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AN ASSUMPTION-FREE FRAMEWORK FOR MEASURING PRODUCTIVITY CHANGE

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The measurement of productivity change (or difference) is usually based on models that make use of strong assumptions such as competitive behavior and constant returns to scale. This survey discusses the basics of productivity measurement and shows that one can dispense with most if not all of the usual, neoclassical assumptions. By virtue of its structural features, the measurement model is applicable to individual establishments and aggregates such as industries, sectors, or economies.

1. Introduction

The methodological backing of productivity measurement and growth accounting usually goes like this. The (aggregate) production unit considered has an input side and an output side, and there is a production function that links output quantities to input quantities. This production function includes a time variable, and the partial derivative of the production function with respect to the time variable is called technological change (or, in some traditions, multi-factor or total factor productivity change). Further, it is assumed that the production unit acts in a competitive environment; that is, input and output prices are assumed as given. Next, it is assumed that the production unit acts in a profit maximizing manner (or, it is said to be "in equilibrium"), and that the production function exhibits constant returns to scale. Under these assumptions it then appears that output quantity growth (defined as the output-share-weighted mean of the individual output quantity growth rates) is equal to input quantity growth (defined as the input-share-weighted mean of the individual input quantity growth rates) plus the rate of technological change (or multi-factor or total factor productivity growth).

For the empirical implementation one then turns to National Accounts, census, and/or survey data, in the form of nominal values and deflators (price indices). Of course, one cannot avoid dirty hands by making various imputations where direct observations failed or were impossible (as in the case of labor input of self-employed workers). In the case of capital inputs the prices, necessary for the computation of input shares, cannot be observed, but must be computed as unit user costs. The single degree of freedom that is here available, namely the rate of return, is used to ensure that the restriction implied by the assumption of constant

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returns to scale, namely that profit equals zero, is satisfied. This procedure is usually rationalized by the assumption of perfect foresight, which in this case means that the *ex post* calculated capital input prices can be assumed as *ex ante* given to the production unit, so that they can be considered as exogenous data for the unit's profit maximization problem.

This account is, of course, somewhat stylized, since there occur many, smaller or larger, variations on this theme in the literature. Recurring, however, are a number of so-called neo-classical assumptions: (1) a technology that exhibits constant returns to scale; (2) competitive input and output markets; (3) optimizing behavior; and (4) perfect foresight. A fine example from academia is provided by Jorgenson *et al*. (2005, pp. 23, 37), while the Sources and Methods publication of Statistics New Zealand (2006) shows that the neo-classical model has also deeply invaded official statistical agencies.[1] Another interesting example where neo-classical assumptions have invaded the measurement system is the World Productivity Database of the United Nations Industrial Development Organisation (UNIDO); see Isaksson (2009).

An interesting position is taken by the EU KLEMS Growth and Productivity Accounts project. Though in their main text Timmer *et al*. (2007) adhere to the Jorgenson *et al*. framework, there is a curious footnote:

> Under strict neo-classical assumptions, MFP [multifactor productivity] growth measures disembodied technological change. *In practice* [my emphasis], MFP is derived as a residual and includes a host of effects such as improvements in allocative and technical efficiency, changes in returns to scale and mark-ups as well as technological change proper. All these effects can be broadly summarized as "improvements in efficiency," as they improve the productivity with which inputs are being used in the production process. In addition, being a residual measure MFP growth also includes measurement errors and the effects from unmeasured output and inputs.

There are more examples of authors who exhibit similar concerns, without, however, feeling the need to adapt their conceptual framework.

I believe that for an official statistical agency, whose main task it is to provide statistics to many different users for many different purposes, it is discomforting to have such strong and often empirically refuted, assumptions built into the methodological foundations of productivity and growth accounting statistics. This especially applies to the behavioral assumptions numbered 2, 3, and 4. There is ample evidence that, on average, markets are not precisely competitive; that producers' decisions frequently turn out to be less than optimal; and that managers almost invariably lack the magical feature of perfect foresight. Moreover, the environment in which production units operate is not so stable as the assumption of a fixed production function seems to claim.

But I also believe that it is possible, and even advisable, to avoid making such assumptions. In a sense I propose to start where the usual story ends, namely at the

[1]The neo-classical model figured already prominently in the 1979 report of the U.S. National Research Council's Panel to Review Productivity Statistics (Rees, 1979). An overview of national and international practice is provided by the regularly updated *OECD Compendium of Productivity Indicators*, available at www.oecd.org/statistics/productivity.

empirical side.[2] For any production unit, the total factor productivity index is then *defined* as an output quantity index divided by an input quantity index. There are various options here, depending on what one sees as input and output, but the basic feature is that, given price and quantity (or value) data, this is simply a matter of index construction. There appear to be no behavioral assumptions involved, and this even applies—as will be demonstrated—to the construction of capital input prices. Surely, a number of imputations must be made (as in the case of the self-employed workers) and there is fairly large number of more or less defendable assumptions involved (for instance on the depreciation rates of capital assets), but this belongs to the daily bread and butter of economic statisticians.

In my view, structural as well as behavioral assumptions enter the picture as soon as it comes to the *explanation* of productivity change. Then there are, depending on the initial level of aggregation, two main directions: (1) to explain productivity change at an aggregate level by productivity change and other factors operating at lower levels of aggregation; (2) to decompose productivity change into factors such as technological change, technical efficiency change, scale effects, input- and output-mix effects, and chance. In this case, to proceed with the analysis one cannot sidestep a technology model with certain specifications.

The contents of this paper are as follows. Section 2 outlines the architecture of the basic, KLEMS-Y, input–output model, with its total and partial measures of productivity change. This section also links productivity measurement and growth accounting. Section 3 proceeds with the KL-VA and K-CF models. Four additional input–output models are briefly introduced in Section 4. This section also contains a comparison of all the models. Section 5 introduces the capital utilization rate. Section 6 concludes by discussing the main decomposition methods.

2. The Basic Input–Output Model

Let us consider a single production unit. This could be an establishment or plant, a firm, an industry, a sector, or even an entire economy. I will simply speak of a "unit." For the purpose of productivity measurement, such a unit is considered as a (consolidated) input–output system. What does this mean?

For the output side as well as for the input side there is some list of commodities (according to some classification scheme). A commodity is thereby defined as a set of closely related items (goods or services) which, for the purpose of analysis, can be considered as "equivalent," either in the static sense of their quantities being additive or in the dynamic sense of displaying equal relative price or quantity changes. Ideally, then, for any accounting period considered (*ex post*), say a year, each commodity comes with a value (in monetary terms) and a price and/or a quantity. If value and price are available, then the quantity is obtained by dividing the value by the price. If value and quantity are available, then the price is obtained

[2]There is another, minor, difference between my approach and the usual story. The usual story runs in the framework of continuous time in which periods are of infinitesimal short duration. When it then comes to implementation several approximations must be assumed. My approach does not need this kind of assumptions either, because this approach is entirely based on accounting periods of finite duration, such as years.

by dividing the value by the quantity. If both price and quantity are available, then value is defined as price times quantity. In any case, for every commodity it must be so that value equals price times quantity, the magnitudes of which of course must pertain to the same accounting period. Technically speaking, the price concept used here is the unit value. At the output side, the prices must be those received by the unit, whereas at the input side, the prices must be those paid. Consolidation (also called net-sector approach) means that the unit does not deliver to itself. Put otherwise, all the intra-unit deliveries are netted out.

The situation as pictured in the preceding paragraph is typical for a unit operating on a (output) market. The question how to deal with non-market units will be considered where appropriate.

The inputs are customarily classified according to the KLEMS format. The letter K denotes the class of owned, reproducible capital assets. The commodities here are the asset-types, sub-classified by age category. Cohorts of assets are assumed to be available at the beginning of the accounting period and, in deteriorated form (due to ageing, wear and tear), still available at the end of the period. Investment during the period adds entities to these cohorts, while desinvestment, breakdown, or retirement remove entities. Examples include buildings and other structures, land, machinery, transport, ICT equipment, and tools. Theory implies that quantities sought are just the quantities of all these cohorts of assets (together representing the productive capital stock), whereas the relevant prices are their unit user costs (per type–age combination), constructed from imputed interest rates, depreciation profiles, (anticipated) revaluations, and tax rates. The sum of quantities times prices then provides the capital input cost of a production unit.[3]

The letter L denotes the class of labor inputs; that is, all the types of work that are important to distinguish, cross-classified for instance according to educational attainment, gender, and experience (which is usually proxied by age categories). Quantities are measured as hours worked (or paid), and prices are the corresponding wage rates per hour. Where applicable, imputations must be made for the work executed by self-employed persons. The sum of quantities times prices provides the labor input cost (or the labor bill, or labor compensation, as it is sometimes called).[4]

The classes K and L concern so-called primary inputs. The letters E, M, and S denote three, disjunct classes of so-called intermediate inputs. First, E is the class of energy commodities consumed by a production unit: oil, gas, electricity, and water. Second, M is the class of all the (physical) materials consumed in the production process, which could be sub-classified into raw materials, semi-fabricates, and auxiliary products. Third, S is the class of all the business services which are consumed for maintaining the production process. This includes the services of leased capital assets and outsourced activities. Though it is not at all a

[3]The productive capital stock may be underutilized, which implies that not all the capital costs are incurred in actual production. See Schreyer (2001, section 5.6) for a general discussion of this issue. For a treatment in the neo-classical framework the reader is referred to Berndt and Fuss (1986), Hulten (1986), and Morrison Paul (1999). We return to this issue later.

[4]The utilization rate of the labor input factors is assumed to be 1. Over- or under-utilization from the point of view of jobs or persons is reflected in the wage rates.

trivial task to define precisely all the intermediate inputs and to classify them, it can safely be assumed that at the end of each accounting period there is a quantity and a price associated with each of those inputs.

Then, for each accounting period, production cost is defined as the sum of primary and intermediate input cost. Though this is usually not executed, there are good reasons to exclude R&D expenditure from production cost, the reason being that such expenditure is not related to the current production process but to a future one. Put otherwise, by performing R&D, production units try to shift the technology frontier. When it then comes to explaining productivity change, the non-exclusion of R&D expenditure might easily lead to a sort of double-counting error.[5]

At the output side, the letter Y denotes the class of commodities, goods, and/or services, which are produced by the unit. Though in some industries, such as services industries or industries producing mainly unique goods, definitional problems are formidable, it can safely be assumed that for each accounting period there are data on quantities produced. For units operating on the market there are also prices. The sum of quantities times prices then provides the production revenue, and, apart from taxes on production, revenue minus cost yields profit.

There is, of course, discussion possible about what to include or exclude at the input and output sides. We are here more or less tacitly assuming a broad production viewpoint, where for instance marketing services are included in the set S. A broader viewpoint would take into account sales and uses from inventories.

Profit is an important financial performance measure. A somewhat less obvious, but equally useful, measure is "profitability," defined as revenue *divided* by cost. Profitability gives, in monetary terms, the quantity of output per unit of input, and is thus a measure of return to aggregate input (and in some older literature called "return to the dollar").

Monitoring the unit's performance over time is here understood to mean monitoring the development of its profit or its profitability. Both measures are, by nature, dependent on price and quantity changes, at the two sides of the unit. If there is (price) inflation and the unit's profit has increased, then that mere fact does not necessarily mean that the unit has been performing better. Also, though general inflation does not influence the development of profitability, differential inflation does. If output prices have increased more than input prices, then any increase of profitability does not necessarily imply that the unit has been performing better. Thus, for measuring the economic performance of the unit one wants to remove the effect of price changes, irrespective of whether those prices are within or beyond the unit's control.

Profit and profitability are different concepts. The first is a difference measure, the second is a ratio measure. Change of a variable through time, which will be our main focus, can also be measured by a difference or a ratio. It is important to realize that, apart from technical details—such as, that a ratio does not make sense if the variable changes sign or becomes equal to zero—these two ways of measur-

[5]See Diewert and Huang (2008) for more on this issue. A big problem seems to be the separation of the R&D part of labor input.

ing change are equivalent. Thus there appear to be a number of ways of mapping the same reality in numbers, but differing numbers do not necessarily imply differing realities.[6]

Profit change stripped of its price component will be called *real* profit change, and profitability change stripped of its price component will be called *real* profitability change.[7] Another name for real profit (-ability) change is (total factor) productivity change. Thus, productivity change can be measured as a ratio (namely as real profitability change) or as a difference (namely as real profit change). At the economy level, productivity change can be related to some measure of overall welfare change. A down-to-earth approach would use the National Accounts to establish a link between labor productivity change and real-income-per-capita change. A more sophisticated approach, using economic models and assumptions, was provided by Basu and Fernald (2002).

For a non-market unit the story must be told somewhat differently. For such a unit there are no output prices; hence, there is no revenue. Though there is cost, like for market units, there is no profit or profitability. National accountants usually resolve the problem here by *defining* the revenue of a non-market unit to be equal to its cost, thereby setting profit equal to 0 or profitability equal to 1.[8] But this leaves the problem that there is no natural way of splitting revenue change through time in real and monetary components. This can only be done satisfactorily when there is some output quantity index that is independent from the input quantity index.[9]

It is useful to remind the reader that the notions of profit and profitability, though conceptually rather clear, are difficult to operationalize. One of the reasons is that cost includes the cost of owned capital assets, the measurement of which exhibits a substantial number of degrees of freedom, as we will see in the remainder of this paper. Also, labor cost includes the cost of self-employed persons, for which wage rates and hours of work usually must be imputed. It will be clear that all these, and many other uncertainties spill over to operational definitions of the profit and profitability concepts.

2.1. *Notation*

Let us now introduce some notation to define the various concepts we are going to use. As stated, at the output side we have M items, each with their price (received) p_m^t and quantity y_m^t, where $m = 1, \ldots, M$, and t denotes an accounting period (say, a year). Similarly, at the input side we have N items, each with their price (paid) w_n^t and quantity x_n^t, where $n = 1, \ldots, N$. To avoid notational clutter, simple vector notation will be used throughout. All the prices and quantities are assumed to be positive, unless stated otherwise. The *ex post* accounting point-of-view will be used; that is, quantities and monetary values of the so-called flow

[6]It is easy to see, for example, that increasing profit can occur simultaneously with decreasing profitability.

[7]Note that real change means nominal change deflated by some price index, not necessarily being a (headline) CPI. "Stripping" is of course a vague term, and a more precise definition will be given later.

[8]This approach goes back to Hicks (1940).

[9]See the insightful paper by Douglas (2006). Though written from a New Zealand perspective, its theme is generic.

variables (output and labor, energy, materials, services inputs) are realized values, complete knowledge of which becomes available after the accounting period has expired. Similarly, the cost of capital input is calculated *ex post*. This is consistent with official statistical practice.

The unit's revenue, that is, the value of its (gross) output, during the accounting period t is defined as

$$R^t \equiv p^t \cdot y^t \equiv \sum_{m=1}^{M} p_m^t y_m^t, \tag{1}$$

whereas its production cost is defined as

$$C^t \equiv w^t \cdot x^t \equiv \sum_{n=1}^{M} w_n^t x_n^t. \tag{2}$$

The unit's profit (disregarding taxes on production) is then given by its revenue minus its cost; that is,

$$\Pi^t \equiv R^t - C^t = p^t \cdot y^t - w^t \cdot x^t. \tag{3}$$

The unit's profitability (also disregarding taxes on production) is defined as its revenue divided by its cost; that is,

$$R^t / C^t = p^t \cdot y^t / w^t \cdot x^t. \tag{4}$$

Notice that profitability expressed as a percentage ($R^t/C^t - 1$) equals the ratio of profit to cost (Π^t/C^t). In some circles this is called the margin of the unit. Given positive prices and quantities, it will always be the case that $R^t > 0$ and $C^t > 0$. Thus, profitability R^t/C^t is always positive, but profit Π^t can be positive, negative, or zero.

As stated, we are concerned with intertemporal comparisons. Moreover, in this paper only bilateral comparisons will be considered, say comparing a certain period t to another, adjacent or non-adjacent, period t'. Without loss of generality it may be assumed that period t' precedes period t. To further simplify notation, the two periods will be labeled by $t = 1$ (which will be called the comparison period) and $t' = 0$ (which will be called the base period).

2.2. *Productivity Index*

The development over time of profitability is, rather naturally, measured by the ratio $(R^1/C^1)/(R^0/C^0)$. How to decompose this into a price and a quantity component? By noticing that

$$\frac{R^1/C^1}{R^0/C^0} = \frac{R^1/R^0}{C^1/C^0} \tag{5}$$

we see that the question reduces to the question how to decompose the revenue ratio R^1/R^0 and the cost ratio C^1/C^0 into two parts. The natural answer is to grab from the economic statistician's toolkit a pair of price and quantity indices that satisfy the Product Test:

$$(6) \qquad \frac{p^1 \cdot y^1}{p^0 \cdot y^0} = P(p^1, y^1, p^0, y^0)Q(p^1, y^1, p^0, y^0).$$

A good choice is the Fisher price and quantity index, since these indices satisfy not only the basic axioms of price and quantity measurement, but also a number of other relatively important requirements (such as the Time Reversal Test). Thus we are using here the "instrumental" or "axiomatic" approach for selecting measures for aggregate price and quantity change, an approach that goes back to Fisher (1922); see Balk (1995) for a survey and Balk (2008) for an up-to-date treatment. When the temporal distance between periods 1 and 0 is not too large, any index that is a second-order differential approximation to the Fisher index may instead be used.[10]

Throughout this paper, when it comes to solving problems such as (6), we will assume that Fisher indices are used. Thus, in particular,

$$(7) \qquad \begin{aligned} \frac{R^1}{R^0} &= P^F(p^1, y^1, p^0, y^0)Q^F(p^1, y^1, p^0, y^0) \\ &\equiv P_R(1,0)Q_R(1,0), \end{aligned}$$

where the second line serves to define our shorthand notation. In the same way we decompose

$$(8) \qquad \begin{aligned} \frac{C^1}{C^0} &= P^F(w^1, x^1, w^0, x^0)Q^F(w^1, x^1, w^0, x^0) \\ &\equiv P_C(1,0)Q_C(1,0). \end{aligned}$$

Of course, the dimensionality of the indices in expressions (7) and (8) will usually be different. The subscripts R and C are used because, as will appear later, there are more output and input concepts.

The number of items distinguished at the output side (M) and the input side (N) of a production unit can be very high. To accommodate this, (detailed) classifications are used, by which all the items are allocated to hierarchically organized (sub-)aggregates. The calculation of output and input indices then proceeds in stages. Theoretically, it suffices to distinguish only two stages. At the first stage one calculates indices for the subaggregates at some level, and at the second stage these subaggregate indices are combined to aggregate indices.

Consequentially, in expressions (7) and (8) instead of one-stage, two-stage Fisher indices may also be used; that is, Fisher indices of Fisher indices for

[10]Note, however, that this is not unproblematic. For instance, when the Törnqvist price index $P^T(.)$ is used, then the implicit quantity index $(p^1 \cdot y^1/p^0 \cdot y^0)/P^T(.)$ does not necessarily satisfy the Identity Test. The Identity Test for a quantity index prescribes that such an index equals unity whenever quantities have not changed.

subaggregates (see Appendix A for precise definitions). Since the Fisher index is not consistent-in-aggregation, a decomposition by two-stage Fisher indices will in general numerically differ from a decomposition by one-stage Fisher indices. Fortunately, one-stage and two-stage Fisher indices are second-order differential approximations of each other (as shown by Diewert, 1978).

Using the two relations (7) and (8), the profitability ratio can be decomposed as

$$\frac{R^1/C^1}{R^0/C^0} = \frac{R^1/R^0}{C^1/C^0} = \frac{P_R(1,0)}{P_C(1,0)}\frac{Q_R(1,0)}{Q_C(1,0)}. \tag{9}$$

The (total factor) productivity index ($IPROD$), for period 1 relative to period 0, is now *defined* by

$$IPROD(1,0) \equiv \frac{Q_R(1,0)}{Q_C(1,0)}. \tag{10}$$

Thus $IPROD(1, 0)$ is the real or quantity component of the profitability ratio. Put otherwise, it is the ratio of an output quantity index to an input quantity index; $IPROD(1, 0)$ is the factor with which the output quantities on average have changed relative to the factor with which the input quantities on average have changed. If the ratio of these factors is larger (smaller) than 1, there is said to be productivity increase (decrease).[11]

Notice that, using (7) and (8), there appear to be three other, equivalent representations of the productivity index, namely

$$IPROD(1,0) = \frac{(R^1/R^0)/P_R(1,0)}{(C^1/C^0)/P_C(1,0)} \tag{11}$$

$$= \frac{(R^1/R^0)/P_R(1,0)}{Q_C(1,0)} \tag{12}$$

$$= \frac{Q_R(1,0)}{(C^1/C^0)/P_C(1,0)}. \tag{13}$$

Put in words, we are seeing here respectively a deflated revenue index divided by a deflated cost index, a deflated revenue index divided by an input quantity index, and an output quantity index divided by a deflated cost index. We will return to these expressions shortly.

Further, if the revenue change equals the cost change, $R^1/R^0 = C^1/C^0$ (for which zero profit in the two periods is a sufficient condition), then it follows that

[11]This approach follows Diewert (1992), Diewert and Nakamura (2003), and Balk (2003a).

$$IPROD(1,0) = \frac{P_C(1,0)}{P_R(1,0)}; \tag{14}$$

that is, the productivity index is equal to an input price index divided by an output price index. In general, however, the *dual* productivity index $P_C(1, 0)/P_R(1, 0)$ will differ from the *primal* one, $Q_R(1, 0)/Q_C(1, 0)$.

For a non-market unit expression (10) cannot be used because there are no output prices available for use in the output quantity index. But if there is some prices-free output quantity index $Q(y^1, y^0)$, then the (total factor) productivity index, for period 1 relative to period 0, is naturally defined by $Q(y^1, y^0)/Q_C(1, 0)$. An alternative expression is obtained by replacing the input quantity index by the deflated cost index, $Q(y^1, y^0)/[(C^1/C^0)/P_C(1, 0)]$.

2.3. *Growth Accounting*

The foregoing definitions are already sufficient to provide examples of simple but useful analysis. Consider relation (12), and rewrite this as

$$R^1/R^0 = IPROD(1,0) \times Q_C(1,0) \times P_R(1,0). \tag{15}$$

Taking logarithms, one obtains

$$\ln(R^1/R^0) = \ln IPROD(1,0) + \ln Q_C(1,0) + \ln P_R(1,0). \tag{16}$$

This relation, implemented with Fisher and Törnqvist indices, was used by Dumagan and Ball (2009) for an analysis of the U.S. agricultural sector.

Recall that revenue change through time is only interesting in so far as it differs from general inflation. Hence, it makes sense to deflate the revenue ratio, R^1/R^0, by a general inflation measure such as the (headline) Consumer Price Index (CPI). Doing this, the last equation can be written as

$$\ln\left(\frac{R^1/R^0}{CPI^1/CPI^0}\right) = \ln IPROD(1,0) + \ln Q_C(1,0) + \ln\left(\frac{P_R(1,0)}{CPI^1/CPI^0}\right). \tag{17}$$

Lawrence *et al.* (2006) basically used this relation to decompose "real" revenue change into three factors: productivity change, input quantity change (which can be interpreted as measuring change of the unit's size), and "real" output price change respectively.

Our second example follows from rearranging expression (13) and taking logarithms. This delivers the following relation:

$$\ln(C^1/C^0) = \ln P_C(1,0) + \ln Q_R(1,0) - \ln IPROD(1,0). \tag{18}$$

This relation was also used by Dumagan and Ball (2009).[12] A further rearrangement gives

[12]Note that it is not necessary to assume that $R^t = C^t$ ($t = 0, 1$) as Dumagan and Ball did.

(19) $$\ln\left(\frac{C^1/C^0}{Q_R(1,0)}\right) = \ln P_C(1,0) - \ln IPROD(1,0).$$

We see here that the growth rate of average cost can be decomposed into two factors, namely the growth rate of input prices and a residual which is the negative of productivity growth. Put otherwise, in the case of stable input prices the growth rate of average cost is equal to minus the productivity growth rate.

Our third example follows from rearranging expression (11) and taking logarithms. This delivers the following interesting relation:

(20) $$\ln P_R(1,0) = \ln\left(\frac{1+\mu^1}{1+\mu^0}\right) + \ln P_C(1,0) - \ln IPROD(1,0),$$

where $\mu^t \equiv R^t/C^t - 1$. This relation analyses output price change as resulting from three factors: change of the margin $1 + \mu^t$, input price change, and, with a negative sign, productivity change.

All these are examples of what is called *growth accounting*. The relation between index number techniques and growth accounting techniques can, more generally, be seen as follows. Recall the generic definition (10), and rewrite this expression as follows:

(21) $$Q_R(1,0) = IPROD(1,0) \times Q_C(1,0).$$

Taking logarithms, this multiplicative expression can be rewritten as

(22) $$\ln Q_R(1,0) = \ln IPROD(1,0) + \ln Q_C(1,0).$$

For index numbers in the neighborhood of 1 the logarithms thereof reduce to percentages, and the last expression can be interpreted as saying that the percentage change of output volume equals the percentage change of input volume plus the percentage change of productivity. Growth accounting economists like to work with equations expressing output volume growth in terms of input volume growth plus a residual that is interpreted as productivity growth, thereby suggesting that the last two factors cause the first. However, productivity change cannot be considered as an independent factor since it is *defined* as output quantity change minus input quantity change. Put otherwise, a growth accounting table is nothing but an alternative way of presenting productivity growth and its contributing factors. And decomposition does not imply anything about causality.[13]

2.4. *Productivity Indicator*

Let us now turn to profit and its development through time. This is naturally measured by the difference $\Pi^1 - \Pi^0$. Of course, such a difference only makes sense

[13]Thus, saying that output growth outpaced input growth because TFP increased is "like saying that the sun rose because it was morning," to paraphrase Friedman (1988, p. 58). Of course, when TFP change is decomposed into factors such as technological change or efficiency change, and one is able to measure such factors independently, more can be said.

when the two money amounts involved, profit from period 0 and profit from period 1, are deflated by some general inflation measure (such as the headline CPI). In the remainder of this paper, when discussing difference measures, such a deflation is tacitly presupposed.

How to decompose the profit difference into a price and a quantity component? By noticing that

$$\Pi^1 - \Pi^0 = (R^1 - R^0) - (C^1 - C^0), \tag{23}$$

we see that the question reduces to the question how to decompose revenue change $R^1 - R^0$ and cost change $C^1 - C^0$ into two parts. We now grab from the economic statistician's toolkit a pair of price and quantity indicators that satisfy the analog of the Product Test:

$$p^1 \cdot y^1 - p^0 \cdot y^0 = \mathcal{P}(p^1, y^1, p^0, y^0) + \mathcal{Q}(p^1, y^1, p^0, y^0). \tag{24}$$

A good choice is the Bennet (1920) price and quantity indicator, since these indicators satisfy not only the basic axioms (see Appendix A), but also a number of other relatively important requirements (such as the Time Reversal Test) (see Diewert 2005). But any indicator that is a second-order differential approximation to the Bennet indicator may instead be used. Thus,

$$\begin{aligned} R^1 - R^0 &= \mathcal{P}^B(p^1, y^1, p^0, y^0) + \mathcal{Q}^B(p^1, y^1, p^0, y^0) \\ &\equiv \mathcal{P}_R(1, 0) + \mathcal{Q}_R(1, 0), \end{aligned} \tag{25}$$

and similarly,

$$\begin{aligned} C^1 - C^0 &= \mathcal{P}^B(w^1, x^1, w^0, x^0) + \mathcal{Q}^B(w^1, x^1, w^0, x^0) \\ &\equiv \mathcal{P}_C(1, 0) + \mathcal{Q}_C(1, 0). \end{aligned} \tag{26}$$

Notice that the dimensionality of the Bennet indicators in these two decompositions is in general different.

The Bennet indicators are difference analogs to Fisher indices. Their aggregation properties, however, are much simpler. The Bennet price or quantity indicator for an aggregate is equal to the sum of the subaggregate indicators.

Using indicators, the profit difference can be written as

$$\begin{aligned} \Pi^1 - \Pi^0 &= \mathcal{P}_R(1, 0) + \mathcal{Q}_R(1, 0) - [\mathcal{P}_C(1, 0) + \mathcal{Q}_C(1, 0)] \\ &= \mathcal{P}_R(1, 0) - \mathcal{P}_C(1, 0) + \mathcal{Q}_R(1, 0) - \mathcal{Q}_C(1, 0). \end{aligned} \tag{27}$$

The first two terms at the right-hand side of the last equality sign provide the price component, whereas the last two terms provide the quantity component of the profit difference. Thus, based on this decomposition, the (total factor) productivity indicator (*DPROD*) is *defined* by

$$DPROD(1, 0) \equiv \mathcal{Q}_R(1, 0) - \mathcal{Q}_C(1, 0); \tag{28}$$

that is, an output quantity indicator minus an input quantity indicator. Notice that productivity change is now measured as an amount of money. An amount larger (smaller) than 0 indicates productivity increase (decrease).[14]

The equivalent expressions for difference-type productivity change are

(29) $$DPROD(1,0) = [R^1 - R^0 - \mathcal{P}_R(1,0)] - [C^1 - C^0 - \mathcal{P}_C(1,0)]$$

(30) $$= [R^1 - R^0 - \mathcal{P}_R(1,0)] - \mathcal{Q}_C(1,0)$$

(31) $$= \mathcal{Q}_R(1,0) - [C^1 - C^0 - \mathcal{P}_C(1,0)],$$

which can be useful in different situations. Notice further that, if $R^1 - R^0 = C^1 - C^0$ then

(32) $$DPROD(1,0) = \mathcal{P}_C(1,0) - \mathcal{P}_R(1,0).$$

For a non-market production unit, a productivity indicator is difficult to define. Though one might be able to construe an output quantity indicator, it is hard to see how, in the absence of output prices, such an indicator could be given a money dimension.

Any productivity indicator can be transformed into an index and vice versa by using the logarithmic mean (see Balk, 2008, pp. 128–9). For example, the index corresponding to (28) is given by

(33) $$\exp\left\{\frac{\mathcal{Q}_R(1,0)}{L(R^1,R^0)} - \frac{\mathcal{Q}_C(1,0)}{L(C^1,C^0)}\right\},$$

where $L(a, b)$ is the logarithmic mean.[15] Note, however, that this index differs from $IPROD(1, 0)$ as a ratio of Fisher indices.

2.5. *Partial Productivity Measures*

The productivity index $IPROD(1, 0)$ and indicator $DPROD(1, 0)$ bear the adjective "total factor" because all the inputs are taken into account. To define partial productivity measures, in ratio or difference form, additional notation is necessary.

All the items at the input side of our production unit are assumed to be allocatable to the five, mutually disjunct, categories mentioned earlier, namely capital (K), labor (L), energy (E), materials (M), and services (S). The entire input price and quantity vectors can then be partitioned as $w^t = (w_K^t, w_L^t, w_E^t, w_M^t, w_S^t)$ and $x^t = (x_K^t, x_L^t, x_E^t, x_M^t, x_S^t)$, respectively. Energy, materials, and services together form the category of intermediate inputs, that is, inputs which are acquired from other production units or imported. Capital and labor are called primary inputs.

[14]This approach follows Balk (2003a).

[15]For any two strictly positive real numbers a and b their logarithmic mean is defined by $L(a, b) = (a - b)/\ln(a/b)$ if $a \neq b$ and $L(a, a) = a$. The properties of this mean are discussed in Balk (2008).

Consistent with this distinction the price and quantity vectors can also be partitioned as $w^t = (w^t_{KL}, w^t_{EMS})$ and $x^t = (x^t_{KL}, x^t_{EMS})$, or as $w^t = (w^t_K, w^t_L, w^t_{EMS})$ and $x^t = (x^t_K, x^t_L, x^t_{EMS})$. Since monetary values are additive, total production cost can be decomposed in a number of ways, such as

$$
\begin{aligned}
(34) \qquad C^t &= \sum_{n\in K} w^t_n x^t_n + \sum_{n\in L} w^t_n x^t_n + \sum_{n\in E} w^t_n x^t_n + \sum_{n\in M} w^t_n x^t_n + \sum_{n\in S} w^t_n x^t_n \\
&\equiv C^t_K + C^t_L + C^t_E + C^t_M + C^t_S \\
&\equiv C^t_K + C^t_L + C^t_{EMS} \\
&\equiv C^t_{KL} + C^t_{EMS}.
\end{aligned}
$$

Now, using as before Fisher indices, the labor cost ratio can be decomposed as

$$
\begin{aligned}
(35) \qquad \frac{C^1_L}{C^0_L} &= P^F(w^1_L, x^1_L, w^0_L, x^0_L) Q^F(w^1_L, x^1_L, w^0_L, x^0_L) \\
&\equiv P_L(1,0) Q_L(1,0).
\end{aligned}
$$

Then the labor productivity index (*ILPROD*) for period 1 relative to period 0 is defined by

$$(36) \qquad ILPROD(1,0) \equiv \frac{Q_R(1,0)}{Q_L(1,0)};$$

that is, the ratio of an output quantity index to a labor input quantity index. Notice that usually the labor productivity index is defined by specifying the labor input quantity index to be the Dutot or simple sum quantity index $Q^D(w^1_L, x^1_L, w^0_L, x^0_L) \equiv \sum_{n\in L} x^1_n / \sum_{n\in L} x^0_n$. The ratio $Q^F(w^1_L, x^1_L, w^0_L, x^0_L) / Q^D(w^1_L, x^1_L, w^0_L, x^0_L)$ is then said to measure the shift in labor quality or composition.

In precisely the same way one can define the capital productivity index

$$(37) \qquad IKPROD(1,0) \equiv \frac{Q_R(1,0)}{Q_K(1,0)}$$

and the other partial productivity indices *IkPROD* for $k = E, M, S$. The ratio

$$(38) \qquad \frac{ILPROD(1,0)}{IKPROD(1,0)} = \frac{Q_K(1,0)}{Q_L(1,0)}$$

is called the index of "capital deepening." Loosely speaking, this index measures the change of the quantity of capital input per unit of labor input.

The relation between total factor and partial productivity indices is as follows. Let $Q_C(1, 0)$ be a two-stage Fisher index, that is,

$$(39) \qquad Q_C(1,0) \equiv Q^F(Q_k(1,0), C^1_k, C^0_k; k = K, L, E, M, S)$$

where all the $Q_k(1, 0)$ are Fisher indices. It is straightforward to check that then

$$(40)\quad IPROD(1,0) = \frac{Q_R(1,0)}{Q_C(1,0)}$$
$$= \frac{Q_R(1,0)}{\left(\sum_k Q_k(1,0)C_k^0/C^0\right)^{1/2}\left(\sum_k Q_k(1,0)^{-1}C_k^1/C^1\right)^{-1/2}}$$
$$= \left(\sum_k \frac{Q_k(1,0)}{Q_R(1,0)}\frac{C_k^0}{C^0}\right)^{-1/2}\left(\sum_k \frac{Q_R(1,0)}{Q_k(1,0)}\frac{C_k^1}{C^1}\right)^{1/2}$$
$$= \left(\frac{\sum_k C_k^0(IkPROD(1,0))^{-1}}{C^0}\right)^{-1/2}\left(\frac{\sum_k C_k^1\, IkPROD(1,0)}{C^1}\right)^{1/2},$$

which is not a particularly simple relation. If instead as second-stage quantity index the Cobb–Douglas functional form was chosen, that is,

$$(41)\qquad Q_C(1,0) \equiv \prod_k Q_k(1,0)^{\alpha_k} \text{ where } \prod_k \alpha_k = 1\,(\alpha_k > 0),$$

then it appears that

$$(42)\qquad \ln IPROD(1,0) \equiv \prod_k \alpha_k \ln IkPROD(1,0).$$

This is a very simple relation between total factor productivity change and partial productivity change. Notice, however, that this simplicity comes at a cost. Definition (41) implies for the relation between aggregate and subaggregate input price indices that

$$(43)\qquad P_C(1,0) = \prod_k P_k(1,0)^{\alpha_k}\frac{C^1/C^0}{\prod_k\left(C_k^1/C_k^0\right)^{\alpha_k}}.$$

Such an index does not necessarily satisfy the fundamental Identity Test; that is, if all the prices in period 1 are the same as in period 0 then $P_C(1, 0)$ does not necessarily deliver as outcome 1.

Let us now turn to partial productivity *indicators*. Using the Bennet indicators, the labor cost difference between periods 0 and 1 is decomposed as

$$(44)\qquad C_L^1 - C_L^0 = \mathcal{P}^B\left(w_L^1, x_L^1, w_L^0, x_L^0\right) + \mathcal{Q}^B\left(w_L^1, x_L^1, w_L^0, x_L^0\right) \equiv \mathcal{P}_L(1,0) + \mathcal{Q}_L(1,0).$$

In the same way one can decompose the capital, energy, materials, and services cost difference. However, since costs are additive, it turns out that the total factor productivity indicator can be written as

$$(45)\qquad DPROD(1,0) = \mathcal{Q}_R(1,0) - \sum_{k=K,L,E,M,S}\mathcal{Q}_k(1,0).$$

By definition, the left-hand side is real profit change. The right-hand side gives the contributing factors. The contribution of category k to real profit change is simply measured by the amount $Q_k(1, 0)$. A positive amount, which means that the aggregate quantity of input category k has increased, means a negative contribution to real profit change.

3. Different Models, Similar Measures

The previous section laid out the basic features of what is known as the KLEMS model of production. This framework is currently used by the U.S. Bureau of Labor Statistics and Statistics Canada for productivity measures at the industry level of aggregation (see Dean and Harper, 2001, and Harchaoui *et al.*, 2001, respectively). The KLEMS model, or, as I will denote it, the KLEMS-Y model delivers gross-output based total or partial productivity measures. However, there are more models in use, differing from the KLEMS-Y model by their input and output concepts. Since these models presuppose revenue as measured independently from cost, they are not applicable to non-market units.

3.1. *The KL-VA Model*

The first of these models uses value added (VA) as its output concept. The production unit's value added (VA) is defined as its revenue minus the costs of energy, materials, and services; that is,

$$(46)\qquad \begin{aligned} VA^t &\equiv R^t - C^t_{EMS} \\ &= p^t \cdot y^t - w^t_{EMS} \cdot x^t_{EMS}. \end{aligned}$$

The value-added concept subtracts the total cost of intermediate inputs from the revenue obtained, and in doing so essentially conceives the unit as producing value added (that is, money) from the two primary input categories capital and labor. It is assumed that $VA^t > 0$.[16]

Although gross output, represented by y^t, is the natural output concept, the value-added concept is important when one wants to aggregate single units to larger entities. Gross output consists of deliveries to final demand and intermediate destinations. The split between these two output categories depends very much on the level of aggregation. Value added is immune to this problem. It enables one to compare (units belonging to) different industries. From a welfare-theoretic point of view the value-added concept is important because value added can be conceived as the income (from production) that flows into society.[17]

In this input–output model the counterpart to *profitability* is the ratio of value added to primary inputs cost, VA^t/C^t_{KL}, and the natural starting point for defining

[16]An early advocate of the value added output concept was Burns (1930). Specifically, he favored what later in this paper will be defined as net value added. Burns was aware of the possibility that for very narrowly defined production units and small time periods value added may become non-positive.

[17]In between the KLEMS-Y model and the KL-VA model figures the KLE″M″S-Margin model, applicable to distributive trade units. Here the set of material inputs M is split into two parts, M' denoting the goods for resale and M'' the auxiliary materials. Likewise E, the set of energy inputs, is split into E' and E''. The Margin is then defined as $R^t - C^t_{E' \cup M'}$. See Inklaar and Timmer (2008).

a productivity index is to consider the development of this ratio through time. Since $(VA^1/C^t_{KL})/(VA^0/C^0_{KL}) = (VA^1/VA^0)/(C^1_{KL}/C^0_{KL})$, we need a decomposition of the value-added ratio and a decomposition of the primary inputs cost ratio.

The question how to decompose a value-added ratio in a price and a quantity component cannot be answered unequivocally. There are several options here, the technical details of which are deferred to Appendix B. Suppose, however, that a satisfactory decomposition is somehow available; that is,

$$\text{(47)} \qquad \frac{VA^1}{VA^0} = P_{VA}(1,0)Q_{VA}(1,0).$$

Using one- or two-stage Fisher indices, the primary inputs cost ratio is decomposed as

$$\text{(48)} \qquad \begin{aligned} \frac{C^1_{KL}}{C^0_{KL}} &= P^F(w^1_{KL}, x^1_{KL}, w^0_{KL}, x^0_{KL})Q^F(w^1_{KL}, x^1_{KL}, w^0_{KL}, x^0_{KL}) \\ &\equiv P_{KL}(1,0)Q_{KL}(1,0). \end{aligned}$$

The value-added based (total factor) productivity index for period 1 relative to period 0 is then defined as

$$\text{(49)} \qquad IPROD_{VA}(1,0) \equiv \frac{Q_{VA}(1,0)}{Q_{KL}(1,0)}.$$

This index measures the "quantity" change of value added relative to the quantity change of primary input; or, can be seen as the index of real value added relative to the index of real primary input.

This is by far the most common model. It is used by the U.S. Bureau of Labor Statistics, Statistics Canada, Australian Bureau of Statistics, Statistics New Zealand, and the Swiss Federal Statistical Office in their official productivity statistics.

In the KL-VA model the counterpart to *profit* is the difference of value added and primary inputs cost, $VA^t - C^t_{KL}$, and the natural starting point for defining a productivity indicator is to consider the development of this difference through time. However, since costs are additive, we see that, by using definition (46),

$$\text{(50)} \qquad \begin{aligned} VA^t - C^t_{KL} &= R^t - C^t_{EMS} - C^t_{KL} \\ &= R^t - C^t. \end{aligned}$$

Thus, profit in the KL-VA model is the same as profit in the KLEMS-Y model, and the same applies to the price and quantity components of profit differences. Using Bennet indicators, one easily checks that

$$\text{(51)} \qquad \begin{aligned} DPROD_{VA}(1,0) &= \mathcal{Q}_{VA}(1,0) - \mathcal{Q}_{KL}(1,0) \\ &= \mathcal{Q}_R(1,0) - \mathcal{Q}_C(1,0) \\ &= DPROD(1,0); \end{aligned}$$

that is, the productivity indicators are the same in the two models. This, however, does not hold for the productivity indices. One usually finds that $IPROD_{VA}(1, 0) \neq IPROD(1, 0)$. Balk (2003b) showed that if profit is zero in both periods, that is, $R^t = C^t$ ($t = 0, 1$), then, for certain two-stage indices which are second-order differential approximations to Fisher indices,

$$\ln IPROD_{VA}(1, 0) = D(1, 0) \ln IPROD(1, 0), \tag{52}$$

where $D(1, 0) \geq 1$ is the (mean) Domar-factor (= ratio of revenue over value added). Usually expression (52) is, in a continuous-time setting, derived under a set of strong neo-classical assumptions (see, for instance, Gollop (1979), Jorgenson *et al.* (2005, p. 298), or Schreyer (2001, p. 143)), so that it seems to be some deep economic-theoretical result. From the foregoing it may be concluded, however, that the inequality of the value-added based productivity index and the gross-output based productivity index is only due to the mathematics of ratios and differences. There is no underlying economic phenomenon.

The value-added based labor productivity index for period 1 relative to period 0 is defined as

$$ILPROD_{VA}(1, 0) \equiv \frac{Q_{VA}(1, 0)}{Q_L(1, 0)}, \tag{53}$$

where $Q_L(1, 0)$ was defined by expression (35). The index defined by expression (53) measures the "quantity" change of value added relative to the quantity change of labor input; or, can be seen as the index of real value added relative to the index of real labor input.

Recall that the labor quantity index $Q_L(1, 0)$ is here defined as a Fisher index, acting on the prices and quantities of all the types of labor that are being distinguished. Suppose that the units of measurement of the various types are in some sense the same; that is, the quantities of all the types are measured in hours, or in full-time equivalent jobs, or in some other common unit. Then one frequently considers, instead of the Fisher quantity index, the Dutot or simple sum quantity index,

$$Q_L^D(1, 0) \equiv \sum_{n \in L} x_n^1 \Big/ \sum_{n \in L} x_n^0. \tag{54}$$

The simple value-added based labor productivity index, defined as

$$ILPROD_{VA}^D(1, 0) \equiv \frac{Q_{VA}(1, 0)}{Q_L^D(1, 0)}, \tag{55}$$

has the alternative interpretation as an index of real value added per unit of labor. As such this measure frequently figures at the left-hand side (thus, as *explanandum*) in a growth accounting equation. However, for deriving such a relation nothing spectacular is needed, as will now be shown.

Consider the definition of the value-added based total factor productivity index, (49), and rewrite this as

$$Q_{VA}(1,0) = IPROD_{VA}(1,0) \times Q_{KL}(1,0). \tag{56}$$

Dividing both sides of this equation by the Dutot labor quantity index, and applying definition (55), one obtains[18]

$$ILPROD_{VA}^{D}(1,0) = IPROD_{VA}(1,0) \times \frac{Q_{KL}(1,0)}{Q_{L}(1,0)} \times \frac{Q_{L}(1,0)}{Q_{L}^{D}(1,0)}. \tag{57}$$

Taking logarithms and, on the assumption that all the index numbers are in the neighborhood of 1, interpreting these as percentages, the last equation can be interpreted as: (simple) labor productivity growth equals total factor productivity growth plus "capital deepening" plus "labor quality" growth. Again, productivity change is measured as a residual and, thus, the three factors at the right-hand side of the last equation can in no way be regarded as causal factors.

If, continuing our previous example, the primary inputs quantity index was defined as a two-stage index of the form

$$Q_{KL}(1,0) \equiv Q_{K}(1,0)^{\alpha} Q_{L}(1,0)^{1-\alpha} \ (0 < \alpha < 1), \tag{58}$$

where the reader recognizes the simple Cobb–Douglas form, then the index of "capital deepening" reduces to the particularly simple form

$$\frac{Q_{KL}(1,0)}{Q_{L}(1,0)} = \left[\frac{Q_{K}(1,0)}{Q_{L}(1,0)}\right]^{\alpha}. \tag{59}$$

The "labor quality" index, $Q_{L}(1,0)/Q_{L}^{D}(1,0)$, basically measures compositional shift or structural change among the labor types in the class L, since it is a ratio of two quantity indices.

3.2. *The K-CF Model*

The next model uses cash flow (CF) as its output concept. The unit's cash flow is defined as its revenue minus the costs of labor and intermediate inputs; that is

$$\begin{aligned} CF^{t} &\equiv R^{t} - C_{LEMS}^{t} \\ &= p^{t} \cdot y^{t} - w_{LEMS}^{t} \cdot x_{LEMS}^{t} \\ &= VA^{t} - C_{L}^{t}. \end{aligned} \tag{60}$$

[18]This is a discrete time version of expression (23) of Baldwin *et al.* (2007).

This input–output model basically sees cash flow as the return to capital input. It is assumed that $CF^t > 0$. Of course, if there is no owned capital (that is, all capital assets are leased), then $C_K^t = 0$, and this model does not make sense.[19]

The counterpart to *profitability* is now the ratio of cash flow to capital input cost, CF^t / C_K^t, and the natural starting point for defining a productivity index is to consider the development of this ratio through time. Since $(CF^1/CF_K^1)/(CF^0/C_K^0) = (CF^1/CF^0)/(C_K^1/C_K^0)$, we need a decomposition of the cash-flow ratio and a decomposition of the capital input cost ratio.

Decomposing a cash-flow ratio in a price and a quantity component is structurally similar to decomposing a value-added ratio (see Appendix B). Thus, suppose that a satisfactory decomposition is somehow available; that is,

$$\frac{CF^1}{CF^0} = P_{CF}(1,0)Q_{CF}(1,0). \tag{61}$$

Using Fisher indices, the capital input cost ratio is decomposed as

$$\begin{aligned} \frac{C_K^1}{C_K^0} &= P^F\left(w_K^1, x_K^1, w_K^0, x_K^0\right)Q^F\left(w_K^1, x_K^1, w_K^0, x_K^0\right) \\ &\equiv P_K(1,0)Q_K(1,0). \end{aligned} \tag{62}$$

The cash-flow based (total factor) productivity index for period 1 relative to period 0 is then defined as

$$IPROD_{CF}(1,0) \equiv \frac{Q_{CF}(1,0)}{Q_K(1,0)}. \tag{63}$$

This index measures the change of the quantity component of cash flow relative to the quantity change of capital input; or can be seen as the index of real cash flow relative to the index of real capital input.

In the K-CF model the counterpart to *profit* is the difference of cash flow and capital input cost, $CF^t - C_K^t$, and the natural starting point for defining a productivity indicator is to consider the development of this difference through time. However, since costs are additive, we see that

$$\begin{aligned} CF^t - C_K^t &= R^t - C_{LEMS}^t - C_K^t \\ &= R^t - C^t. \end{aligned} \tag{64}$$

Thus, profit in the K-CF model is the same as profit in the KLEMS-Y model, and the same applies to the price and quantity components of profit differences. Using Bennet indicators, one easily checks that

[19]Cash flow is also called gross profit. The National Accounts term is "gross operating surplus." In some sectors it occasionally occurs that production units exhibit negative cash flows during certain periods. An example of such a sector is agriculture.

$$(65) \quad DPROD_{CF}(1,0) \equiv Q_{CF}(1,0) - Q_K(1,0) = Q_R(1,0) - Q_C(1,0) = DPROD(1,0);$$

that is, the productivity indicators are the same in the two models. This, however, does not hold for the productivity indices. In general it will be the case that $IPROD_{CF}(1, 0) \neq IPROD(1, 0)$. Following the reasoning of Balk (2003b) it is possible to show that, if profit is zero in both periods, that is, $R^t = C^t$ ($t = 0, 1$), then, for certain two-stage indices which are second-order differential approximations to Fisher indices,

$$(66) \quad \ln IPROD_{CF}(1,0) = E(1,0) \ln IPROD(1,0),$$

where $E(1, 0) \geq 1$ is the ratio of mean revenue over mean cash flow. Since $CF_t \leq VA^t$, it follows that $E(1, 0) \geq D(1, 0)$.

4. More Models

The K-CF model provides a good point of departure for a discussion of the measurement of capital input cost. Cash flow, as defined in the foregoing, is the (*ex post* measured) monetary balance of all the flow variables. Capital input cost is different, since capital is a stock variable. Basically, capital input cost is measured as the difference between the book values of the production unit's owned capital stock at beginning and end of the accounting period considered. The theory, for which no behavioral or other far-reaching assumptions appear to be required, was developed by Balk (2009a).

In the framework of this theory it appears that capital input cost can rather naturally be split into four meaningful components,

$$(67) \quad C_K^t = C_{K,w}^t + C_{K,e}^t + C_{K,u}^t + C_{K,tax}^t,$$

respectively denoting the aggregate cost of waiting, anticipated time-series depreciation, unanticipated revaluation, and tax. This leads to four additional input–output models.

The first two models are variants of the KL-VA model. The idea here is that the (*ex post*) cost of time-series depreciation plus tax should be treated like the cost of intermediate inputs, and subtracted from value added. Hence, the output concept is called net value added, and defined by

$$(68) \quad NVA^t \equiv VA^t - \left(C_{K,e}^t + C_{K,u}^t + C_{K,tax}^t\right).$$

The remaining input cost is the sum of labor cost, C_L^t, and waiting cost of capital, $C_{K,w}^t$.

Some argue that this model is to be preferred from a welfare-theoretic point of view. If the objective is to hold owned capital (including investments during the accounting period) in terms of money intact, then depreciation—whether expected or not—and tax should be treated like intermediate inputs (Spant, 2003). This model was also strongly defended by Rymes (1983). Apart from land, he considered labor and waiting as the only primary inputs, and connected this with a Harrodian model of technological change.

Diewert *et al*. (2005), Diewert and Lawrence (2006), and Diewert and Wykoff (forthcoming) suggested consideration of unanticipated revaluation, which is the unanticipated part of time-series depreciation, as a monetary component that must be added to profit. The result could be called "profit from normal operations of the production unit." Following this suggestion, the output concept becomes

$$NNVA^t \equiv VA^t - \left(C^t_{K,e} + C^t_{K,tax}\right), \quad (69)$$

which could be called normal net value added. As inputs are considered labor, C^t_L, and waiting cost of capital, $C^t_{K,w}$.

The last two models are variants of the K-CF model. Here also the idea is that the (*ex post*) cost of time-series depreciation plus tax should be treated like the cost of intermediate inputs, and subtracted from cash flow. Hence, the output concept is called net cash flow, and defined by

$$NCF^t \equiv CF^t - \left(C^t_{K,e} + C^t_{K,u} + C^t_{K,tax}\right). \quad (70)$$

The remaining input cost is the waiting cost of capital, $C^t_{K,w}$.

A variant of the K-NCF model is obtained by considering unanticipated revaluation, which is the unanticipated part of time-series depreciation, as a component that must be added to profit. Hence, the output concept becomes

$$NNCF^t \equiv CF^t - \left(C^t_{K,e} + C^t_{K,tax}\right), \quad (71)$$

which could be called normal net cash flow. The only input category is the waiting cost of capital, $C^t_{K,w}$.[20]

A number of observations can now be made. First, all the input–output models (KLEMS-Y, KL-VA, KL-NVA, KL-NNVA, K-CF, K-NCF, and K-NNCF) lead to different (total factor) productivity *indices*. However, most of these differences are artefacts, caused by a different mixing of subtraction and division.[21] Thus, it depends on purpose and context of a study which particular model is chosen for the presentation of results. When productivity *indicators* are compared, the real difference turns up, namely between the KL-NNVA and K-NNCF models on the one hand and the rest on the other hand. The reason is that the KL-NNVA and K-NNCF are based on a different profit concept, namely $\Pi^{*t} = \Pi^t + C^t_{K,u}$.

Second, the waiting cost of capital is determined by an interest rate r^t. Setting in the accounting relation of the K-NCF model,

[20]In the model of Hulten and Schreyer (2006), total (= unanticipated plus anticipated) revaluation is added to profit. This is consistent with *SNA93*'s prescription for non-market units.

[21]Rymes (1983) would single out the KL-NVA model as the "best" one, but this is clearly not backed by the argument presented here.

$$C^t_{K,w} + \Pi^t = NCF^t, \tag{72}$$

profit Π^t equal to zero, and solving this equation then for r^t delivers the so-called "endogenous," or "internal," or "balancing" rate of return. However, one could do the same with the accounting relation of the K-NNCF model,

$$C^t_{K,w} + \Pi^{*t} = NNCF^t. \tag{73}$$

This would lead to a thing one could call the "normal endogenous" rate of return. The important point to stress here is that there appears to be no single concept of the endogenous rate of return. There is rather a continuum of possibilities, depending on the way one wants to deal with unanticipated revaluations.

Third, an endogenous rate of return, of whatever variety, can only be calculated *ex post*. Net cash flow as well as normal net cash flow require for their computation that the accounting period has expired.

Fourth, as the name suggests, a total factor productivity index or indicator suggests that all the inputs and outputs are correctly observed. Unobserved inputs and outputs and measurement errors lead to a distorted profit figure and have impact on the interpretation of total factor productivity change. Since an endogenous rate of return can be said to absorb profit, the extent of undercoverage has also implications for the interpretation of the rate of return (see also Schreyer, forthcoming). Put otherwise, since an endogenous rate of return closes the gap between the input and the output side of the production unit, it is influenced by all sorts of measurement errors.

The question whether to use, for a certain production unit, an endogenous or an exogenous rate of return belongs, according to Diewert (2008), to the list of still unresolved issues. The practice of official statistical agencies is varied, as a brief survey learns.

The U.S. Bureau of Labor Statistics uses endogenous rates (see Dean and Harper, 2001),[22] as does Statistics Canada (see Harchaoui *et al.*, 2001). The Australian Bureau of Statistics uses, per production unit considered, the maximum of the endogenous rate and a certain exogenous rate (set equal to the annual percentage change of the CPI plus 4 percent) (see Roberts, 2006). Statistics New Zealand uses endogenous rates (according to their Sources and Methods 2006 publication). The Swiss Federal Statistical office has the most intricate system: per production unit the simple mean of the endogenous rate and a certain exogenous rate is used as the final exogenous rate (see Rais and Sollberger, 2008). Concerning the endogenous rates, however, these sources are not clear as to which concept is used precisely.

Statistics Netherlands sets the interest rate equal to the so-called Internal Reference Rate, which is the interest rate that banks charge to each other, plus 1.5 percent (see van den Bergen *et al.*, 2007).

For the Netherlands, interesting empirical results were obtained by Vancauteren *et al.* (2009). Over the years 1995 to 2007 these authors calculated total factor

[22]It seems to me that Jorgenson (2009) is also proposing endogenous rates of return for the four sectors considered.

productivity changes according to the KLEMS-Y, KL-VA, KL-NVA, K-CF, and K-NCF models, with exogenous and endogenous interest rates, for nine industrial sectors and their aggregate.

5. Capital Utilization

Until now it was tacitly assumed that the productive capital stock was fully used in actual production. We want to make this assumption explicit. For introducing the capital utilization rate, let us return to the K-CF model, which is governed by the equation

$$C_K^t + \Pi^t = CF^t. \tag{74}$$

Cash flow, if positive, is seen as the return to the productive capital stock. But what if this stock is only partly used in productive operations? I sketch the simplest approach.

Let θ_K^t $(0 < \theta_K^t \leq 1)$ denote the fraction of the productive capital stock (averaged over all the type-age classes) that is actually used during period t. Then one easily checks that the foregoing equation can be written as

$$\theta_K^t C_K^t + (1-\theta_K^t)C_K^t + \Pi^t = CF^t, \tag{75}$$

where $\theta_K^t C_K^t$ and $(1-\theta_K^t)C_K^t$ are the user costs of the used and unused parts of the capital stock, respectively.[23]

Now, like unanticipated revaluation, one can argue that the cost of unused capital should be added to profit and that the measurement of productivity change should be based on the equation

$$\theta_K^t C_K^t + \Pi^{**t} = CF^t, \tag{76}$$

with $\Pi^{**t} \equiv (1-\theta_K^t)C_K^t + \Pi^t$ being the profit adjusted for underutilization of capital. Put otherwise, in this model the (total factor) productivity index for period 1 relative to period 0 is defined as

$$IPROD_{CFU}(1,0) \equiv \frac{Q_{CF}(1,0)}{(\theta_K^1/\theta_K^0)Q_K(1,0)}. \tag{77}$$

This is the index of real cash flow divided by the index of real capital input multiplied by the change of the capital utilization rate.

It is straightforward to check that the utilization rate can be introduced in any of the models discussed in this article. This exercise is left to the reader. On the empirical relevance of the capital utilization rate in productivity measurement, see Coremberg (2008).

[23]It is implicitly assumed here that the unit user costs of used and unused assets are the same. See Hulten (2009) for a brief discussion of this issue.

6. Conclusion

After measurement comes explanation. Depending on the initial level of aggregation, there appear to be two main directions. The first is disaggregation: the explanation of productivity change at an aggregate level (economy, sector, industry) by productivity change at a lower level (firm, plant) and other factors, collectively subsumed under the heading of reallocation (expansion, contraction, entry, and exit of units). This topic was reviewed by Balk (2003a, section 6). As the example of Balk and Hoogenboom-Spijker (2003) demonstrates, this type of research is of economic–statistical nature, and there are no neoclassical assumptions involved. This was shown more formally by Balk (2009b).

The second direction is concerned with the decomposition of productivity change into factors such as technological change, technical efficiency change, scale effects, input- and output-mix effects, and chance. The basic idea can be explained as follows.

To start with, for each time period t the technology to which the production unit under consideration has access is defined as the set S^t of all the input–output quantity combinations which are feasible during t. Such a set is assumed to have nice properties like being closed, bounded, and convex. Of particular interest is the subset of S^t, called its frontier, consisting of all the efficient input–output combinations. An input–output quantity combination is called efficient when output cannot be increased without increasing some input and input cannot be decreased without decreasing some output.

From base period to comparison period our production unit moves from $(x^0, y^0) \in S^0$ to $(x^1, y^1) \in S^1$, and these two input–output combinations are not necessarily efficient. Decomposition of productivity change means that between these two points some hypothetical path must be constructed, the segments of which can be given a distinct interpretation.

In particular, we consider the projection of (x^0, y^0) on the frontier of S^0, and the projection of (x^1, y^1) on the frontier of S^1. Comparing the base period and comparison period distance between the original points and their projections provides a measure of efficiency change.

Two more points are given by projecting (x^0, y^0) also on the frontier of S^1, and (x^1, y^1) also on the frontier of S^0. The distance between the two frontiers at the base and comparison period projection points provides a (local) measure of technological change. And, finally, moving over each frontier (which is a surface in $N + M$-dimensional space) from a base period to a comparison period projection point provides measures of the scale and input–output mix effects.

The construction of all those measures is discussed by Balk (2004). Since there is no unique path connecting the two observations, there is no unique decomposition either.

And here come the neoclassical assumptions, at the end of the day rather than at its beginning. Suppose that the production unit always stays on the frontier, that its input- and output-mix is optimal at the, supposedly given, input and output prices, and that the two technology sets exhibit constant returns to scale, then productivity change reduces to technological change (see Balk (1998, section 3.7) for a formal proof). The technology sets are thereby supposed to reflect the true state of nature, which rules out chance as a factor also contributing to productivity change.[24]

Appendix A: Indices and Indicators

The basic measurement tools used are price and quantity indices and indicators. The first are ratio-type measures, and the second are difference-type measures. What, precisely, are the requirements for good tools? The following just serves to introduce and illustrate some concepts used in the main text of this article. For a complete treatment, the reader is referred to Balk (2008).

Indices

A *price index* is a positive, continuously differentiable function $P(p^1, y^1, p^0, y^0): \Re^{4N}_{++} \to \Re_{++}$ that correctly indicates any increase or decrease of the elements of the price vectors p^1 or p^0, conditional on the quantity vectors y^1 and y^0. A *quantity index* is a positive, continuously differentiable function of the same variables $Q(p^1, y^1, p^0, y^0): \Re^{4N}_{++} \to \Re_{++}$ that correctly indicates any increase or decrease of the elements of the quantity vectors y^1 or y^0, conditional on the price vectors p^1 and p^0. The number N is called the dimension of the price or quantity index.

The basic requirements on price and quantity indices comprise: (1) that they exhibit the correct monotonicity properties; (2) that they are linearly homogeneous in comparison period prices (quantities, respectively); (3) that they satisfy the Identity Test; (4) that they are homogeneous of degree 0 in prices (quantities, respectively); and (5) that they are invariant to changes in the units of measurement of the commodities. The Product Test requires that price index times quantity index equals the value ratio.

Any function $P(p^1, y^1, p^0, y^0)$ or $Q(p^1, y^1, p^0, y^0)$, invariant to changes in the units of measurement, can be written as a function of only $3N$ variables, namely the price relatives p^1_n/p^0_n or the quantity relatives y^1_n/y^0_n, the comparison period values $v^1_n \equiv p^1_n y^1_n$, and the base period values $v^0_n \equiv p^0_n y^0_n$ $(n = 1, \ldots, N)$.

Some simple examples might be useful to illustrate this. Consider the Laspeyres price index as function of prices and quantities,

$$P^L(p^1, y^1, p^0, y^0) \equiv p^1 \cdot y^0 / p^0 \cdot y^0,$$

and notice that this index can be written as a function of price relatives and (base period) values,

$$P^L(p^1, y^1, p^0, y^0) = \sum_{n=1}^{N} (p^1_n/p^0_n) v^0_n \Big/ \sum_{n=1}^{N} v^0_n.$$

Similarly, the Paasche price index

[24]On stochastic productivity measurement, see Chambers (2008).

$$P^P(p^1, y^1, p^0, y^0) \equiv p^1 \cdot y^1 / p^0 \cdot y^1$$

can be written as a function of price relatives and (comparison period) values,

$$P^P(p^1, y^1, p^0, y^0) = \left(\sum_{n=1}^{N} (p_n^0/p_n^1) \upsilon_n^1 \Big/ \sum_{n=1}^{N} \upsilon_n^1 \right)^{-1}.$$

Finally, the Fisher price index, defined as the geometric mean of the Laspeyres and Paasche indices, reads

$$P^F(p^1, y^1, p^0, y^0) = \left[\frac{\sum_{n=1}^{N} (p_n^1/p_n^0) \upsilon_n^0 \Big/ \sum_{n=1}^{N} \upsilon_n^0}{\sum_{n=1}^{N} (p_n^0/p_n^1) \upsilon_n^1 \Big/ \sum_{n=1}^{N} \upsilon_n^1} \right]^{1/2}.$$

Such functional forms are useful for the definition of *two-stage indices*. Let the aggregate under consideration be denoted by A, and let A be partitioned arbitrarily into K subaggregates A_k,

$$A = \bigcup_{k=1}^{K} A_k,\ A_k \cap A_{k'} = \emptyset\ (k \neq k').$$

Each subaggregate consists of a number of items. Let $N_k \geq 1$ denote the number of items contained in A_k $(k = 1, \ldots, K)$. Obviously $N = \sum_{k=1}^{K} N_k$. Let $(p_k^1, y_k^1, p_k^0, y_k^0)$ be the subvector of (p^1, y^1, p^0, y^0) corresponding to the subaggregate A_k. Recall that $\upsilon_n^t \equiv p_n^t y_n^t$ is the value of item n at period t. Then $V_n^t \equiv \sum_{n \in A_k} \upsilon_n^t$ $(k = 1, \ldots, K)$ is the value of subaggregate A_k at period t, and $V^t \equiv \sum_{n \in A} \upsilon_n^t = \sum_{k=1}^{K} V_k^t$ is the value of aggregate A at period t.

Let $P(.)$, $P^{(1)}(.)$, . . . , $P^{(K)}(.)$ be price indices of dimension K, $N_1, \ldots, N_K$ respectively that satisfy the basic requirements. Then the price index defined by

$$P^*(p^1, y^1, p^0, y^0) \equiv P\left(P^{(k)}(p_k^1, y_k^1, p_k^0, y_k^0), V_k^1, V_k^0; k = 1, \ldots, K\right) \tag{78}$$

is of dimension N and also satisfies the basic requirements. The index $P^*(.)$ is called a two-stage index. The first stage refers to the indices $P^{(k)}(.)$ for the subaggregates A_k $(k = 1, \ldots, K)$. The second stage refers to the index $P(.)$ that is applied to the subindices $P^{(k)}(.)$ $(k = 1, \ldots, K)$. A two-stage index such as defined by expression (78) closely corresponds to the calculation practice at statistical agencies. All the subindices are usually of the same functional form, for instance Laspeyres or Paasche indices. The aggregate, second-stage index may or may not be of the same functional form. This could be, for instance, a Fisher index.

If the functional forms of the subindices $P^{(k)}(.)$ $(k = 1, \ldots, K)$ and the aggregate index $P(.)$ are the same, then $P^*(.)$ is called a two-stage $P(.)$-index. Continuing the example, the two-stage Laspeyres price index reads

$$P^{*L}(p^1, y^1, p^0, y^0) \equiv \sum_{k=1}^{K} P^L(p_k^1, y_k^1, p_k^0, y_k^0) V_k^0 \Big/ \sum_{k=1}^{K} V_k^0,$$

and one simply checks that the two-stage and the single-stage Laspeyres price indices coincide. However, this is the exception rather than the rule. For most indices, two-stage and single-stage variants do not coincide.

Two-stage quantity indices are defined similarly.

Indicators

Provided that certain reasonable requirements are satisfied, the continuous functions $\mathcal{P}(p^1, y^1, p^0, y^0): \Re^{4N}_{++} \to \Re$ and $\mathcal{Q}(p^1, y^1, p^0, y^0): \Re^{4N}_{++} \to \Re$ will be called *price indicator* and *quantity indicator*, respectively. Notice that these functions may take on negative or zero values. The basic requirements comprise: (1) that the functions exhibit the correct monotonicity properties; (2) that they satisfy the Identity Test; (3) that they are homogeneous of degree 1 in prices (quantities, respectively); and (4) that they are invariant to changes in the units of measurement of the commodities. The analog of the Product Test requires that price indicator plus quantity indicator equals the value difference.

Any function $\mathcal{P}(p^1, y^1, p^0, y^0)$ or $\mathcal{Q}(p^1, y^1, p^0, y^0)$, invariant to changes in the units of measurement, can be written as a function of only $3N$ variables, namely the price relatives p_n^1/p_n^0 or the quantity relatives y_n^1/y_n^0, the comparison period values $\upsilon_n^1 \equiv p_n^1 y_n^1$, and the base period values $\upsilon_n^0 \equiv p_n^0 y_n^0$ $(n = 1, \ldots, N)$.

Some simple examples might also be useful here. Consider the Laspeyres price indicator as function of prices and quantities,

$$\mathcal{P}^L(p^1, y^1, p^0, y^0) \equiv (p^1 - p^0) \cdot y^0,$$

and notice that this indicator can be written as a function of price relatives and (base period) values,

$$\mathcal{P}^L(p^1, y^1, p^0, y^0) = \sum_{n=1}^{N} (p_n^1/p_n^0 - 1)\upsilon_n^0.$$

Similarly, the Paasche price indicator

$$\mathcal{P}^P(p^1, y^1, p^0, y^0) \equiv (p^1 - p^0) \cdot y^1$$

can be written as a function of price relatives and (comparison period) values,

$$\mathcal{P}^P(p^1, y^1, p^0, y^0) = \sum_{n=1}^{N} (1 - p_n^0/p_n^1)\upsilon_n^1.$$

Finally, the Bennet price indicator is usually defined by

$$\mathcal{P}^B(p^1, y^1, p^0, y^0) \equiv (1/2)(p^1 - p^0) \cdot (y^0 + y^1),$$

but can be written as

$$\mathcal{P}^B(p^1, y^1, p^0, y^0) = (1/2)\left[\sum_{n=1}^{N} (p_n^1/p_n^0 - 1)\upsilon_n^0 + \sum_{n=1}^{N} (1 - p_n^0/p_n^1)\upsilon_n^1\right].$$

The Bennet price indicator for an aggregate is a simple sum of Bennet price indicators for its subaggregates:

$$\mathcal{P}^B(p^1, y^1, p^0, y^0) = \sum_{k=1}^{K} \mathcal{P}^B(p_k^1, y_k^1, p_k^0, y_k^0),$$

and a similar relation holds for quantity indicators.

Appendix B: Decompositions of the Value Added Ratio

Value added is defined as revenue minus the cost of intermediate inputs. Let the revenue ratio R^1/R^0 as in expression (7) be decomposed as

(79) $$\frac{R^1}{R^0} = P^F(p^1, y^1, p^0, y^0)Q^F(p^1, y^1, p^0, y^0) \equiv P_R(1,0)Q_R(1,0),$$

and let the intermediate inputs cost ratio C_{EMS}^1/C_{EMS}^0 be decomposed by one- or two-stage Fisher indices as

(80) $$\frac{C_{EMS}^1}{C_{EMS}^0} = P^F(w_{EMS}^1, x_{EMS}^1, w_{EMS}^0, x_{EMS}^0)Q^F(w_{EMS}^1, x_{EMS}^1, w_{EMS}^0, x_{EMS}^0) \equiv P_{EMS}(1,0)Q_{EMS}(1,0).$$

Then $P_{VA}(1, 0)$ could be defined as a Fisher-type index of the subindices $P_R(1, 0)$ and $P_{EMS}(1, 0)$; that is,

(81) $$P_{VA}^F(1,0) \equiv \left[\frac{\frac{R^0}{VA^0}P_R(1,0) - \frac{C_{EMS}^0}{VA^0}P_{EMS}(1,0)}{\frac{R^1}{VA^1}(P_R(1,0))^{-1} - \frac{C_{EMS}^1}{VA^1}(P_{EMS}(1,0))^{-1}}\right]^{1/2}.$$

The numerator is a Laspeyres-type double deflator, and the denominator is the inverse of a Paasche-type double deflator. Similarly, $Q_{VA}(1, 0)$ is defined as a Fisher-type index of the subindices $Q_R(1, 0)$ and $Q_{EMS}(1, 0)$; that is,

(82) $$Q_{VA}^F(1,0) \equiv \left[\frac{\frac{R^0}{VA^0}Q_R(1,0) - \frac{C_{EMS}^0}{VA^0}Q_{EMS}(1,0)}{\frac{R^1}{VA^1}(Q_R(1,0))^{-1} - \frac{C_{EMS}^1}{VA^1}(Q_{EMS}(1,0))^{-1}}\right]^{1/2}.$$

One easily checks that $P_{VA}^F(1,0)Q_{VA}^F(1,0) = VA^1/VA^0$. These indices satisfy the Equality Test, but fail the Consistency-in-Aggregation Test.

The quantity index $Q_{VA}^F(1,0)$ was proposed by Geary (1944), though Karmel (1954) mentions some earlier sources. For studying the behavior of this index the following relation is useful:

$$(83)\qquad \frac{Q_{VA}^{F}(1,0)}{Q_R(1,0)} = \left[\frac{1-\dfrac{C_{EMS}^{0}}{R^{0}}\dfrac{Q_{EMS}(1,0)}{Q_R(1,0)}}{1-\dfrac{C_{EMS}^{0}}{R^{0}}} \frac{1-\dfrac{C_{EMS}^{1}}{R^{1}}}{1-\dfrac{C_{EMS}^{1}}{R^{1}}\left(\dfrac{Q_{EMS}(1,0)}{Q_R(1,0)}\right)^{-1}} \right]^{1/2}.$$

When $VA^t > 0$ then $1 - C_{EMS}^{t}/R^{t} > 0$ $(t = 0, 1)$. However, the function $Q_{VA}^{F}(1,0)$ is undefined when $Q_R(1,0) \le \left(C_{EMS}^{0}/R^{0}\right)Q_{EMS}(1,0)$ or $Q_R(1,0) \ge \left(R^{1}/C_{EMS}^{1}\right)Q_{EMS}(1,0)$. Moreover, equation (83) implies the following relations:

$$(84)\qquad \begin{aligned} Q_R(1,0) > Q_{EMS}(1,0) &\Rightarrow Q_{VA}^{F}(1,0) > Q_R(1,0)\\ Q_R(1,0) = Q_{EMS}(1,0) &\Rightarrow Q_{VA}^{F}(1,0) = Q_R(1,0)\\ Q_R(1,0) < Q_{EMS}(1,0) &\Rightarrow Q_{VA}^{F}(1,0) < Q_R(1,0). \end{aligned}$$

Karmel (1954) showed that in the case of chained index numbers these relations might be violated.

Thus there occur situations where Fisher-type indices are undefined. An alternative decomposition, which does not exhibit this defect, can be developed as follows.

For the logarithm of the value added ratio we get by repeated application of the logarithmic mean $L(a, b)$,

$$(85)\qquad \begin{aligned} \ln\left(\frac{VA^{1}}{VA^{0}}\right) &= \frac{VA^{1}-VA^{0}}{L(VA^{1},VA^{0})} = \frac{R^{1}-R^{0}}{L(VA^{1},VA^{0})} - \frac{C_{EMS}^{1}-C_{EMS}^{0}}{L(VA^{1},VA^{0})}\\ &= \frac{L(R^{1},R^{0})\ln(R^{1}/R^{0})}{L(VA^{1},VA^{0})} - \frac{L(C_{EMS}^{1},C_{EMS}^{0})\ln(C_{EMS}^{1}/C_{EMS}^{0})}{L(VA^{1},VA^{0})}. \end{aligned}$$

Using the decompositions of the revenue ratio and the intermediate inputs cost ratio, the logarithm of the value added ratio can be expressed as

$$(86)\qquad \begin{aligned} \ln\left(\frac{VA^{1}}{VA^{0}}\right) = {} & \frac{L(R^{1},R^{0})\ln(P_R(1,0)Q_R(1,0))}{L(VA^{1},VA^{0})} - {}\\ & \frac{L(C_{EMS}^{1},C_{EMS}^{0})\ln(P_{EMS}(1,0)Q_{EMS}(1,0))}{L(VA^{1},VA^{0})}. \end{aligned}$$

This can simply be rearranged to

$$(87)\qquad \frac{VA^{1}}{VA^{0}} = \frac{P_R(1,0)^{\phi}}{P_{EMS}(1,0)^{\psi}} \frac{Q_R(1,0)^{\phi}}{Q_{EMS}(1,0)^{\psi}},$$

where $\phi \equiv L(R^1, R^0)/L(VA^1, VA^0)$, that is, mean revenue over mean value added, and $\psi \equiv L\left(C_{EMS}^{1}, C_{EMS}^{0}\right)/L\left(VA^{1}, VA^{0}\right)$, that is, mean intermediate inputs cost over mean value added. Thus, value added price and quantity indices can rather naturally be defined by

(88) $$P_{VA}^{MV}(1,0) \equiv \frac{P_R(1,0)^{\phi}}{P_{EMS}(1,0)^{\psi}}$$

(89) $$Q_{VA}^{MV}(1,0) \equiv \frac{Q_R(1,0)^{\phi}}{Q_{EMS}(1,0)^{\psi}}.$$

These indices generalize the conventional Montgomery–Vartia indices (see Balk, 2008, p. 87 for their definition). They are Consistent-in-Aggregation, but fail the Equality Test. The reason is that

(90) $$\phi - \psi = \frac{L(R^1, R^0) - L(C_{EMS}^1, C_{EMS}^0)}{L(VA^1, VA^0)} \leq 1,$$

because $L(a, 1)$ is a concave function.

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