost of living from hundreds of components. For example, the U.S. Bureau of Labor Statistics constructs its Consumer Price Index from 305 "entry-level items" (U.S. Bureau of Labor Statistics, 2007). The U.K.'s Consumer Prices Index and Retail Prices Index have some 650 "items" (Office for National Statistics, 1998, 2006). To estimate the parameters of the QAIDS for 650 products would require over three centuries of annual data, a requirement that is not and is never likely to be met. So when econometricians use time series data to test the theory of demand, they are forced to aggregate the products into a small number of groups; for example, Christensen *et al*. (1975) tested the theory of demand using three product groups over 1929–72. But additional, strong assumptions on separability are needed to justify this aggregation (Deaton and Muellbauer, 1980b, ch. 5; Blackorby *et al*., 2008); to test these assumptions would run into the same problem of insufficient data as just outlined and in practice this is never done. So the "prices" and "quantities" which are the basic data for testing the theory of demand in this kind of study are themselves index numbers.[2] But then the theoretical justification for these index numbers is unclear. Cross section studies of household demand fare better since in any given year it may be reasonable to assume prices are the same for all households (except for regional effects). With typically several thousand observations in any cross section, lack of observations is not such a problem. But then only the effects of income (and of household composition) on demand can be measured, as in, for example, Blow *et al*. (2004).[3]

The upshot is that all the empirical work that economists have done on household demand has had no effect on the measurements actually made by national statistical agencies (although the underlying theory may have been influential). Similar remarks apply to the measurement of other indices, such as the producer price index.

1.2. *Non-Homotheticity*

Actually, none of this matters much *provided that demand (for consumer goods or inputs) is homothetic*. If this condition holds and if we are prepared to accept

[2]Latent separability (Blundell and Robin, 2000) imposes fewer restrictions than weak separability. But it is still necessary to estimate a complete demand system in order to determine which goods belong in which groups.

[3]Cross section studies also often employ highly aggregated data: five product groups in the case of Banks *et al*. (1997), eight in the case of Blundell *et al*. (2007) [they present estimates for 7 groups; the 8th omitted, unnamed group accounts for 30% of expenditure (see their Table V)], both studies of British household budgets, and 11 in the case of Neary (2004), a cross-country study of 1980 PPPs. The panel study on Canadian households of Lewbel and Pendakur (2009) employed nine groups.

that economic theory is true,[4] then we have no need to estimate cost or expenditure functions. We can instead estimate a discrete approximation to a Divisia index (which I show is the ideal measure in this case), using the superlative index numbers of Diewert (1976) with their flexibility improved by chaining.

Unfortunately, an overwhelming body of empirical evidence establishes that consumer demand is *not* homothetic. The most obvious manifestation of this is Engel's Law: the proportion of total household expenditure devoted to food falls as expenditure rises. Since its original publication in 1857, Engel's Law has been repeatedly confirmed. Houthakker (1957) showed that the Law held in some 40 household surveys from about 30 countries.[5] Engel's Law also holds in the much more econometrically sophisticated study of Banks *et al*. (1997) on U.K. household budgets. The prevalence of non-homotheticity is also confirmed by the more disaggregated studies of Blow *et al*. (2004), also on U.K. household budgets, which considered 18 product groups; Oulton (2008), who considered 70 product groups; and Oulton (2012b), 100 product groups.[6]

If demand is not homothetic, then superlative index numbers are not guaranteed to be good approximations to Konüs price indices, even locally. In fact the true price index may lie outside the Laspeyres–Paasche spread. And the true price index is no longer unique but depends on the reference level chosen for utility (or, for the producer price index, on the reference output level). The fact that the Konüs price index generally varies with the reference utility level is sometimes taken as puzzlingly paradoxical. But it can be given a simple intuitive justification. Consider a household with a very low standard of living spending 60 percent of its budget on food (as was the case with the working class households studied by Engel in 1857). Suppose the price of food rises by 20 percent, with other prices constant. Then money income will probably have to rise by close to 12 percent ($0.60 \times 20\%$), to leave utility unchanged, since there are limited possibilities for substituting clothing and shelter for food. Compare this household to a modern day British one, spending 15 percent of its budget on food prepared and served at home (Blow *et al*., 2004). Now the maximum rise in income required to hold utility constant is only 3 percent ($0.15 \times 20\%$), and probably a good bit less as substitution opportunities are greater.

[4]Throughout this paper I adopt the economic approach to index numbers; see Diewert (1981, 2008) for surveys of this and of the alternative axiomatic and stochastic approaches, and Balk (2008) on the axiomatic approach.

[5]Engel's (1857) results for expenditure by households of various income levels in Saxony are described more accessibly in Marshall (1920, ch. IV); see Chai and Moneta (2010) for a modern account of Engel's work. In each of the surveys that he collected, Houthakker (1957) estimated the elasticity of expenditure on food and three other product groups (clothing, housing, and miscellaneous) with respect to total expenditure and to household size. The results for food were clear-cut: demand was inelastic with respect to expenditure in every survey. The results for clothing and miscellaneous were equally clear-cut: demand was expenditure-elastic. The result for housing was more mixed.

[6]An exception to this consensus is Dowrick and Quiggin (1997). They studied the 1980 and 1990 PPPs for 17 OECD countries, using 38 components of GDP, and argued that the data could be rationalized by a homothetic utility function. But their anomalous finding may be due partly to the fact that the per capita incomes of these countries were fairly similar, partly to the fact that some of the 38 components were not household spending, and partly to the low power of their non-parametric test (Neary, 2004). By contrast, Crawford and Neary (2008) found that the cross-country data in Neary (2004)—11 commodity groups in 60 countries from the World Bank's 1980 ICP—are rationalizable by a single non-homothetic utility function, but not by any homothetic utility function.

This leaves the welfare interpretation of conventional consumer price indices and their cross-country cousins, the Purchasing Power Parities (PPPs) constructed by the OECD and the World Bank, somewhat up in the air. If the true price index depends on the reference level of utility, how are we to interpret real world price indices? The answer in the time series context is that a chained, superlative index is likely to be approximately equal to a true price index with reference utility level at the midpoint of the sample period (Diewert, 1976, 1981; Feenstra and Reinsdorf, 2000; Balk, 2010).[7] For a cross-country comparison, the viewpoint will be that of a "middle" country. While there is nothing wrong with this viewpoint, there is no special reason why the midpoint should be so privileged. There is also the disadvantage that when the sample period is extended (or the number of countries in the comparison increased), the viewpoint changes.[8]

A parallel issue arises on the production side and takes the form of input biases in economies of scale: if output is doubled, holding prices and technology constant, does that leave all cost shares unchanged? The possibility that this is *not* the case has certainly been entertained as a matter of theory, though I am not aware of any substantial body of empirical work devoted to this issue. But such a situation may be quite common. Consider a firm which has fixed and variable costs, where the fixed costs are white collar workers and the variable costs are blue collar workers. Then an expansion of output will lower the share of white collar workers in total costs. In this case the cost function is non-homothetic and also non-homogeneous in output. So it would certainly seem desirable to take non-homotheticity into account when trying to measure Total Factor Productivity.

1.3. *The Algorithm*

The proposed algorithm can be summarized as follows. The growth rate of a Konüs consumer price index resembles that of a Divisia index (or the latter's empirical counterpart, a chain index) in that it is an expenditure-share-weighted average of the growth rates of the component prices. But for the Konüs index the shares are not the actual, observed ones, but rather what I call the compensated shares: the shares that would be observed if prices were the actual, observed ones but utility were held constant at some given reference level. I derive a relationship between the compensated and the actual shares: the compensated shares are equal to the actual ones, adjusted for the difference in real income (utility) between the actual situation and the reference level. The adjustment requires us to know, for

[7]Suppose a utility function exists which rationalizes the data but may be non-homothetic. Diewert (1981) showed that there exists a utility level which is intermediate between the levels at the endpoints of the interval under study such that a Konüs price index over this interval, with utility fixed at the intermediate level, is bounded below by the Paasche and above by the Laspeyres. Balk (2010) showed that when the growth of prices is piecewise log linear a chained Fisher price index approximates a Konüs price index over an interval when the reference utility level is fixed at that of some intermediate point in the interval. More precise results are available for specific functional forms. Diewert (1976) showed that a Törnqvist price index is exact for a non-homothetic translog cost function when the reference utility level is the geometric mean of the utility levels at the endpoints; see also Diewert (2009) for extensions. For the AIDS, Feenstra and Reinsdorf (2000) showed that, if prices are growing at constant rates, the Divisia index between two time periods equals the Konüs price index when the reference utility level is a weighted average of utility levels along the path.

[8]Though adding more time periods (countries) may also change the estimates in the econometric approach advocated here.

each product, the consumer's response to real income changes but not the response to price changes. This is why the algorithm can be implemented at a very disaggregated level, since the number of parameters needed to describe the consumer's response to income changes is quite small: in the case of the QAIDS only two parameters for each product need to be known. These income response parameters can be estimated econometrically, provided we do not try at the same time to estimate the responses to individual price changes. This can be done by estimating a flexible demand system such as the QAIDS but with the price variables replaced by a much smaller number of principal components. In this way the data limitation problem can be overcome.

It is important to note that the algorithm proposed here is not designed as a test of whether the theory of consumer (or producer) demand is true. Rather it seeks to *use* demand theory to construct better measures of living standards and productivity. In fact, the algorithm *assumes* that demand theory is true and hence that the consumer's or producer's responses can be approximated by a flexible system like the QAIDS.

1.4. *Plan of the Paper*

I start in Section 2 with the homothetic case. I show that a Divisia index provides an ideal measure and that this can be well approximated by a chained, superlative index number. In Section 3 I go on to consider the non-homothetic case and present a general algorithm for estimating a true (Konüs) price index for a representative consumer. The algorithm requires just the same data (and no more) as would be required to estimate a conventional index number. This algorithm is illustrated more specifically for the QAIDS. I argue that it can be applied both to time series and to cross section (e.g. cross country studies). In Section 4 the analysis is extended by dropping the assumption of a representative consumer. I show how the QAIDS can be adapted to allow for inequality in the distribution of income. It turns out that this just requires adding two additional variables, both statistics of the income distribution, to the share equations of the QAIDS. The algorithm derived for the simpler case of a representative consumer can then be applied much as before. This section also discusses including household characteristics as additional determinants of demand. Section 5 shows how the general method applies, after some adaptation, to the estimation of a true input price index for producers, in the case where (dis)economies of scale may exist and may be input-biased. The algorithm enables input biases in economies of scale and in technical change to be estimated simultaneously. Section 6 then illustrates the method by applying it to the problem of estimating true PPPs for 141 countries and 100 products, using the underlying data from the World Bank's most recently published International Comparison Program (ICP). Finally, Section 7 concludes.

2. Price Indices: The Homothetic Case

In this section I argue that chained, superlative index numbers have solved the problem of measuring the true cost of living for a single, representative consumer in the case where demand is homothetic.

Let the consumer's expenditure function be

$$x = E(\mathbf{p}, u), \quad \partial x/\partial u > 0.$$

This shows the minimum expenditure x needed to reach utility level u when $\mathbf{p} = (p_1 p_2 \ldots p_N)$ is the $N \times 1$ price vector faced by the consumer; $x = \sum_i p_i q_i$ where the q_i are the quantities purchased. Expenditure at time t is therefore a function of prices at time t and the utility level. The expenditure function is assumed to possess derivatives of all orders. Suppose that, hypothetically, utility were held at its level at time b while the consumer faced the prices of time t. Let $x(t, b)$ denote the minimum expenditure at the prices of time t required to achieve the utility level of time b. Then

$$x(t, b) = E(\mathbf{p}(t), u(b)). \tag{1}$$

For brevity write the right-hand side as

$$E(t, b) = E(\mathbf{p}(t), u(b))$$

where the first argument of $E(t, b)$ is the time period for prices and the second is the time period for utility. The Konüs price index at time t relative to time r, with time b as the base period for utility, is defined as the ratio of the minimum expenditure required with the prices of time t to attain the utility level of time b, to the minimum expenditure required to attain this same utility level, when the consumer faces the prices of time r:

$$P^K(t, r, b) = E(t, b)/E(r, b). \tag{2}$$

In other words, period r is the *reference* period and period b is the *base* period. (Clearly, $P^K(r, r, b) = 1$.) The base period b might be the same as the reference period ($b = r$), or the same as the current period ($b = t$), or it might be some other period. In general, the Konüs price index depends on both the prices and the specified utility level. However, as is well known, the index is independent of the utility level and depends only on the prices if and only if demand is homothetic, i.e. if all income elasticities are equal to one (Konüs, 1939; Samuelson and Swamy, 1974; Deaton and Muellbauer, ch. 7, 1980b).

Let s_i denote the share of product i in total expenditure. Applying Shephard's Lemma to the expenditure function, we obtain the *share functions*:

$$s_i = \frac{\partial \ln E(\cdot,\cdot)}{\partial \ln p_i}, \quad i = 1, \ldots, N. \tag{3}$$

The expenditure shares clearly depend on both prices and utility. Let the share of product i in total expenditure at time t, if utility were fixed at the level of period v, be $s_i(t, v)$. Evaluating this function with the prices of time t and the utility level of time b we have

$$s_i(t,b) = \frac{\partial \ln E(t,b)}{\partial \ln p_i(t)}, \quad i = 1, \ldots, N.$$

These can be called the hypothetical or *compensated* (Hicksian) shares, the shares that would be observed if utility were held constant at some base level (here, the level prevailing in period b), while prices followed their observed path. The actual, observed shares in period t are

$$s_i(t,t) = \frac{\partial \ln E(t,t)}{\partial \ln p_i(t)}, \quad i = 1, \ldots, N.$$

Note that the compensated shares in the base period b, $s_i(b, b)$, are the same as the actual shares in that period.

By totally differentiating the Konüs price index of equation (2) with respect to time, we obtain

$$(4) \qquad \frac{d \ln P^K(t,r,b)}{dt} = \sum_{i=1}^{i=N} \frac{\partial \ln E(t,b)}{\partial \ln p_i(t)} \frac{d \ln p_i(t)}{dt} = \sum_{i=1}^{i=N} s_i(t,b) \frac{d \ln p_i(t)}{dt}.$$

So the level of the Konüs price index in some period T, relative to its level in the reference period r, is found by integration:

$$(5) \qquad \ln P^K(T,r,b) = \int_r^T \left[\sum_{i=1}^{i=N} s_i(t,b) \left(\frac{d \ln p_i(t)}{dt} \right) \right] dt, \quad P^K(r,r,b) = 1.$$

The Konüs price index resembles a Divisia index (P^D) which is defined as:

$$(6) \qquad \ln P^D(T,r) = \int_r^T \left[\sum_{i=1}^{i=N} s_i(t,t) \left(\frac{d \ln p_i(t)}{dt} \right) \right] dt, \quad P^D(r,r) = 1.$$

The only difference between them is that the Konüs index employs the compensated, not the actual, shares as weights (Balk, 2005; Oulton, 2008).[9] However, in the homothetic case the compensated and the actual shares are always the same: $s_i(t, b) = s_i(t, t)$, $\forall i,b$, since shares depend only on prices, not on utility (or real income); that is, the Konüs and Divisia indices are identical. So in this case the task of index number theory is to find the best discrete approximation to the continuous Divisia index of equation (6).

In fact in the homothetic case the problem of estimating true cost-of-living indices and indices of the standard of living, together with their counterparts on the production side, has been solved, at least within the limit of what is empirically possible. The solution was in fact provided by Diewert's superlative index numbers, index numbers which are exact for some flexible functional form

[9]Since it is a line integral, the Divisia index is in general path-dependent unless demand is homothetic, as its inventor Divisia (1925–26) was well aware; see Hulten (1973) for detailed discussion and Apostol (1957, ch. 10), for the underlying mathematics. But the Konüs price index, the right-hand side of equation (5), is not path-dependent since by definition utility is being held constant along the path (Oulton, 2008).

(Diewert, 1976). In the homothetic case, the true index is bounded by the Laspeyres and Paasche indices (Konüs, 1939). But superlative index numbers are only guaranteed to be good approximations locally, so they need to be chained together in order to approximate better the continuously changing weights in the Divisia index (6).[10,11]

Unfortunately, the assumption of homotheticity is a very dubious one for consumer demand. As argued earlier, there is overwhelming evidence from household surveys that income elasticities are not all equal to one. Economists have been somewhat readier to accept the assumption of constant returns to scale in the case of producers, but even so this assumption should ideally be tested. The next section therefore turns to the non-homothetic case.

3. Estimating a True Cost-of-Living Index Over Time: The Non-Homothetic Case

3.1. *The Taylor Series Approach*

In this section I consider the problem of how to estimate a true cost-of-living index over time when demand is non-homothetic and there are insufficient time series observations available to estimate the consumer's expenditure function.[12] This might be called the "large N, small T" problem: there are a large number of products but only a small number of time periods. This is the typical situation faced by national statistics agencies when, for example, estimating the consumer price index. Throughout this section I assume a single, representative consumer. In the next section this assumption will be relaxed.

Equation (5) shows that in order to calculate the Konüs price index in practice, we need to know the compensated shares, which differ in general from the actual ones in the non-homothetic case. We seek a way of at least approximating the compensated shares, which cannot of course be directly observed (except for the $s_i(b, b)$ which are both the actual and the compensated shares in period b). We can do this by expressing the actual shares $s_i(t, t)$ in terms of a Taylor series expansion of the compensated shares $s_i(t, b)$ in equation (3) around the point $\ln x = \ln E(t, b)$, i.e. holding prices constant at their levels at time t and varying expenditure (utility). When this is done we can establish the following proposition:

Proposition 1. The differences between the compensated and the actual shares depend on: (a) the difference in real expenditure between the base period and the

[10]Diewert (1976) was well aware of the need for chaining: see his footnote 16. For more on superlative indices, including discussion of the critique of them by Hill (2006), see section A.1 of the appendix to Oulton (2012a).

[11]Using an axiomatic approach, van Veelen (2002) has proved an impossibility theorem which purports to rule out an economically acceptable solution to the problem of measuring the standard of living, both internationally and intertemporally. However, his 4th and final axiom, "Independence of irrelevant countries" (or irrelevant time periods), would rule out the use of chain indices. On the economic approach the latter are essential to derive good approximations to Divisia indices.

[12]The argument of this section is a generalization of the one set out in Oulton (2008).

current period; and (b) the consumer's response to real expenditure changes. The differences do *not* depend on the consumer's response to price changes. More precisely,

$$(7)\quad s_i(t,b) = s_i(t,t) - \eta_{i1}(t,b)\ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b,b)}\right] - \frac{\eta_{i2}(t,b)}{2!}\left\{\ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b,b)}\right]\right\}^2 - \frac{\eta_{i3}(t,b)}{3!}\left\{\ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b,b)}\right]\right\}^3 - \ldots,\quad i = 1,\ldots,N;\ t \in [0,T]$$

where

$$(8)\quad \eta_{ik}(t,b) = \left(\frac{\partial^k s_i(\cdot,\cdot)}{\partial \ln E(\cdot,\cdot)^k}\right)_{\substack{\mathbf{p}=\mathbf{p}(t),\\ x=E(t,b)}},\quad k = 1,2,\ldots;\ i = 1,\ldots,N.$$

Proof. Take a Taylor series expansion of the share function $s_i(t, t)$ with respect to its second argument around the point $s_i(t, b)$: see Section A.1 of the Appendix.

The partial derivative $\eta_{i1}(t, b)$ is the semi-elasticity of the budget share of the i-th product with respect to expenditure, with prices held constant; it is evaluated at base year utility and at the prices of time t. It measures the consumer's response to expenditure changes, as asserted in Proposition 1. These semi-elasticities and the higher order derivatives in (7) measure basic aspects of consumer behavior. The terms in square brackets measure the proportionate difference between real expenditure at time t and at time b. Note that if the expenditure function is a K-th order polynomial in log expenditure, then the Taylor series effectively terminates after K terms, since $\eta_{i,K+1} = \eta_{i,K+2} = \ldots = 0$. So equation (7) with terms higher than powers of K in log expenditure omitted is then exact and not an approximation.

The system of equations (7) might not appear to take us very much further if our goal is to estimate the Konüs price index, since the latter appears on the right-hand side. But in fact this system, together with (4), is the basis for a practical method. Suppose that the $\eta_{i1}(t, b)$ and the higher order derivatives $\eta_{i2}(t, b)$, $\eta_{i3}(t, b)$, etc, that are required for a good approximation, were somehow known or could be estimated (see the next section on ways to do this). Then we could estimate the Konüs price index using equations (4) and (7). This is because these equations constitute a set of equations for $P^K(t, b, b)$ and hence for $P^K(t, r, b)$,[13] in which the compensated shares and the Konüs price index are the only unknowns; the actual shares $s_i(t, t)$, the nominal expenditures $x(t, t)$ and $x(b, b)$, and (by assumption) the semi-elasticities ($\eta_{i1}(t, b)$, $\eta_{i2}(t, b)$, etc) are all known.

The general procedure for solving these equations is straightforward in principle. First, we need to take discrete approximations. Equations (7) must be understood to hold in discrete not continuous time, i.e. for $t = 0,1,\ldots,T$. We must also decide how many terms in the Taylor series are required. If the utility function

[13]From the definition of the Konüs in equation (2), $P^K(t, r, b) = P^K(t, b, b)/P^K(r, b, b)$.

is quadratic in log expenditure, then only the first two terms of the Taylor series are needed; see the next section. Equation (4) must be replaced by a discrete approximation, e.g. a chained Törnqvist or chained Fisher formula.

Let us define the following chained, *compensated* index numbers. Each index number is for period t relative to period r, with utility held constant at the level of period b.

Compensated Törnqvist:

$$\ln P^{CT}(t,r,b) = \sum_{i=1}^{i=N} \left(\frac{s_i(t,b) + s_i(r,b)}{2} \right) \ln\left(\frac{p_i(t)}{p_i(r)} \right). \tag{9}$$

Compensated Laspeyres:

$$P^{CL}(t,r,b) = \sum_{i=1}^{i=N} s_i(r,b) \frac{p_i(t)}{p_i(r)}. \tag{10}$$

Compensated Paasche:[14]

$$P^{CP}(t,r,b) = \left[\sum_{i=1}^{i=N} s_i(t,b) \frac{p_i(r)}{p_i(t)} \right]^{-1}. \tag{11}$$

Compensated Fisher:

$$P^{CF}(t,r,b) = [P^{CL}(t,r,b) \cdot P^{CP}(t,r,b)]^{1/2}. \tag{12}$$

Each of these index numbers is defined in the same way as its empirical counterpart, except that compensated, not actual, shares are used. If $r = t - 1$ these compensated indices are the links in the corresponding chained index. The natural choices for discrete approximations to the continuous Konüs price index are either the compensated Törnqvist, equation (9), or the compensated Fisher, equation (12). We now have:

Proposition 2. The true index is bounded by the compensated Laspeyres and the compensated Paasche. This is the case when we are looking at links in a chain index, i.e. when we are comparing two adjacent years (or countries):

$$P^{CL}(t,t-1,b) \geq P^{K}(t,t-1,b) \geq P^{CP}(t,t-1,b). \tag{13}$$

It is also true when we are looking at a bilateral (two-period or two-country) index, comparing year (country) t with reference year (country) r, with year (country) b as the base:

$$P^{CL}(t,r,b) \geq P^{K}(t,r,b) \geq P^{CP}(t,r,b). \tag{14}$$

[14]The formula for the Paasche is not the usual one but is mathematically equivalent to the usual one.

Proof. Since utility is being held constant at its level in period b, the proof of Proposition 2 follows similar lines to that of the well-known Konüs (1939) inequalities: see section A.1 of the Appendix for the details.

We also need to take account of the Konüs (1939) inequalities relating *actual* Laspeyres and Paasche price indices to Konüs indices. Denote the *actual* Laspeyres and Paasche price indices for year (country) t relative to year (country) r by $P^L(t, r)$ and $P^P(t, r)$, respectively. (So the Laspeyres index uses the weights of year (country) r and the Paasche uses the weights of year (country) t.) Then the Konüs (1939) inequalities state that

$$P^L(t,r) \geq P^K(t,r,r) \text{ and } P^P(t,r) \leq P^K(t,r,t). \tag{15}$$

A Konüs index is only guaranteed to lie within the *actual* Laspeyres–Paasche spread if demand is homothetic so that $P^K(t, r, r) = P^K(t, r, t)$.

The Laspeyres–Paasche spreads, calculated using either compensated or actual shares, can be used as a check on the accuracy of whatever index number formula is adopted.[15]

Equations (7) now constitute a system of $(N - 1)(T + 1)$ independent equations since the N shares sum to one in each period.[16] Together with (4), this system can be solved iteratively:[17]

1. Start with an initial guess at $P^K(t, b, b)$: this could be derived as a chained Törnqvist or chained Fisher index which uses actual not compensated shares.
2. Substitute this estimate of $P^K(t, b, b)$ into (7) to get estimates of the compensated shares for each of $N - 1$ products and for each of $T + 1$ time periods; the share of the N-th product can be derived as a residual.
3. Use these estimates of the compensated shares to obtain a new estimate of $P^K(t, b, b)$ from either of the two discrete approximations to (4), the Törnqvist (equation (9)) or the Fisher (equation (12)).
4. Check whether the estimate of $P^K(t, b, b)$ has converged. If not, return to step 2.

The intuition behind this result is as follows. In the homothetic case it turns out that we do not need to know the individual parameters of the expenditure function: the observed shares encapsulate all the required information. In the non-homothetic case, we need to know the compensated shares. These can be thought of as like the actual shares, but contaminated by the effects of changes in real income (expenditure). What is needed is to purge the actual shares of income effects.[18]

[15]Of all superlative index numbers, only the Fisher is guaranteed to lie within the Laspeyres–Paasche spread (Hill, 2006), assuming all use compensated or all use actual shares, and all are chained or all are bilateral. But a *chained* Fisher is not guaranteed to lie within a *bilateral* Laspeyres–Paasche spread.

[16]The actual shares of course sum to one and since they derive from the expenditure function, so do the compensated shares: see equation (3).

[17]If the Engel curves are log-linear, i.e. all the η_{ik} are zero except the η_{i1}, then the whole system is linear and an explicit solution for the compensated shares is available; see section A.2 of the appendix to Oulton (2012a).

[18]The algorithm is not guaranteed to converge; the convergence issue is discussed in section A.2 of the appendix to Oulton (2012a).

So given knowledge of the η_{ik} up to the required order, we can estimate the Konüs price index. Estimating the η_{ik} themselves may still seem a difficult task but notice that only the response of demand to changes in real income needs to be known, not the response to price changes. This is a very significant reduction in the complexity of the task empirically.

It is possible that estimates of the η_{ik} are available "off the shelf," in which case the problem is solved. The response to expenditure changes can be estimated from cross-section data since prices can usually be assumed to be the same for all households in a given region (see, e.g., Blow *et al.*, 2004). But cross-section estimates may not be available[19] or, even if they are, the product classification may be different. In the absence of ready-made estimates, is it possible to estimate these parameters from the aggregate data available to national statistical agencies—the same data that they use to construct conventional index numbers? The answer is yes. To make further progress I turn now to consider systems of demand which are consistent with economic theory and also seem capable of fitting the data reasonably well.

3.2. *Specifying the Demand System*

If we want to implement the algorithm set out in the previous sub-section in the absence of off-the-shelf estimates of the semi-elasticities η_{ik}, then we need to choose a specific model of consumer demand. The PIGLOG demand system, introduced by Muellbauer (1976) (see also Deaton and Muellbauer, 1980a, 1980b, ch. 3) has found wide application empirically.[20] The PIGLOG expenditure function is:

$$\ln x = \ln A(\mathbf{p}) + B(\mathbf{p}) \ln u. \tag{16}$$

Here $A(\mathbf{p}) \geq 0$ and $B(\mathbf{p}) > 0$ (non-satiation). Also, $A(\mathbf{p})$ is assumed homogeneous of degree one and $B(\mathbf{p})$ homogeneous of degree zero in prices. An example of the PIGLOG is the Almost Ideal Demand System (AIDS) in which case $A(\mathbf{p})$ takes the translog form.

This expenditure function gives rise to Engel curves which are linear in the log of expenditure. However, a linear relationship does not fare well empirically (Banks *et al.*, 1997; Blow *et al.*, 2004; Oulton, 2008) and it is found necessary to add a squared term in the log of expenditure to the share equations. A squared term arises if the utility function takes the following form, known as the generalized PIGLOG:

[19]The latest round of the World Bank's International Comparison Program has generated prices and expenditures for 106 products classified to "actual individual consumption," for each of 146 countries. But there are no corresponding micro data for these countries.

[20]Other approaches are possible. Balk (1990) employed the Rotterdam demand system to estimate an approximate Konüs price index for an aggregate time series of Dutch data. His method depended on the marginal budget shares ($p_i \partial q_i / \partial x$) being constant within the sample, a property which does not hold in the PIGLOG demand system. See also Balk (1995) for a survey of other methods of approximating a cost-of-living index.

$$\text{(17)} \qquad \ln u = \left\{ \left[\frac{\ln x - \ln A(\mathbf{p})}{B(\mathbf{p})} \right]^{-1} + \lambda(\mathbf{p}) \right\}^{-1} = \frac{\ln[x/A(\mathbf{p})]}{B(\mathbf{p}) + \ln[x/A(\mathbf{p})]\lambda(\mathbf{p})}$$

where $\lambda(\mathbf{p})$ is a differentiable, homogeneous function of degree zero in prices $\mathbf{p}$ and $\lambda(\mathbf{p}) \geq 0$. The generalized PIGLOG retains the exact aggregation property of the simple PIGLOG (see Section 4). The corresponding expenditure function is:

$$\text{(18)} \qquad \ln x = \ln A(\mathbf{p}) + \frac{B(\mathbf{p})\ln u}{1 - \lambda(\mathbf{p})\ln u}.$$

(This reduces to the simple PIGLOG system (16) when $\lambda(\mathbf{p}) = 0$.)

Applying Shephard's Lemma, and after substituting for u from (17), the expenditure shares in this demand system are:

$$\text{(19)} \qquad s_i = \frac{\partial \ln A(\mathbf{p})}{\partial \ln p_i} + \frac{\ln[x/A(\mathbf{p})]}{B(\mathbf{p})}\frac{\partial B(\mathbf{p})}{\partial \ln p_i} + \frac{[\ln[x/A(\mathbf{p})]]^2}{B(\mathbf{p})}\frac{\partial \lambda(\mathbf{p})}{\partial \ln p_i}.$$

I now follow Banks *et al.* (1997) and adopt the following specification for $B(\mathbf{p})$ and $\lambda(\mathbf{p})$:

$$\text{(20)} \qquad B(\mathbf{p}) = \prod_{k=1}^{k=N} p_k^{\beta_k}, \sum_{k=1}^{k=N} \beta_k = 0$$

and

$$\text{(21)} \qquad \lambda(\mathbf{p}) = \sum_{k=1}^{k=N} \lambda_k \ln p_k, \sum_{k=1}^{k=N} \lambda_k = 0.$$

Under this specification,

$$\frac{\partial \ln B(\mathbf{p})}{\partial \ln p_i} = \beta_i$$

$$\frac{\partial \lambda(\mathbf{p})}{\partial \ln p_i} = \lambda_i$$

so the system of share equations (19) becomes

$$\text{(22)} \qquad s_i = \frac{\partial \ln A(\mathbf{p})}{\partial \ln p_i} + \beta_i \ln\left[\frac{x}{A(\mathbf{p})}\right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left\{ \ln\left[\frac{x}{A(\mathbf{p})}\right] \right\}^2.$$

What is the relationship between compensated and actual shares in this demand system? In equation (7) above we found a Taylor series expansion for the compensated shares which involved the semi-elasticity of the shares with respect to real income, $\partial s_i/\partial \ln E$, and higher order derivatives, $\partial^2 s_i/\partial \ln E^2$, etc. Now from (19) we get that

(23)
$$\frac{\partial s_i}{\partial \ln x} = \beta_i + \frac{2\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \ln\left[\frac{x}{A(\mathbf{p})}\right]$$
$$\frac{\partial^2 s_i}{\partial[\ln x]^2} = \frac{2\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}}$$

and higher order derivatives are zero.

These derivatives have to be evaluated when $x = E(t, b)$. The simplest way to do this is to adopt the normalization that $\ln u(b) = 0$. This is always possible by appropriate choice of utility units. It now follows also from (18) that

(24)
$$\ln x(t, b) = \ln A_b(\mathbf{p}(t)) + \frac{B(\mathbf{p}(t))\ln u(b)}{1-\lambda(\mathbf{p}(t))\ln u(b)} = \ln A_b(\mathbf{p}(t)).$$

Here and from now on, I write $A_b(\mathbf{p})$ rather than just $A(\mathbf{p})$, to mark the fact that this normalization changes the function $A(\mathbf{p})$.[21]

We can now use these results to evaluate the derivatives in (23) at the point $x = E(t, b)$, $\mathbf{p} = \mathbf{p}(t)$:

$$\eta_{i1}(t, b) = \left[\frac{\partial s_i}{\partial \ln x}\right]_{\substack{\mathbf{p}=\mathbf{p}(t)\\ x=E(t,b)}} = \beta_i + \frac{2\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}(t)} \ln\left[\frac{x(t, b)}{A_b(\mathbf{p}(t))}\right]$$
$$= \beta_i$$

using (24) and

$$\eta_{i2}(t, b) = \left[\frac{\partial^2 s_i}{\partial[\ln x]^2}\right]_{\substack{\mathbf{p}=\mathbf{p}(t)\\ x=E(t,b)}} = \frac{2\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}(t)}.$$

Substituting these results into (7) we obtain

(25)
$$s_i(t, b) = s_i(t, t) - \beta_i \ln\left[\frac{x(t, t)/x(b, b)}{P^K(t, b, b)}\right] - \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}}\left\{\ln\left[\frac{x(t, t)/x(b, b)}{P^K(t, b, b)}\right]\right\}^2,$$
$$i = 1, 2, \ldots, N; \quad t = 0, 1, \ldots, T$$

and this Taylor series expansion is not an approximation but is exact for the generalized PIGLOG with the specification of (20) and (21).

[21]It is simplest to see this is in the log-linear PIGLOG case when $\lambda(\mathbf{p}) = 0$. Add and subtract $B(\mathbf{p}(t))u(b)$ from the right-hand side of the expenditure function (16) to obtain:

$$x(t, t) = \ln A(\mathbf{p}(t)) + B(\mathbf{p}(t))\ln u(b) + B(\mathbf{p}(t))[\ln u(t) - \ln u(b)]$$
$$= \ln A_b(\mathbf{p}(t)) + B(\mathbf{p}(t))\ln u_b(t)$$

putting $\ln A_b(\mathbf{p}(t)) = \ln A(\mathbf{p}(t) + B(\mathbf{p}(t))\ln u(b)$ and $\ln u_b(t) = \ln u(t) - \ln u(b)$. Note that $\ln u_b(b) = 0$ and that $A_b(\mathbf{p}(t))$ is homogeneous of degree one in prices. So the new expenditure function with rebased utility has the same properties as the original one.

As a further step toward putting the demand system into a form which can be estimated in practice, it is helpful to use (22) and (24) to write the equations for the observed shares at time t as:

$$(26)\qquad s_i(t,t) = \frac{\partial \ln A_b(\mathbf{p}(t))}{\partial \ln p_i(t)} + \beta_i \ln\left[\frac{x(t,t)/x(b,b)}{A_b(\mathbf{p}(t))/A_b(\mathbf{p}(b))}\right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}}\left\{\ln\left[\frac{x(t,t)/x(b,b)}{A_b(\mathbf{p}(t))/A_b(\mathbf{p}(b))}\right]\right\}^2.$$

Here we have used the fact that, from (24), $x(b, b) = A_b(\mathbf{p}(b))$.

One further result involving the interpretation of the Konüs price index is also needed. From the definition of the Konüs price index, equation (2), and equation (24), we find that for the generalized PIGLOG system:

$$(27)\qquad \ln P^K(t,b,b) = \ln E(t,b) - \ln E(b,b) = \ln x(t,b) - \ln x(b,b) = \ln A_b(\mathbf{p}(t) - \ln A_b(\mathbf{p}(b))$$

Substituting this into the share equations (26),

$$(28)\qquad s_i(t,t) = \frac{\partial \ln A_b(\mathbf{p}(t))}{\partial \ln p_i(t)} + \beta_i \ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b,b)}\right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}(t)}\left\{\ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b,b)}\right]\right\}^2.$$

The compensated shares can now be written as

$$(29)\qquad s_i(t,b) = \frac{\partial \ln A_b(\mathbf{p}(t))}{\partial \ln p_i(t)} + \beta_i \ln\left[\frac{x(t,b)/x(b,b)}{P^K(t,b,b)}\right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}(t)}\left\{\ln\left[\frac{x(t,b)/x(b,b)}{P^K(t,b,b)}\right]\right\}^2 = \frac{\partial \ln A_b(\mathbf{p}(t))}{\partial \ln p_i(t)}$$

where use has been made of (24) and (27). The notation can be simplified by putting

$$z(t,b) = \ln\left[\frac{x(t,t)/x(b,b)}{P^K(t,b,b)}\right] \text{ and } y(t,b) = \frac{[z(t,b)]^2}{\prod_{k=1}^{k=N} p_k^{\beta_k}(t)}$$

after which (28) and (29) (or (24)) become

$$(30)\qquad s_i(t,t) = \frac{\partial \ln A_b(\mathbf{p}(t))}{\partial \ln p_i(t)} + \beta_i z(t,b) + \lambda_i y(t,b)$$

$$(31)\qquad s_i(t,b) = s_i(t,t) - \beta_i z(t,b) - \lambda_i y(t,b), \quad i = 1, 2, \ldots, N; t = 0, 1, \ldots, T.$$

The QAIDS specification of the real income terms, as in (30), will now be used to show how the Konüs price index can be estimated in practice, when there are too few observations to estimate all the parameters of the expenditure function.[22]

3.3. *The Estimation Procedure*

In order to implement the procedure outlined above for estimating the Konüs price index, we need to estimate only the N β_i parameters and the N λ_i parameters of equations (28); in both cases only $N - 1$ of these are independent because these coefficients each sum to zero across the products. That is, $2(N - 1)$ parameters in total need to be estimated or just two per share equation. These parameters determine the consumer's response to changes in real expenditure. We do *not* need to estimate the much more numerous parameters which determine the response to price changes. This is a huge reduction in the difficulty of the task.

Even if we need only the expenditure response parameters, how can we estimate these while avoiding estimating all the other parameters of the system at the same time? After all, if we just estimate the share equations with the price variables omitted, then our estimates of the expenditure response will undoubtedly be biased, since relative prices and real expenditures are likely to be correlated over time (and across countries). The answer is to collapse the $N - 1$ relative prices in the system into a smaller number of variables using principal components.[23] We can collapse the relative prices into (say) M principal components, where $M < N - 1$ is to be chosen empirically.

The share equations (30) can now be written in a form suitable for econometric estimation by replacing the individual price variables by principal components and adding an error term:

$$s_i(t,t) = \alpha_i^b + \sum_{k=1}^{M} \theta_{ik} PC_k(t) + \beta_i z(t,b) + \lambda_i y(t,b) + \varepsilon_i(t), \quad i = 1,\ldots,N;\ t = 0,\ldots,T \tag{32}$$

Here α_i^b is the base-year-dependent constant term $\left(\sum_i \alpha_i^b = 1\right)$; $PC_k(t)$ is the k-th principal component of the $N - 1$ relative prices; the θ_{ik} are coefficients subject to the cross-equation restrictions $\sum_i \theta_{ik} = 0$, $\forall k$; $\varepsilon_i(t)$ is the error term. The presence of the principal components in equation (32) means that the estimates of the coefficients on z and y need not be biased as they would be if prices were simply omitted.[24]

We have now reduced the problem to estimating a system of $N - 1$ independent equations, each of which contains only $M + 3$ coefficients—the

[22]In principle the method developed here could be applied to the EASI demand system recently proposed by Lewbel and Pendakur (2009). However, I have not been able to develop tractable expressions for the semi-elasticities (the η_{ik}). From the point of view of the present paper, the EASI system suffers from the disadvantage that exact aggregation does not hold: see Section 4 for discussion of aggregation over consumers who may differ in income and in other ways.

[23]See Johnson and Wichern (2002) for a textbook exposition of principal components.

[24]The empirical flexibility of equation (32) could be increased by adding cubic and higher order terms in $z(t, b)$. But then the property of exact aggregation would no longer hold. See Section 4 for more on aggregation.

θ_{ik} (M in number), α_i^b, β_i and λ_i.[25] The success of this strategy will depend on whether the variation in relative prices can be captured by a fairly small number of principal components—small that is in relation to the number of time series observations, $T + 1$. This is obviously an empirical matter. At one extreme, if there is little or no correlation between the prices over time (or space), then the use of principal components yields no benefit. At the other extreme, suppose that the demand system is specified in terms of the logs of prices and that all relative prices are just loglinear time trends, though the growth rate varies between prices. The evolution of relative prices can be written as:

$$\ln[p_j(t)/p_1(t)] = \mu_j t, \quad j = 2, \ldots, N,$$

where the μ_j are the growth rates and the first product is taken as the numeraire. Assume too that the matrix $A(\mathbf{p})$ takes the AIDS form:

$$\ln A(\mathbf{p}) = \alpha_0 + \sum_i \alpha_i \ln p_i + (1/2)\sum_i \sum_j \gamma_{ij} \ln p_i \ln p_j,$$
$$\sum_i \alpha_i = 1, \sum_i \gamma_{ij} = \sum_j \gamma_{ij} = 0, \gamma_{ij} = \gamma_{ji}$$

Then in the i-th share equation (28) the price effects are

$$\frac{\partial \ln A(\mathbf{p}(t))}{\partial \ln p_i} = \alpha_i + \sum_{j=1}^{N} \gamma_{ij} \ln p_j(t) = \alpha_i + \sum_{j=2}^{N} \gamma_{ij} \ln[p_j(t)/p_1(t)]$$
$$= \alpha_i + t\left[\sum_{j=2}^{N} \gamma_{ij}\mu_j\right] = \alpha_i + \delta_i t, \text{ say}$$

(Here we have used the fact that $\sum_j \gamma_{ij} = 0$.) In this case the effect of changing relative prices is captured entirely by a time trend, with a different coefficient in each share equation (subject to the cross-equation restriction that $\sum_i \delta_i = 0$). So just one principal component captures the whole variation in relative prices (i.e., in this case $M = 1$). This is an extreme case and in practice we must expect that more than one principal component will be required to capture the variation in relative prices.[26]

The specification of the principal components depends on the demand system chosen. If we chose the AIDS (and QAIDS) form for $A(\mathbf{p})$, then it would be natural to estimate the principal components in terms of log relative prices, e.g. $\ln(p_j/p_1)$, $j = 2, \ldots, N$, taking the first product as the numeraire. Alternatively, we might use the normalized quadratic of Diewert and Wales (1988), in

[25]This is not quite true since all the β_i appear in each equation via the denominator of y. We can handle this by an iterative procedure: see below.

[26]In Oulton (2008), I applied the method to 70 products covering the whole of the U.K.'s Retail Prices Index over 1974–2004. I found that six principal components were sufficient to capture 97.8 percent of the variation in the 69 log relative prices.

which case the principal components would be estimated in terms of relative prices (not in logs).

In estimating equations (32) econometrically, it is straightforward to impose the adding-up and homogeneity restrictions on the coefficients; homogeneity is imposed by using relative prices and adding-up is imposed by cross-section restrictions on the coefficients (these restrictions are automatically imposed by OLS). But there is one loss from using principal components: we can no longer impose the symmetry restrictions.[27]

Equations (32) are non-linear in the parameters of interest, since to measure both z and y correctly it is necessary to know the Konüs price index, the object of the whole exercise; in addition, to measure y we also need to know all the β_i and λ_i. The solution is an iterative process, similar to the one described in the previous section. Here the unknown parameters, the β_i and λ_i, are estimated jointly with the compensated shares and the Konüs price index. The system consists of equations (25) and (32), and the equation for the Konüs price index, either equation (9) if we use a compensated Törnqvist to approximate the Konüs, or equation (12) if we use a compensated Fisher. The iterative process for some particular choice of the base period is as follows:

1. Obtain initial estimates of the Konüs price index $P^K(t, b, b)$ and of the β_i and λ_i coefficients. An initial estimate of $P^K(t, b, b)$ can be obtained from equation (9) or equation (12) by using actual instead of compensated shares (i.e., replace $s_i(t, b)$ by $s_i(t, t)$ in the formulas). And for an initial estimate of the β_i, set $\beta_i = 0$, $\forall i$.
2. Derive estimates of $z(t, b) = \ln[x(t, t)/P^K(t, b)]$ and of $y(t,b) = [z(t,b)]^2 \big/ \prod_k p_k^{\beta_k}(t)$, using the latest estimates of $P^K(t, b, b)$ and of the β_i. Using these new estimates of z and y, estimate equation (32) econometrically, to obtain new estimates of the β_i and the λ_i.
3. Using the new estimates of the β_i and λ_i, estimate the compensated shares from equation (25). Then use the compensated shares to derive a new estimate of the Konüs price index $P^K(t, b, b)$ from equation (9) or equation (12).
4. If the estimate of the Konüs price index has changed by less than a preset convergence condition, stop. If not, go back to step 2.[28]

The algorithm can be rerun to generate estimates for any other base year. Alternatively, the estimates of the β_i and λ_i produced by the first run can be plugged into the simpler algorithm of Section 3.1 to generate Konüs price indices for any other base year.

[27]For example, suppose that $N = 3$ and that the special case of all relative prices changing at constant rates applies. Then, dropping the third equation, taking the first product as the numeraire, and imposing all the constraints, the relationship between the δ_i and the γ_{ij} is as follows: $\delta_1 = \gamma_{12}\mu_2 - (\gamma_{11} + \gamma_{12})\mu_3$, $\delta_2 = \gamma_{22}\mu_2 - (\gamma_{12} + \gamma_{22})\mu_3$. These relationships imply no further restrictions on δ_1 and δ_2. So we cannot test whether $\gamma_{12} = \gamma_{12}$.

[28]This is the same as the Iterated Linear Least squares Estimator (ILLE) proposed by Blundell and Robin (1999). They prove that the limit values of these parameter estimates are consistent.

3.4. *Comparisons Across Space*

The analysis carries over unchanged to the problem of estimating a cost of living index and hence the standard of living across countries at a point in time.[29] The solution for the Konüs price index given by equations (7) and (5) can be applied directly in the cross-country context. Initially we must imagine a continuum of countries indexed by t, just as in Section 3 we imagined a continuum of time periods. Then we consider discrete approximations; i.e. as before, equation (5) can be approximated by either (9) or (12).

One problem which is often said to arise in the cross-country but not the inter-temporal context is that, unlike time, countries have no natural order. In the present case this objection does not apply. Here the natural order for countries is the ranking by real income (or real expenditure) per capita. Adopting this order minimizes the gap between country t and country $t-1$and so should improve the discrete approximation. It is true that the rank order is not known for certain in advance, since the whole point of the exercise is to estimate the true standard of living. But in practice the rank order is very similar whatever the deflator employed (Oulton, 2012b). Alternatively, the ordering of countries could be determined by the minimum-spanning-tree method suggested by and implemented on cross-country data by Hill (1999). Then the links in the chain would be selected so as to minimize the (compensated) Laspeyres–Paasche spread.

4. Extensions to the Basic Analysis

Section 3 offered a solution to the problem of estimating a true cost-of-living index over time for a single representative consumer. I now consider two extensions to the analysis. First, I consider the effect of relaxing the assumption of a single representative consumer. I now assume that the aggregate data is generated by heterogeneous consumers who differ in income. If the degree of inequality were constant, the preceding analysis could stand unchanged. This may or may not be a reasonable approximation in a time series context over a few decades. But in a cross-country context the assumption is certainly problematic: countries differ widely in the extent of inequality (Anand and Segal, 2008). So we need to extend our framework to encompass this. Second, I consider aggregation over different types of household.

4.1. *Aggregation Over Rich and Poor Consumers*

Let the population be composed of G groups. The groups are assumed to be of equal size (e.g., percentiles, deciles, or quintiles), with the first group being the poorest and the G-th group the richest. The fraction of households in each group

[29]See Hill (1997) for a survey of methods of making international comparisons; a general overview was provided by Balk (2009). Caves *et al*. (1982) have applied chained superlative index numbers to cross-country comparisons. Hill (2004) also estimates a chain superlative index but employs the minimum-spanning tree approach to find the best links in the chain. Neary (2004) employed the World Bank's 1980 PPPs for 60 countries and 11 commodity groups to estimate a QAIDS; he then derived a measure of real GDP per capita for the 60 countries. The World Bank's current methodology for deriving PPPs at the aggregate level is set out in World Bank (2008).

is then $1/G$. Let x_g be mean expenditure per household in the g-th group. Within a group, each household's expenditure is the same, namely the group mean. The share of product i in the expenditure of the g-th group, s_{ig}, is then

$$s_{ig} = \frac{p_i q_{ig}}{x_g},$$

where q_{ig} is the quantity per capita of the i-th product purchased by each member of the g-th group. The share of the i-th product in aggregate expenditure is therefore

$$(33) \qquad s_i = \frac{p_i q_i}{x} = \frac{\sum_{g=1}^{g=G} p_i q_{ig}}{Gx} = \sum_{g=1}^{g=G} \left[\frac{x_g}{Gx} \frac{p_i q_{ig}}{x_g} \right] = \sum_{g=1}^{g=G} w_g s_{ig},$$

where w_g is the share of the g-th group in aggregate expenditure:

$$(34) \qquad w_g = \frac{x_g}{Gx}, \quad \sum_{g=1}^{g=G} w_g = 1.$$

We assume that preferences have the Ernest Hemingway property: the rich are different from the poor but only because the rich have more money.[30] So the parameters of the expenditure function are the same for all households. All consumers are assumed to face the same prices. So from (22) and adopting the QAIDS formulation, the share of the i-th product in expenditure by the g-th group is:

$$s_{ig} = \frac{\partial \ln A(\mathbf{p})}{\partial \ln p_i} + \beta_i \ln \left[\frac{x_g}{A(\mathbf{p})} \right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left\{ \ln \left[\frac{x_g}{A(\mathbf{p})} \right] \right\}^2.$$

Using (33), the aggregate share equations are weighted averages of the underlying equations for each group:

$$(35) \quad s_i = \sum_{g=1}^{g=G} w_g s_{ig} = \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \sum_{g=1}^{g=G} w_g \ln x_g - \beta_i \ln A_b(\mathbf{p}) + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left[\sum_{g=1}^{g=G} w_g (\ln x_g)^2 - 2 \ln A_b(\mathbf{p}) \sum_{g=1}^{g=G} w_g \ln x_g + [\ln A_b(\mathbf{p})]^2 \right]$$

The difference between this and our previous equation (22) is that instead of the log of aggregate expenditure per capita, $\ln x = \ln \left[\sum_{g=1}^{g=G} x_g / G \right]$, appearing on the right-hand side, we now have the share-weighted average of log expenditure per capita in each group, $\sum_{g=1}^{g=G} w_g \ln x_g$; and instead of $(\ln x)^2$, we now have $\sum_{g=1}^{g=G} w_g (\ln x_g)^2$. The relationship between $\sum_{g=1}^{g=G} w_g \ln x_g$ and $\ln x$ is, from (34),

[30]The well-known (though apparently fictional dialogue; Clark, 2008) runs as follows. Fitzgerald: "The rich are different from us, Ernest." Hemingway: "Yes, Scott, they have more money than we do."

$$\sum_{g=1}^{g=G} w_g \ln x_g = \sum_{g=1}^{g=G} w_g \ln(w_g Gx) = \sum_{g=1}^{g=G} w_g \ln w_g + \ln G + \ln x.$$

The first term on the right-hand side, $\sum_g w_g \ln w_g$, is the negative of entropy (ignoring an unimportant scale constant); it was suggested as a measure of inequality by Theil (1967, ch. 4). Define $I = -\sum_{g=1}^{g=G} w_g \ln w_g$ as entropy and define also the related inequality statistic $J = \sum_{g=1}^{g=G} w_g (\ln w_g)^2$. Substituting these into (35), we find after some manipulation (see section A.3 of the Appendix) that

$$(36) \qquad s_i = \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \left\{ W_1 + \ln\left[\frac{x}{A_b(\mathbf{p})}\right]\right\} + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left\{ W_2 + 2\left\{ W_1 \ln\left[\frac{x}{A_b(\mathbf{p})}\right]\right\} + \left\{ \ln\left[\frac{x}{A_b(\mathbf{p})}\right]\right\}^2 \right\}$$

where we have set $W_1 = \ln G - I$ and $W_2 = J - 2I \ln G + (\ln G)^2$. In the case of a perfectly equal distribution (when $w_g = 1/G$), note that $I = \ln G$, $J = (\ln G)^2$, and $W_1 = W_2 = 0$, so that (36) then reduces back down to the original QAIDS formulation, equation (22). Compared to (22), there are two additional variables in (36), W_1 and W_2, though no additional parameters. These additional variables may help to explain changes in shares, to the extent that inequality varies either over time or across countries. Note too that in the simpler AIDS case (i.e., when all the λ_i are zero), equation (36) simplifies to

$$(37) \qquad s_i = \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \left\{ (\ln G - I) + \ln\left[\frac{x}{A_b(\mathbf{p})}\right]\right\},$$

which contains just one additional variable (I).[31]

By analogy with equation (32), equation (36) can be written in a form suitable for econometric estimation as:

$$(38) \qquad s_i(t,t) = \alpha_i^b + \sum_{k=1}^{M} \theta_{ik} PC_k(t) + \beta_i w_1(t,b) + \lambda_i w_2(t,b) + \varepsilon_i(t), \quad i = 1,\ldots,N;\ t = 0,\ldots,T$$

where w_1 and w_2 are the expenditure variables corrected for income distribution effects:

$$w_1(t,b) = W_1 + \ln\left[\frac{x}{A_b(\mathbf{p})}\right] = W_1 + z(t,b)$$

and

[31]The role of Theil's inequality measure, entropy (I), was discussed in Deaton and Muellbauer (1980b, ch. 6, section 6.2). They derived a result equivalent to (37).

$$w_2(t,b) = \frac{1}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left\{ W_2 + 2\left\{ W_1 \ln\left[\frac{x}{A_b(\mathbf{p})}\right]\right\} + \left\{\ln\left[\frac{x}{A_b(\mathbf{p})}\right]\right\}^2 \right\}$$
$$= \frac{W_2 + 2W_1 z(t,b) + y(t,b)}{\prod_{k=1}^{k=N} p_k^{\beta_k}}.$$

The upshot is that the QAIDS can be parsimoniously extended to capture the effect of income inequality. The additional empirical requirement is fairly modest: we need to know the shares of different groups in aggregate expenditure, at a reasonable level of detail.

4.2. *Aggregation Over Different Household Types*

Suppose there are a set of H characteristics that influence demand, in addition to income and prices. These could include household characteristics such as number of children, average age, and educational level, and also environmental characteristics such as climate. Now the share equations of the QAIDS for the g-th income group could be written as:

$$\text{(39)}\quad s_{ig} = \alpha_i + \frac{\partial \ln A_b(\mathbf{p})}{\partial \ln p_i} + \beta_i \ln\left[\frac{x_g}{A(\mathbf{p})}\right] + \frac{\lambda_i}{\prod_{k=1}^{k=N} p_k^{\beta_k}} \left\{\ln\left[\frac{x_g}{A(\mathbf{p})}\right]\right\}^2 + \sum_{h=1}^{h=H} \theta_{ih} K_{hg},$$

where K_{hg} is the level of the h-th characteristic in the g-th group; I assume that each household in the g-th group has the same level of each of the K_{hg} as all the other households in that group (this entails no loss of generality if there is only one household in each group). The θ_{ih} coefficients must satisfy the adding-up restrictions:

$$\sum_{i=1}^{i=N} \theta_{ih} = 0, \quad h = 1, 2, \ldots, H.$$

(At some cost to parsimony, the model could be extended by interacting the characteristic variables with income.) Again, underlying preferences are assumed to be the same but people's situations differ for various reasons, in the spirit of Stigler and Becker (1977):[32] at the same incomes and prices, people in cold climates buy more winter clothes. We can aggregate equation (39) over the income groups to obtain the same result as (36), but with an additional term:

$$+\sum_{h=1}^{h=H} \theta_{ih} K_h,$$

where $K_h = \sum_{g=1}^{g=G} w_g K_{hg}$. Now K_h is a weighted average of the level of the h-th characteristic in a particular country (time period). The only difficulty from an empirical point of view is that it is an income-weighted, not a population-weighted, average. So, for example, if the rich have fewer children than the poor nowadays,

[32]This approach seems likely to be more fruitful in the present context than assuming that tastes may differ; the latter approach is taken by van Veelen and van der Weide (2008).

then using the mean number of children per household as a measure would be a misspecification when estimating share equations from aggregate data.

5. Cost Functions: Estimating Input-Biased Scale Economies and Technical Change

In this section I consider the parallel problem of estimating an input price index and technical change when the cost function is not homothetic. Now both economies of scale and technical change may be input-biased. I assume that the typical firm is a price taker in input markets and wishes to minimize costs. We can write the cost function in general as:

$$x = C(\mathbf{p}, Y, t). \tag{40}$$

Here output (Y) plays the role of utility in the expenditure function. While formally this makes no difference, there is a big difference empirically since output is objectively and directly measurable (at least in principle) while utility is only indirectly measurable. The presence of time (t) as an indicator of technical change in the cost function also has no counterpart in the theory of demand.

By analogy with equation (18), we can use a generalized PIGLOG formulation:

$$\ln x = \ln C(\mathbf{p}, Y, t) = \ln A(\mathbf{p}) + \frac{B(\mathbf{p})\ln Y}{1-\lambda(\mathbf{p})\ln Y} + \beta_Y \ln Y + \mu(\mathbf{p})t + \mu_t t, \tag{41}$$

where Y is output, $x = \sum_i p_i q_i$ is total expenditure on the inputs q_i, and as before $B(\mathbf{p}) > 0$ is homogeneous of degree one in prices and $\lambda(\mathbf{p}) \geq 0$ is homogeneous of degree zero in prices. There are two new elements here. First, the parameter β_Y measures overall economies of scale. When there are no input biases, i.e. $B(\mathbf{p}) = 1$ and $\lambda(\mathbf{p}) = 0$, then $\beta_Y = 0$ implies constant returns to scale and $\beta_Y < 0$ implies increasing returns. In this case the cost function is homothetic but not necessarily homogeneous of degree one in output. Second, the last two terms on the right-hand side of (41) measure technical change. Neutral technical change is measured by the parameter μ_t ($\mu_t < 0$ implies that technical change is positive); input-biased technical change is measured by the function $\mu(\mathbf{p})$. By analogy with $\lambda(\mathbf{p})$, $\mu(\mathbf{p})$ could be specified as

$$\mu(\mathbf{p}) = \sum_{k=1}^{k=N} \mu_k \ln p_k, \quad \sum_{k=1}^{k=N} \mu_k = 0. \tag{42}$$

Under this specification, and with $B(\mathbf{p})$ and $\lambda(\mathbf{p})$ defined as earlier for the generalized PIGLOG (see (20) and (21)), the share equations are now given by:[33]

[33]These are cost shares, not revenue shares. In the presence of economies of scale there may be monopoly power, so profit is above the competitive level. I assume that the competitive rate of return to capital is known so that it is possible to calculate competitive rental prices for capital inputs; see Oulton (2007) for alternative ways of doing this.

$$s_i = \frac{\partial \ln C}{\partial \ln p_i} = \frac{\partial \ln A(\mathbf{p})}{\partial \ln p_i} + \beta_i \left[\frac{B(\mathbf{p}) \ln Y}{1 - \lambda(\mathbf{p}) \ln Y} \right] + \frac{\lambda_i}{B(\mathbf{p})} \left[\frac{B(\mathbf{p}) \ln Y}{1 - \lambda(\mathbf{p}) \ln Y} \right]^2 + \mu_i t. \tag{43}$$

The parameters β_i and λ_i now measure input bias in scale economies. If they are all zero there is no bias and the degree of returns to scale is measured just by β_Y. The parameter μ_i measures the bias in technical change against input i: $\mu_i < 0$ would imply that technical change is biased in favor of input i.

If our goal is to estimate the degree of economies of scale and the rate of technical change, the parameters of interest in the cost function can be estimated by a simpler method than in the case of the expenditure function. We can consider equation (43) as a regression equation by adding an error term, in the same way as we did to obtain equation (32) above for the expenditure shares. After replacing the price variables in (43) by principal components, we can then estimate the β_i, λ_i and μ_i by a similar iterative process to the one set out in Section 3, while imposing the appropriate cross-equation restrictions. Next, the degree of scale economies and the rate of neutral technical change can be estimated by differentiating the cost function (41) totally with respect to time, using (42), applying Shephard's Lemma, and rearranging:

$$\begin{aligned} &\frac{d \ln x(t,t)}{dt} - \sum_{i=1}^{i=N} s_i(t,t) \left(\frac{d \ln p_i(t)}{dt} \right) - \sum_{i=1}^{i=N} \mu_i \ln p_i(t) \\ &- \left[\frac{B(\mathbf{p}(t))}{[1 - \lambda(\mathbf{p}(t) \ln Y(t)]^2} \right] \left(\frac{d \ln Y(t)}{dt} \right) = \mu_t + \beta_Y \left(\frac{d \ln Y(t)}{dt} \right) \end{aligned} \tag{44}$$

Everything on the left-hand side is now measurable and the only unknowns are the coefficients μ_t and β_Y on the right-hand side. So (44) can be considered as a regression equation and used to estimate these remaining unknowns.[34]

The compensated shares, holding output constant at its level in period b, are

(45)

$$\begin{aligned} s_i(t,b) &= \frac{\partial \ln A(\mathbf{p}(t))}{\partial \ln p_i(t)} + \beta_i \left[\frac{B(\mathbf{p}(t)) \ln Y(b)}{1 - \lambda(\mathbf{p}(t)) \ln Y(b)} \right] + \frac{\lambda_i}{B(\mathbf{p}(t))} \left[\frac{B(\mathbf{p}(t)) \ln Y(b)}{1 - \lambda(\mathbf{p}(t)) \ln Y(b)} \right]^2 + \mu_i t \\ &= \frac{\partial \ln A(\mathbf{p}(t))}{\partial \ln p_i(t)} + \mu_i t \end{aligned}$$

setting $\ln Y(b) = 0$. So the relationship between the actual and the compensated shares is

$$s_i(t,b) = s_i(t,t) - \beta_i \left[\frac{B(\mathbf{p}(t)) \ln Y(t)}{1 - \lambda(\mathbf{p}(t)) \ln Y(t)} \right] - \frac{\lambda_i}{B(\mathbf{p}(t))} \left[\frac{B(\mathbf{p}(t)) \ln Y(t)}{1 - \lambda(\mathbf{p}(t)) \ln Y(t)} \right]^2 \tag{46}$$

and the compensated shares can be used to construct a Konüs index of input prices.

[34]Actually, overall technical change is not separately identifiable from biased technical change. Any non-zero estimate for μ_t can be absorbed into the μ_i by relaxing the constraint that $\sum_i \mu_i = 0$.

The analysis of inequality in Section 4 can also be applied to the cost functions of firms, if the size distribution varies over time or across countries. Entropy (I) and the related statistic J would now appear in the share equations (43), just as they do in (36).

6. Applying the Method: True PPPs for 141 Countries and 100 Products

In this section I outline briefly how the algorithm developed in previous sections was employed to measure true (Konüs) PPPs for 141 of the 146 countries included in the latest, 2005 round of the World Bank's International Comparison Program (ICP): see Oulton (2012b) for a full account. The aim was to construct a measure of the standard of living for each country, defined as real household consumption per head, in other words household consumption per head measured in local currency units deflated by a true (Konüs) PPP.

6.1. *The 2005 Round of the ICP*

The 2005 round of the ICP was the most comprehensive to date. It included 146 states or territories comprising 95 percent of the world's population (6.128 billion people); see World Bank (2008). The ICP gathered price data and corresponding expenditures for 129 "Basic Headings" (products) covering the whole of GDP. An example of a Basic Heading is "Rice"; another is "Bread," and a third is "Cultural services"; see Oulton (2012b, table A.1) for the full list of Basic Headings.

The Final Report (World Bank, 2008) contains estimates of PPPs for GDP as a whole and for various major aggregates like "Individual Consumption Expenditure by Households," for each of the 146 countries. These high level PPPs were derived as multilateral index numbers over the lower level PPPs which were at the Basic Heading level. The PPPs at the Basic Heading level and the corresponding expenditures were not published and are unfortunately confidential. However the World Bank kindly made these data available to me. This dataset consisted of expenditures (in local currency units) and corresponding PPPs (expressed as local currency units per U.S. dollar) for each of the 146 countries and for each of the 129 Basic Headings which make up GDP, with no missing values; population in 2005 was also supplied.

Here we are concerned only with expenditures which fall within the category dubbed "Individual Consumption Expenditure by Households," comprising 106 Basic Headings. For some Basic Headings a large number of countries reported zero expenditure. Examples are Net Purchases Abroad and FISIM. These were excluded from the definition of household consumption. A few other Basic Headings were aggregated together. The final number of Basic Headings included in the demand system was exactly 100. Each country's household consumption was defined to be the total of spending on these 100 products. Five countries had an implausibly large number of Basic Headings where expenditure was reported to be zero, even though PPPs were supplied in all cases. I therefore excluded these five, leaving 141 countries.

6.2. *Models*

Based on equations (32) and (38), four models were fitted to the data. The models are distinguished by the expenditure variables included on the right-hand side and by whether or not allowance is made for within-country inequality:

Model I. Simple (linear) PIGLOG: equation (32) with $\lambda_i = 0, \forall i$; z is the only real income variable, i.e. no allowance for within-country inequality.

Model II. Simple (linear) PIGLOG with allowance for within-country inequality: equation (38) with $\lambda_i = 0, \forall i$; w_1 is the only real income variable.

Model III. Generalized (quadratic) PIGLOG: equation (32); z and y are the real income variables, i.e. no allowance for within-country inequality.

Model IV. Generalized (quadratic) PIGLOG with allowance for within-country inequality: equation (38); w_1 and w_2 are the real income variables.

In addition to the expenditure variables, each model included 24 principal components of the 99 log relative prices; these principal components accounted for 91 percent of the variation in the log relative prices. (The first principal component accounted for 36 percent and the first ten accounted for 80 percent of the variation.)

6.3. *Background Variables*

Because of the huge variety of countries included, it was thought necessary to gather a large number of background variables, in fact all variables which might conceivably influence spending patterns apart from prices and incomes. But a constraint was the need for the variables to be available for all the 141 countries. The variables which I was able to find fell into nine categories: climate (5 variables); religion (4 variables); hegemony and culture (7 dummy variables); health (3 variables); urbanization (1 variable); openness to trade (1 variable); demography (2 variables); inequality (2 variables); and World Bank ICP region (5 dummy variables). Preliminary testing found that the background variables were collectively significant and possessed considerable explanatory power. So these 30 variables were included in all the results reported below.

6.4. *The Results*

The demand system was estimated by OLS which automatically imposes the adding-up restrictions on the coefficients. Since the same variables are included in all equations, OLS is equivalent to SUR here. The algorithm set out in Section 3.3 was applied with 60 iterations. For all four models, the mean R^2 across the 100 share equations was about 0.58. The number of estimates of the coefficients on expenditure which were significant at the 5% level was considerably greater than would be expected on a chance basis; e.g. for model IV, w_1 is significant at the 5% level in 18 cases and w_2 in 12 cases, when by chance (at the 5% level) we would expect only 5 cases for each. This confirms the empirical importance of non-homotheticity in consumer demand.

Chained Törnqvist price indices were then calculated using compensated shares. The compensated shares are derived from equation (31), modified by the inclusion of the effects of background variables. The background variables can be set to the levels of the base country or to the world average levels. But given the model, there was little difference between the results under these two choices, so only results for the first choice are presented here.

Any one or all of the 141 countries could be chosen as the base country. In fact, two countries were chosen, the poorest and the richest. The poorest country as measured by real household consumption per head (whatever deflator is used for nominal consumption) is the Democratic Republic of Congo (DRC) and the richest is the United States.

The results are summarized in Table 1 which compares Konüs estimates of the standard of living in these 141 countries for each of the four models with estimates based on conventional chained, bilateral, or multilateral index numbers. In all cases the reference country is the DRC, but for the Konüs indices the base country is either the DRC or the U.S. The Konüs estimates are constructed as nominal household consumption per capita deflated by a chained Törnqvist price index (PPP); the latter employs *compensated* shares estimated by one of models I–IV. Though the results are similar for all four models, more attention will be paid to model IV which, being more general, is preferred on theoretical grounds.

All the conventional indices use actual shares. There are two multilateral indices. The first uses the World Bank's own deflator for Individual Consumption Expenditure by Households (an aggregate which is very similar but not identical to my household consumption); this is a multilateral index number but one which preserves "regional fixity" among the six ICP regions: i.e. the relative position of countries within an ICP region is not affected by the results for countries outside that region. The second is my own estimate: an EKS Fisher index for household consumption, which turns out to be very similar to the World Bank's index. The two bilateral indices use first, U.S. *actual* budget shares as weights to construct the deflator (a Laspeyres price index), or second, each country's own *actual* budget shares as weights (a Paasche price index); recall that the U.S. is the reference country for PPPs.

The first point to note from Table 1 is that all the Konüs measures of the standard of living show greater inequality across the world than do the multilateral indices, whether the World Bank's or my own. The standard deviation and the maximum are in all cases higher. By considering the ratio of the maximum to the median and of the median to the minimum, we can see that on the Konüs measures, the upper half of the global income distribution is stretched out compared to the conventional multilateral measures. The rank order however is very similar whichever deflator is used.

Second, summary statistics for all the Konüs measures lie close to or above those of the bilateral measure using actual U.S. weights (Q^L, where deflation is by a Laspeyres price index), and all are substantially above those of the bilateral actual measure using each country's own actual weights (Q^P, where deflation is by a Paasche price index). By contrast the two multilateral indices can be seen to lie squarely within the bilateral Laspeyres–Paasche spread (where all indices use actual shares).

Third, irrespective of the base, the Konüs measures display greater inequality and a wider gap between the richest and the poorest, than does a conventional chained Törnqvist index (Q^{ChT}), or a conventional chained Fisher (Q^{ChF}). This is

TABLE 1

Real Household Consumption Per Capita for 141 Countries: Comparison of Konüs PPPs and Conventional PPPs as Deflators, Summary Statistics (poorest country is reference, i.e. DRC = 1; base country is *either* the DRC *or* the U.S.)

(A) Konüs Indices: Poorest Country (DRC) is Base and Reference						
HC per Head	Background Variables Level	Mean	Median	S.D.	Minimum	Maximum
Model I: $Q^K_{I,DRC}$	DRC	54.6	24.6	60.5	1.0	243.6
Model II: $Q^K_{II,DRC}$	DRC	58.4	26.4	64.8	1.0	261.5
Model III: $Q^K_{III,DRC}$	DRC	52.9	24.4	58.5	1.0	237.5
Model IV: $Q^K_{IV,DRC}$	DRC	56.0	26.0	61.8	1.0	249.6
(B) Konüs Indices: Richest Country (U.S.) is Base; DRC is Reference						
HC per Head	Background Variables Level	Mean	Median	S.D.	Minimum	Maximum
Model I: $Q^K_{I,US}$	U.S.	66.4	27.1	75.2	1.0	307.7
Model II: $Q^K_{II,US}$	U.S.	59.3	24.0	67.3	1.0	277.5
Model III: $Q^K_{III,US}$	U.S.	69.0	28.1	78.3	1.0	320.5
Model IV: $Q^K_{IV,US}$	U.S.	59.8	24.5	67.3	1.0	275.6
(C) Conventional Indices: DRC is Reference						
HC per Head		Mean	Median	S.D.	Minimum	Maximum
World Bank (multilateral): Q^{WB}	–	51.8	29.0	54.9	1.0	235.9
Conventional multilateral (EKS Fisher): Q^{EKS}	–	51.5	28.2	53.8	1.0	223.9
Bilateral (own weights): Q^P	–	48.0	28.0	46.9	1.0	190.6
Bilateral (U.S. weights): Q^L	–	51.1	26.8	56.8	1.0	247.8
Chained Törnqvist: Q^{ChT}	–	33.5	21.3	30.9	1.0	135.5
Chained Fisher: Q^{ChF}	–	50.1	28.6	51.5	1.0	217.2

Notes: Each index (Q) uses the same nominal total, household consumption in local currency units (HC) per capita in 2005, but deflators vary. In symbols, $Q = HC/P$, where the deflator P is either Konüs or conventional; superscripts indicate the type of index and subscripts the model and base, e.g. $Q^K_{IV,US} = HC\big/P^K_{IV,US}$. HC is the sum of expenditure on Basic Headings 1–102, 104 and 105, 100 products in all: see appendix table A1 of Oulton (2012b) for the list of Basic Headings. Memo item: HC per capita in the U.S. in 2005 was $29,024.

Konüs indices. Konüs indices are HC per capita deflated by a Konüs price index estimated by a chained Törnqvist index using *compensated* shares as weights; the compensated shares use the utility level of either the *poorest* country (the DRC) or the *richest* country (the U.S.). Alternative estimates of the compensated shares are derived from four regression models, I–IV, with background variables included: see text for full description.

Conventional indices. "World Bank (multilateral)" is HC per capita deflated by the World Bank's PPP for Individual Consumption by Households. "Bilateral (own weights)" is HC per capita deflated by a Paasche price index which uses each country's *actual* shares in turn to weight the PPPs. "Bilateral (U.S. weights)" is HC per capita deflated by a Laspeyres price index which uses *actual* U.S. shares. "Conventional multilateral" is HC per capita deflated by my own estimate of a multilateral (EKS Fisher) price index for HC, again employing *actual* shares.

Source: Unpublished World Bank spreadsheet from the 2005 ICP and own calculations: see Oulton (2012b) for full details of sources and methods and individual results for each of the 141 countries.

surprising: we might have expected that if inequality on the Konüs measure with the poorest country as base (e.g. $Q^K_{IV,DRC}$) is greater than on the conventional measure, then it will be lower when the richest country is the base (e.g. $Q^K_{IV,DRC}$). This guess would be correct if the Konüs measures were bilateral ones. But here they are chain indices and the result in Table 1 is quite possible; see Oulton (2012b) for a discussion.

The Konüs measures generally lie within the spread between a compensated Laspeyres and a compensated Paasche, as required by Proposition 2 above. For model IV there are no violations of either the chain or bilateral bounds when the DRC is the base. When the U.S. is the base, 10 countries violate the bilateral Laspeyres lower bound, by an average of 7 percent; 3 countries violate the Paasche upper bound by an average of only 1 percent.

But we must also take account of the restrictions implied by the well-known Konüs (1939) inequalities relating the Konüs to *actual* Laspeyres and Paasche indices: these are stated in equation (15) in Section 3. They imply that when household consumption is deflated by a bilateral Laspeyres price index, which weights individual PPPs by actual U.S. shares, the result is a *lower* bound to the Konüs measure with the *poorest* country taken as the base. In the notation of Table 1:

$$Q^L \leq Q^K_{\cdot,DRC}.$$

Also, when household consumption is deflated by a bilateral Paasche price index, which weights individual PPPs by each country's own actual shares, the result is an *upper* bound to the Konüs measure when the *richest* country is taken as the base:

$$Q^P \geq Q^K_{\cdot,US}.$$

Using the theoretically preferred model IV, seven countries (all within the OECD–Eurostat group) infringe the Paasche upper bound, though by an average of only 2 percent. But 49 countries infringe the Laspeyres lower bound, by an average of 8 percent. However, if we adjust for these infringements by setting the Konüs measure equal to the Laspeyres bound in these cases, we find that the mean is now 56.5, the median is 27.0, the standard deviation is 61.9, and the maximum is 247.8, little changed from the unadjusted values. The conclusion then remains that global inequality in living standards is higher than suggested by conventional multilateral indices.

7. Conclusions

An algorithm which generates Konüs price indices when demand is not homothetic has now been presented. We have shown that it can be applied in both time series and cross-section. It is not dependent on the assumption of a representative consumer but can be extended to the case where income levels and other characteristics differ between consumers. The same algorithm can be applied to the parallel problem of estimating a true index of a producer's input prices and of technical change in the presence of input-biased economies of scale. The algorithm involves some econometric estimation but uses exactly the same price and quantity data as are required for conventional index numbers. The advantage of the algorithm is that it does not require the estimation of a complete system of consumer (or producer) demand, but only the consumer's responses to expenditure changes. So it can be applied at a very disaggregated level. And no restrictive assumptions about preferences (such as separability) are needed.

The algorithm has been illustrated by applying it to the problem of measuring the standard of living across 141 countries covered by the World Bank's latest ICP, at the level of 100 products within household consumption. Compared with conventional multilateral indices, the Konüs measures show significantly higher global inequality.

It is now time to consider some limitations of the analysis and some unanswered questions. If we are trying to measure the standard of living, then our maintained hypothesis must be that tastes are identical. Otherwise the relative living standards of (say) Bangladeshi peasants and American investment bankers must be regarded as simply incommensurable. But the assumption of identical tastes might be considered overly strong. Is an intermediate position possible, in which tastes are identical at some comparatively high level, but might differ at a lower one? For example, the taste for hot, non-alcoholic beverages might be universal even though (at identical incomes and prices) some people prefer tea and others coffee.

A related and unanswered question in the theory of demand and production is, at what level of aggregation is the analysis supposed to apply? It is hard to believe that there exists a stable structure of preferences (common to all time periods and all countries) at a very detailed level, such as individual brands of breakfast cereal. Equally, it is not obvious that "food" is the right level either, since food items range from necessities (bread) to luxuries (caviar). In practice, the level of aggregation is often chosen on pragmatic grounds, to obtain sufficient observations to estimate the parameters of interest. Resolution of these issues must await further research.

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Supporting Information

Additional Supporting Information may be found in the online version of this article:

Appendix A.1: Proofs of Propositions in Section 3
Appendix A.2: Aggregating Over Unequal Incomes in the Generalized PIGLOG

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