Review of Income and Wealth
Series 56, Number 1, March 2010

POLARIZATION OF THE POOR: MULTIVARIATE RELATIVE POVERTY MEASUREMENT SANS FRONTIERS

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A major impediment to poverty evaluation in environments with a multiplicity of wellbeing indicators is the difficulty associated with formulating a poverty frontier in many dimensions. This paper proposes two multivariate relative polarization measures which, in appropriate circumstances, serve as multivariate poverty measures which do not require computation of a poverty frontier. As poverty measures they have the intuitive appeal of reflecting the degree to which poor and non-poor societies are polarized. (The measures would also have considerable application in studying multivariate convergence issues in economic growth models.) The measures are exemplified in a poor–non-poor country study over the period 1990–2005, based upon the joint distribution of per capita GNP and Life Expectancy. The results suggest that as a group, the world's poor are experiencing diminished poverty polarization; however, within the world's poor the African nations are experiencing increased poverty polarization.

Introduction

There are a great many uses for simple measures which capture the degree to which two collections of agents are polarized, when those agents are characterized by a number of characteristics. When the collections are the poor and the non-poor, such measures can solve a problem that has bedeviled multivariate relative poverty measurement. Within the long and extensive debate over exactly how the plight of the poor can be measured,[1] issues surrounding the poverty frontier loom large, especially when poverty is assessed in many dimensions. What is to be measured is clear: the sense of disadvantage that a particular group of individuals (referred to as the poor) experience. The difficulty in actually quantifying this sense of disadvantage is defining a boundary between the poor and non-poor. Labeling and sorting is the problem; in essence there is a group of people who are in some sense inherently poor and another who are inherently non-poor, but neither group are labeled or sorted in an identifiable fashion. Poverty frontiers are contrivances to facilitate the labeling and determine how poor is poor; anyone whose measured

Note: Many thanks are due to two anonymous referees and to the participants in the Research Workshop of the Israel Science Foundation on Income Polarization Bar Ilan 2008 for their many helpful comments.

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[1]The debate is extensive. What should be measured as indicators of impoverishment and how should the measure be formulated? (See Zheng, 1997, for a discussion and survey.) Should the comparison be in relative or absolute terms and how should the boundary between the poor and non poor be defined? (See Sen, 1983; Townsend, 1985, for a debate.) Is poverty uni- or multi-dimensional, and if the latter, how should the many dimensions be accommodated? (See Grusky and Kanbur, 2006; Kakwani and Silber, 2008, for a discussion.)

characteristics are below the boundary are rendered poor by the definition, and the extent to which they are below the boundary corresponds to a measure of the extent to which they are poor.

The nature of the frontier will depend upon whether relative or absolute poverty measures are required (a matter of considerable debate itself). With single indicators, in the absolute case the frontier is based upon the amount of income required to purchase a minimalist set of necessities; in the relative case the frontier is based upon some proportion, usually 0.5 or 0.6 of an overall distributional location measure like the median or mean. When multiple indicators are considered, things are much less straightforward; what constitutes "below the frontier" gets more complicated and there are several possibilities. At one extreme poverty can be defined as deficiency with respect to at least one indicator (in essence treating all indicators as perfect complements with each other); at the other, poverty can be defined in terms of deficiency in all indicators (treating indicators as perfect substitutes for each other). In between there will be frontiers which represent trade-offs between indicators that maintain agents at some reference level of disadvantage (see Duclos *et al.*, 2006; Anderson *et al.*, 2008, for a discussion).

Here new relative multivariate bipolarization measures are proposed which, when used in the context of the poor and non-poor, obviate the need to struggle with a poverty frontier definition. Underlying the measures is the supposition that a society contains two classes of people—the poor and the non-poor—each observably characterized by measurable processes they experience (for convenience of discussion and the ultimate application, assume them to be income and life expectancy). In each case the measurable processes are partially random consequences of their unobservable circumstances or the functionings and capabilities with which they are endowed (their health, intellect, environment, education, freedom of action, location, etc). It is these functionings and capabilities that truly determine the extent of impoverishment. Unfortunately they are often latent in the sense of not being observable, all that is observed being the measurable processes they engender. Some poor people will do relatively well in observed characteristics in spite of being poor in circumstance, i.e. they get a good draw from the poor income–life expectancy distribution. Some rich people will do observably badly in spite of being rich in circumstance, i.e. they will get a bad draw from the rich income–life expectancy distribution.[2] When the characteristic distributions of the poor and non-poor are aligned in a particular fashion (essentially when the non-poor distribution stochastically dominates the poor distribution), it can be argued that such measures also reflect a sense of relative ill-being of a representative agent of the relatively impoverished group. It is the direct comparison of these poor and non-poor distributions that provides a somewhat different approach to measuring the plight of the poor which is unencumbered by the need for defining poverty frontiers and readily accommodates many indicators. It also turns out that a decomposition of one of the measures presents some useful and interesting insights into the nature of the poverty experience.

[2]If a poverty frontier is employed, almost inevitably some truly poor people will be counted as rich and some rich people will be counted as poor. The extent to which this occurs depends on the arbitrarily chosen poverty boundary (construed in measurable terms) and the nature of the poor and non-poor distributions.

In Section 2, two bipolarization measures are introduced: an "overlap" measure and a "trapezoidal" measure. Distributional overlap measures perform quite well as depolarization indicators (Anderson, 2008b; Anderson *et al.*, 2009a), especially when there is a multiplicity of indicators. Unfortunately if distributions of characteristics do not intersect in every dimension, or if the separate distributions are not identifiable, the overlap measure is of little use. However, the "trapezoidal" indicator proposed here does not depend on overlap in any dimension and can be readily applied with many indicators in situations where the separate distributions are not identified, provided they engender sufficient "bumps" in the mixture distribution. Section 2 illustrates the use of these measures in the context of population weighted comparisons of rich and poor countries in terms of their per capita GNP and life expectancy over the period 1990–2005; Section 4 draws some conclusions.

1. Bipolarization Measurement: A Review

The multivariate poverty measures presented here are founded upon the notion of polarization of the poor group from the rest of society; the proposed measures are very much relative poverty measures. Esteban and Ray (1994), Duclos *et al.* (2004), and Wang and Tsui (2000) posited a collection of propositions with which a polarization measure should be consistent, and proposed a collection of univariate measures appropriate for a variety of circumstances that would reflect such polarization between potentially many groups. The propositions are based upon a so-called Identification–Alienation nexus, wherein notions of polarization are fostered jointly by an agent's sense of increasing within-group identity or association and between-group distance or alienation.

There have been several proposed univariate polarization indices which focus on an arbitrary number of groups and *a fortiori* two groups (Esteban and Ray, 1994; Esteban *et al.*, 1998; Zhang and Kanbur, 2001; Duclos *et al.*, 2004), and a similar number that focus on just two groups (Foster and Wolfson, 1992; Wolfson, 1994; Alesina and Spolare, 1997; Wang and Tsui, 2000). Gigliarano and Mosler (2009) develop a family of multivariate polarization measures based upon measures of between- and within-group multivariate variation and relative group size which exploit notions of subgroup decomposability. An excellent summary of the properties of the univariate indices is to be found in Esteban and Ray (2007), wherein the properties of indices are evaluated in terms of their coherence with some basic axioms that reflect three notions:

(1) When there is only one group there is little polarization.
(2) Polarization increases when within-group inequality is reduced.
(3) Polarization increases when between-group inequality increases.

Duclos *et al.* (2004) and Esteban and Ray (2007) have evaluated polarization measures on the basis of the extent to which such measures satisfy certain axioms. The axioms are formed around a notional univariate density that is a mixture of kernels $f(x, a)$ that are symmetric uni-modal on a compact support of $[a, a + 2]$, with $E(x) = \mu = (a + 1)$ also representing the mode. The kernels are subject to slides (location shifts) $g(y) = f(y - x)$ and squeezes (shrinkages) of the form $f^{\lambda}(x) = f(\{x - [1 - \lambda]\mu\}/\lambda)/\lambda$ $(0 < \lambda < 1)$, and potential indices are evaluated in the context of such changes in terms of the extent to which they satisfy the following set of axioms.

Axiom 1. A squeeze of a distribution that consists of a single basic density cannot increase polarization.
Axiom 2. Symmetric squeezes of the two kernels cannot reduce polarization.
Axiom 3. Slides of the two kernels outward increases polarization.
Axiom 4. Common population scaling preserves the ordering.
Axiom 5. Polarization indices have to come from a family where, if x and y are independently distributed with marginal distributions $f(x)$ and $f(y)$, then the index is the expected value of some function $T(f(x),|x-y|)$ which is increasing in its second argument.
Axiom 6. Symmetric squeezes of the sub-distributions weakly increase polarization.
Axiom 7. There is non-monotonicity of the index with respect to outward slides of the sub-distributions.
Axiom 8. Flipping the distribution around its support should leave polarization unchanged.

The general polarization index developed for discrete distributions as a consequence of these axioms (Esteban and Ray, 1994) may be written as:

(1) $$P_\alpha = K\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|\pi_i^{1+\alpha}\pi_j.$$

Here K is a normalizing constant, π_i is the sample weight of the i-th observation, and $\alpha \geq 0$ is a polarization sensitivity factor chosen by the investigator. It may readily be seen that $\alpha = 0$ yields a sample weighted Gini coefficient.

The continuous distribution analogue (Duclos *et al.*, 2004) may be written as:

(2) $$P_\alpha(F) = \int_y f(y)^\alpha \int_x |y-x|\,dF(x)\,dF(y).$$

Again, α is the polarization sensitivity factor which in this case is confined to [0.25,1].

Esteban and Ray (2007) point out that the bipolarization measures they discuss (those of Wolfson, 1994; Alesina and Spolare, 1997; Wang and Tsui, 2000) essentially measure the difference between the empirical distribution and one which has all of the population concentrated at the median. This is most obviously seen in the Wang and Tsui index which is given by:

(3) $$P^{WT} = k\int\left|\frac{x-m}{m}\right|^r f(x)\,dx.$$

The Wolfson Index is given by:

(4) $$P^W = \frac{\mu}{m}\{0.5 - L(0.5) - 0.5G\}$$

where μ is the population mean, m is the population median, $L(0.5)$ is the Lorenz ordinate at median income, and G is the Gini coefficient. The Alesina and Spolare measure is essentially the median distance to the median.

The extent to which these indices cohere with the axioms is discussed in Esteban and Ray (2007) and will not be elaborated here. What should be noted is that they all work with the overall population distribution whether the subgroups are identified or not and whether multimodality is identified in the overall population distribution or not, which, were multivariate analogues of them to exist, would represent a clear advantage over the indices and tests we are proposing here. To the author's knowledge, the only multivariate polarization index is that provided by Gigliarano and Mosler (2009), and it requires the groups to be separately identified. Their index is a function of three measures—within group inequality $W(X)$, between group inequality $B(X)$, and relative group size $S(X)$—where X is the $N \times K$ overall sample matrix of N observations on K characteristics, so:

$$P^{GM} = \Phi(W(X), B(X), S(X)), \tag{5}$$

where Φ is decreasing in its first argument and increasing in its second and third arguments. A variety of multivariate inequality measures could be employed for the first two arguments, and the relative group size index has to increase with the degree of similarity of group sizes. Here two multivariate polarization measures are proposed which work off the anatomy of the subgroup distributions and as such are very natural measures of the notion of polarization.

2. The New Multivariate Polarization Measures

For the purposes of considering bipolarization measures as poverty measures, consider two continuous multivariate uni-modal distributions $f_p(x)$ and $f_r(x)$, where x is a $K \times 1$ vector of agent characteristics such that individual wellbeing is a monotonically non-decreasing in each element of x. Assume that $f_p(x)$ is stochastically dominated by $f_r(x)$ at some order so that the wellbeing of agents under $f_p(x)$ is not preferred to that of agents under $f_r(x)$. Under a first order dominance condition this requires:

$$\int_{-\infty}^{z_1}\int_{-\infty}^{z_2}\cdot\int_{-\infty}^{z_K}\{f_p(x_1, x_2, ., x_K) - f_r(x_1, x_2, ., x_K)\}dx_1dx_2\cdot dx_K \geq 0 \;\; for\ all\ z_1, z_2, ., z_K.$$

2.1. *The Overlap Measure "OV"*

$$\text{OV} = \iint..\int \min\{f_p(x_1, x_2, \ldots, x_K)f_r(x_1, x_2, \ldots, x_K)\}dx_1dx_2, \ldots, dx_K. \tag{6}$$

One way of gauging the extent of polarization between the two groups is to measure how little they have in common. The overlap measure captures the degree of commonality between two distributions so that 1 – OV will measure the degree of dissimilarity. It is a very natural measure (always a number between 0 and 1), used for comparing similarities between multivariate distributions (Anderson *et al.*, 2009a), and is readily calculated in multivariate contexts employing multivariate kernel estimation techniques (see, e.g. Silverman, 1986) which have a well defined asymptotically normal sampling distribution (Anderson *et al.*, 2009b).

Given that the dominance condition is satisfied (so that $f_p(x)$ can be properly thought of as the poor distribution), one approach to relative poverty measurement is to consider the overlap between the poor and non-poor distributions (the greater the overlap the less there is relative poverty). It fails when the distributions f and g do not intersect in any dimension, and it cannot be employed when the groups are not identified (when working with mixtures of the two distributions where the mixing weights are unknown).

2.2. *The Polarization Trapezoid*

Let x_{mp} be the value of the characteristic vector at the modal point of the poor distribution, and x_{mr} the corresponding vector for the non-poor distribution, each characterizing the representative modal agents of those distributions. If $f_r(x)$ stochastically dominates $f_p(x)$, it can be inferred that the poor have lower wellbeing than the rich. In these circumstances the area of the trapezoid formed by the heights of the distributions at their modal points and the mean normalized Euclidean distance between the two modal points provides a measure of the polarization a representative agent of the poor perceives with respect to the rich. Letting μ_k be the mean of the k-th characteristic in the pooled population, in a two-characteristics world this may illustrated as in Figure 1.

Formally, when the poor and non-poor distributions are separately identified in K dimensions, the indicator BIPOL may be written as:

$$BIPOL = 0.5\left(f_p\left(x_{mp}\right) + f_r(x_{mr})\right)\frac{1}{\sqrt{K}}\sqrt{\sum_{k=1}^{K}\frac{\left(x_{mpk} - x_{mrk}\right)^2}{\mu_k}}. \tag{7}$$

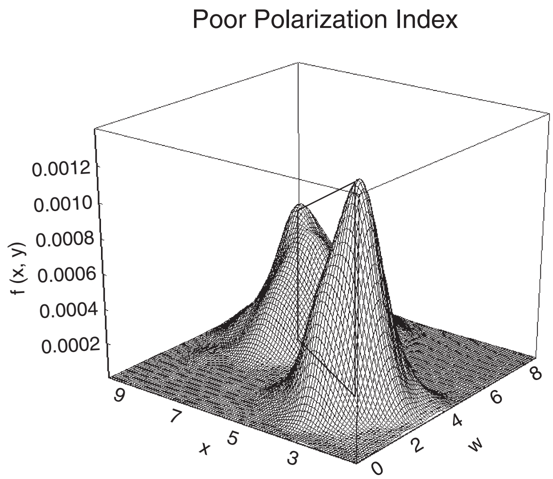


Figure 1.

When the groups are not separately identified (NI) and the index is calculated from the modal points of the mixture distribution, the poor and rich modes may be written in terms of the underlying distributions as:

$$f(x_{mp}) = f_r(x_{mp}) + w(f_p(x_{mp}) - f_r(x_{mp}))$$
$$f(x_{mr}) = f_r(x_{mr}) + w(f_p(x_{mr}) - f_r(x_{mr})).$$

The index may also be written as:

(8) $$BIPOL_{NI} = 0.5[f(x_{mp}) + f(x_{mr})]\frac{1}{\sqrt{K}}\sqrt{\sum_{k=1}^{K}\frac{(x_{mpk} - x_{mrk})^2}{\mu_k}}.$$

For present purposes consider the population distribution to be made up of two multidimensional symmetric uni-modal kernels with mean (modal) vectors μ_1 and μ_2 (with $\mu_1 >> \mu_2$ for convenience) so that x, the μ's and the a's are $k \times 1$ vectors, with λ remaining a scalar. A slide is now defined in terms of $\mu_1 - \mu_2$ becoming larger, and a squeeze increases the value of the density at the mode to $f(\mu)/\lambda$.

Axiom 1. "A squeeze of a distribution that consists of a single basic density cannot increase polarization." In the present context this axiom is not particularly relevant for evaluating the extent to which bipolarization measures capture that phenomenon. Note, however that if such a squeeze is applied to the mixture distribution (whose mean vector will be $(\mu_1 + \mu_2)/2$), the trapezoid measure will only be effective as long as the "bumps" remain identifiable.

Axiom 2. "Symmetric squeezes of the two kernels cannot reduce polarization." Given that the trapezoid index is BIPOL = $0.5(f(\mu_1) + f(\mu_2))|\mu_1 - \mu_2|$, the change in BIPOL will be λBipol/$(1 - \lambda) > 0$. The extent to which the squeeze affects the overlap measure again depends upon the extent of common support; if there is common support, then the overlap measure will reflect the effect of the squeeze appropriately.

Axiom 3. "Slides of the two kernels outward increases polarization." Again the impact on BIPOL is fairly straightforward since BIPOL is a positive linear function of $|\mu_1 - \mu_2|$ which will simply be increased by such a slide the effect. With regard to the overlap measure, as long as there is common support in the two distributions, this too will reflect polarization in the desired fashion.

Axiom 4. "Common population scaling preserves the ordering." Neither the overlap nor the trapezoidal measure are affected by common scaling, so ordering will be preserved in both cases.

Axiom 5. "Polarization indices have to come from a family where if x and y are independently distributed with marginal distributions $f(x)$ and $f(y)$, then the index is the expected value of some function $T(f(x),|x - y|)$ which is increasing in its second argument." While this is true for the trapezoidal measure it is not demonstrably true for the overlap measure.

Axiom 6. "Symmetric squeezes of the sub-distributions weakly increases polarization." This is axiom is much like Axiom 2 in the present context and the same comments apply.

Axiom 7. "There is non-monotonicity of the index with respect to outward slides of the sub-distributions." Neither the trapezoidal nor the overlap measure satisfy this axiom.
Axiom 8. "Flipping the distribution around its support should leave polarization unchanged." This is satisfied by both the trapezoidal and the overlap measures. Note that polarization measures which satisfy this axiom in the present context reflect the degree of advantage an agent from the rich group perceives from his/her position.

As a measure of polarization, the area of the trapezoid reflects "wellbeing deficiency" perceived by the poor representative agent only if the dominance condition is satisfied, since otherwise the "rich" modal point may not be deemed preferred to the poor modal point and the agents would only perceive themselves as different as opposed to poorer or richer. In order to correspond to a societal poverty measure, it should be scaled by some monotonic increasing function of the relative size of the poor group.

With respect to polarization, the intensity or within-group association is represented by the averaged heights of the modal points $f_p(x_{mp})$ and $f_r(x_{mr})$, following the intuition that the greater the mass within a region close to the modal point, the greater will be the height of the pdf. That the mean normalized Euclidean distance between the two modal points represents the sense of alienation between the two groups is somewhat more obvious. It is interesting to speculate how the identity components could be interpreted. If I am poor, the poor modal height ($f_p(x_{mp})$) tells me the extent to which there are others like me or close to me; the higher it is, the more identification with my group will I perceive. The rich modal height $f_r(x_{mr})$ tells me how easily I can identify "the other club" and reflects how strongly I may perceive the other group from whom I am alienated. The higher the rich modal height, the more closely associated the agents in that club are; the lower it is, the more widely dispersed they are. The symmetry property attaches equal importance to them in the index reflecting its "relative" nature. If, as will be discussed below, an absolute poverty measure is desired, the rich modal height should have no play and the Euclidean distance from the nearest point on a poverty frontier (rather than the modal point of the rich distribution) would correspond to a measure of alienation from the non-poor group.

Many variants of this index are possible. Note that the weights given to either the within group association or the between group alienation components could be varied if such emphasis is desired. Thus a general form of BIPOL could be $(\text{Height}^{\alpha}\text{Base}^{1-\alpha})^2$, where $0 < \alpha < 1$ represents the relative importance of the self-identification component. Similarly the modal point height components could be individually re-weighted to reflect the different importance of the identification component of the rich and poor groups. Note also that if indices based upon different numbers of characteristics are being compared, the identification component of the index should be scaled by the number of characteristics being contemplated, based upon the fact that the peak of the joint density of K independent variables distributed as $N(0,1)$ is $1/\sqrt{K}$ times the height of one $N(0,1)$.

BIPOL represents the degree of polarization a typical agent in the poverty group experiences, but it says little about the degree to which such polarization is prevalent in society. Multiplying BIPOL by w_p, the relative size of the poor group represented by $f_p(x_{mp})$, the societal poverty polarization index SPPI becomes SPPI = w_p. BIPOL will provide such an index. The statistical distribution of the multivariate trapezoidal statistic has not as yet been determined, but a preliminary and somewhat limited Monte Carlo study suggests that it is asymptotically normal (see Appendix 1), and the univariate version of the statistic does appear to be asymptotically normal (Anderson *et al.*, 2009b, 2009c).

When the poor and rich distributions are not identified, life gets a little more complicated but the principles are the same. The mixture distribution is always observed, but the question is whether it is possible to identify the sub-distributions in the mixture (or at least, can the locations and the heights of the sub-distribution peaks be identified?). In the application to be reported later this has not presented a problem, however it is not always so simple. Some discussion of modality detection is contained in a "Bump Hunting" literature reported in Silverman (1986), but it is primarily in a univariate context. Among other approaches extending the Dip test (Hartigan and Hartigan, 1985) to multivariate contexts, alternative search methods (for example, applying the Dip test along the predicted regression line) and parametric methods are all matters of current research.

3. An Application: The Increasing Relative Impoverishment of Africa and Decreasing Relative Poverty in the World

To illustrate these ideas, we consider the progress of the world distribution as an entity in itself and then consider the progress of African nations relative to the rest of the world in terms of the joint distribution of gross national product per capita and life expectancy from 1990 to 2005. Appendix 2 lists the nations included in the sample; the data were drawn from the World Bank World Development Indicators 2007 data file. The focus is the welfare of individuals, so that the GNP per capita and life expectancy are considered to be those of a representative individual for each country. It is thus appropriate to weight the measures by country population in order to reflect individual wellbeing.

To highlight the progress of the "world wellbeing distribution" through the period 1990 to 2005, the joint densities of the natural logarithms of gross national product per capita and life expectancy at five-year periods are presented in contour plots and reported in Appendix 3.[3] The most striking feature is the merging of the three modes in the 1990 distribution into two in the 2005 distribution. This is undoubtedly due to the progress of China and India throughout the period. These two countries, because of their large populations, dominate the world's joint distributions when the data are weighted by population size. For similar reasons African nations are not really apparent in these distributions. Their populations constituted between 10 and 12 percent of the total population in the sample over

[3]All of the densities and overlap measures in this paper were computed using population weighted versions of the multivariate kernel density estimator $K(x) = (4(1 - x'x)^3)/(\pi))$ (see Silverman, 1986, equation 4.6).

TABLE 1

Trapezoidal Calculations Mixture Distributions

Year	Low (Poor) Mode Modal Height	Low (Poor) Mode Location (Income, Life Expectancy)		High (Non-Poor) Mode Modal Height	High (Non-Poor) Mode Location (Income, Life Expectancy)		Relative Poverty Index (equation (7))
1990	0.3598	5.9089	4.1202	0.0966	10.0507	4.3362	0.2531
1995	0.3421	6.2625	4.1607	0.0942	10.1059	4.3478	0.2226
2000	0.3253	6.5184	4.1795	0.0923	10.2002	4.3594	0.2022
2005	0.2960	6.8493	4.2014	0.0902	10.2647	4.3720	0.1429

TABLE 2

Trapezoidal Calculations Sub-Distributions, Africa and the Rest

Year	Africa Dominates Rest of World: Max D	Africa Dominates Rest of World: $(P(\sqrt{n}*D < \lambda)$	Rest of World Dominates Africa: Max D	Rest of World Dominates Africa: $(P(\sqrt{n}*D < \lambda)$
2005	0.000628	7.89e-007	0.526846	0.426004
2000	0.001723	5.94e-006	0.475490	0.363762
1995	7.52e-005	1.13e-008	0.415322	0.291769
1990	7.68e-005	1.18e-008	0.315112	0.180115

the period, and thus they are not obvious in the overall distribution (they can barely be perceived as a bulge in the front of the mound nearest the origin in the 2005 diagram). The "within-distribution" polarization index reported in Table 1 suggests that the world's poorer nations are making considerable progress, with substantive reductions in the association and alienation components of the polarization index over time. Note that the interpretation of this index as a poverty index is sustained by the poor mode being strictly less than the non-poor mode in every dimension every year.

To consider the polarization aspect of African nations in terms of the overlap and poverty polarization measures, African nations were separated out from the rest and their GNP – LE joint density, f_A(GNP, LE), was estimated separately from that of the "Rest of the World" f_R(GNP, LE), again using the population weighted kernel reported in footnote 4. The distributions and their overlaps are illustrated in Figures 2–4. For the polarization index to be construed as a poverty index in the context of Africa, the African distribution must be dominated by the Rest of the World distribution. Table 2 presents Kolmonogorov–Smirnov statistics[4] for the dominance of the joint densities. To establish dominance, the African distribution has to be shown to not dominate the Rest of the World distribution,

[4]The formula used for $(P(\sqrt{n}*D < \lambda)$ is $1 - \exp(-2\lambda^2)$, which is Rayleigh's formula for the univariate statistic ($K = 1$). Kiefer and Wolfowitz (1958) established the existence of a distribution function for the D's when $K > 1$, but found that generally it depended upon F. Kiefer (1961) revisited the bounds issue for situations where $K > 1$, and established a bound which suggests that the formula for the univariate case would provide conservative (i.e. larger) estimates of the true values when $K > 1$, (see Anderson, 2008a).

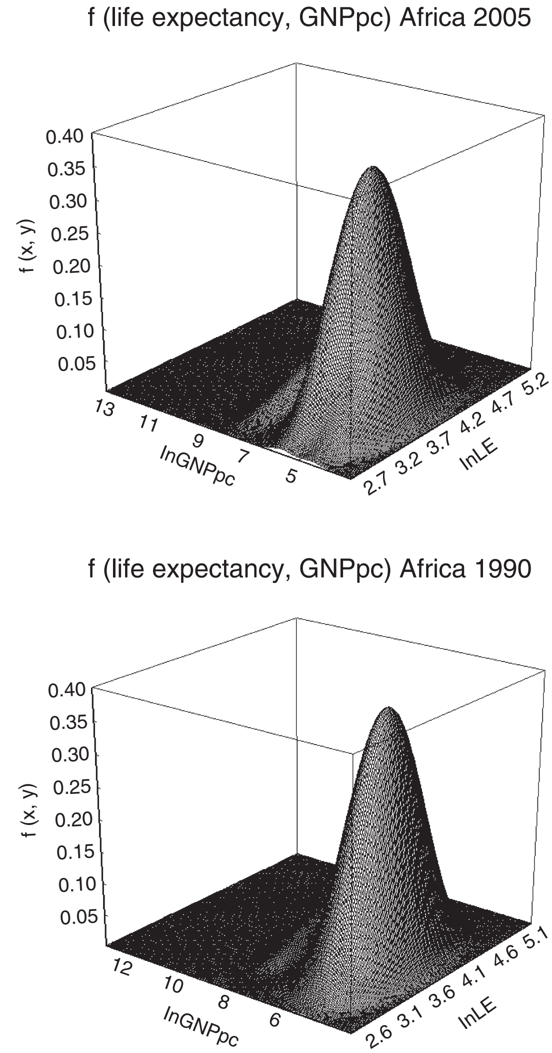


Figure 2.

and the Rest of the World distribution has to dominate that of Africa which, as Table 2 indicates, is the case.

Following Anderson *et al.* (2009b), overlap measures of the form

$$\iint \min\left(f_A(\mathrm{GNP}, \mathrm{LE}), f_R(\mathrm{GNP}, \mathrm{LE})\right) d\mathrm{GNP}d\mathrm{LE}$$

were computed, which are asymptotically normal, thus providing a test of increased polarization (in terms of decreased overlap). The results for the 1990–2005 comparison are reported in Table 3 and indicate a significant reduction in the overlap of the two distributions, suggesting an increased polarization of Africa over the period.

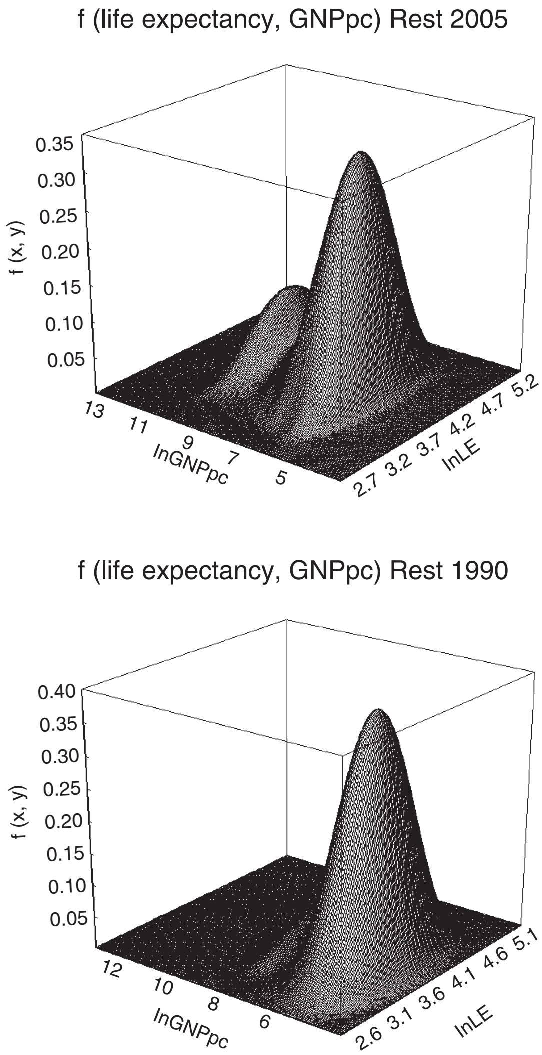


Figure 3.

Table 4 presents the results for the polarization trapezoid. Four points need to be made regarding the results of Table 4. Firstly, the alienation component of the index has increased substantially over the period, whereas the within-group association component has diminished somewhat over the period. Their net effect on BIPOL, however, has seen it increase over the period. This suggests that the representative agent has experienced slightly diminished within-group identity (inequality amongst African countries has increased) together with increased between-group alienation (the gap between African countries and the rest of the world has increased), combining for an increased sense of poverty polarization. Clearly re-weighting the alienation and association components in favor of the

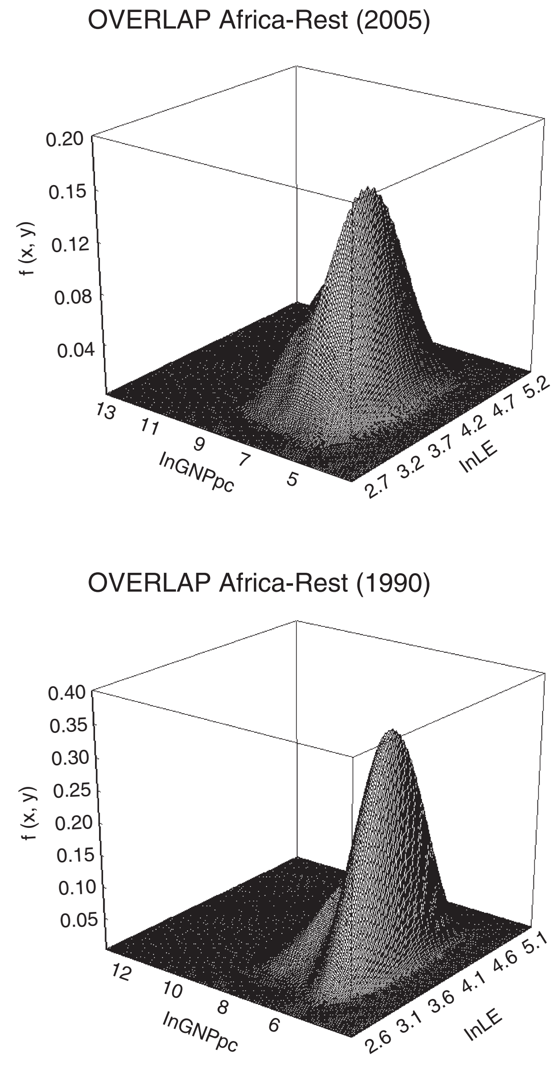


Figure 4.

TABLE 3

Overlap Test: Africa and the Rest

1990 Overlap	Standard Error	2005 Overlap	Standard Error	Standard Normal Test
0.4311	0.0403	0.6443	0.0390	–3.7268

identification component would ultimately yield a reversal of the trend revealed in this particular exercise. Finally, note that variations in Africa's population share in the sample have failed to mask the overall upward trend in Africa's contribution to world poverty.

TABLE 4
Population Weighted BIPOL Poverty Index

Year	Euclidean Distance	$(f_p(x_{mp}) + f_r(x_{mp}))/2$	BIPOL	Population Share of the Poor	SPPI
2005	0.2941	0.3460	0.1018	12.71%	1.2939
2000	0.2489	0.3566	0.0888	13.75%	1.2211
1995	0.2327	0.3617	0.0841	12.94%	1.0270
1990	0.1413	0.3769	0.0533	10.88%	0.5799

4. Conclusions

Probably the greatest difficulty associated with multivariate poverty measurement is the formulation of an appropriate poverty frontier. Here it has been argued that, under certain circumstances, concepts of polarization can be employed to characterize the sense of polarization that the poor experience in a multivariate context, which in turn may be used to reflect a sense of their relative impoverishment, circumventing the need for defining a poverty frontier. The circumstances are that the distribution of the poor outcomes must be stochastically dominated by the distribution of the non-poor outcomes at some order (were this not the case, polarization measures would only reflect perceived differences rather than perceived impoverishment).

Simple to calculate "overlap" and "trapezoidal" polarization measures have been proposed which, in the context of comparisons of poor and rich groups, circumvent the difficulties associated with defining poverty frontiers in relative poverty measures. Both are amenable to calculation in multivariate environments, but the overlap measure only works when the separate distributions are identified and have common points of support in every dimension. The trapezoidal measure has the advantage of not requiring any common points of support and of being applicable in cases where the separate distributions are not identified but are embodied in a mixture distribution in such a way that two modes are engendered. As such, they provide practical alternative relative poverty measures in circumstances which hitherto have been difficult for poverty measurement. It should be noted that these measures rely upon non-parametric kernel density estimation techniques which are notoriously reliant upon large samples, especially as the dimensionality of the problem increases. Here the dimensionality is modest (just two dimensions). The sample is small (just over 150 observations), yet statistically significant results have been obtained.

Application of the techniques to population-weighted world income and life expectancy distributions revealed an overall improvement in the lot of the poor (largely due to the strides made by China and India over those years), but when applied specifically to a comparison of Africa and the Rest of the World it revealed that Africa's relative position is deteriorating. The deterioration is largely attributable to increases in the degree of alienation measured by the Euclidean distance between its distributional mode and that of the Rest of the World, since its measure of within group association decreased, this being somewhat consistent with increased African within-group variation.

Appendix 1: The Distribution of BIPOL

The Normality of the kernel estimate f^e is discussed in Pagan and Ullah (1999). Essentially $(nh)^{0.5}(f^e - f) \sim N(0, f\int K^2(\psi)d\psi)$, if:

(1) x_i $i = 1, \ldots, n$ are i.i.d.

(2) $h_n \to 0$ as $n \to \infty$, $nh_n \to \infty$ as $n \to \infty$ and $n^{0.5}h_n^{2.5} \to 0$ as $n \to \infty$.

(3) $K(\psi)$ is in the class of all Borel measurable bounded real value functions such that $\int K(\psi)d\psi = 1$ and $\int|K(\psi)|d\psi < \infty$.

(4) $f(x)$ and its derivatives up to the second order are continuous and bounded for some neighborhood of x.

However, it is not clear that BIPOL will also be normal since it depends upon the product of a sum of these estimators (which would presumably be normal) and the Euclidean distance measure, which depends upon the modal locations determined by the estimated modes.

A limited Monte Carlo study of BIPOL suggests that it is a normally distributed variate. In the first experiment, samples (of size 150) of two bivariate normal distributions were generated with means which were k standard deviations apart, where $k = 1, 2, 3, 4$, and BIPOL was calculated for each of 200 replications in each case. A Pearson Goodness of Fit Test for normality ($\chi^2(9)$) was calculated based upon a partition into 10 equally likely intervals. The results are presented in Table A1.

In the second experiment, samples (of size 100, 150, and 200) of two bivariate normal distributions were generated with means which were 1 standard deviation apart and BIPOL was calculated for each of 200 replications. A Pearson Goodness of Fit Test for normality ($\chi^2(9)$) was calculated, based upon partition into 10 equally likely intervals. The results are presented in Table A2.

In only one experiment could the hypothesis of normality be rejected.

TABLE A1

Results: First Set of Experiments

	1 Standard Dev	2 Standard Dev	3 Standard Dev	4 Standard Dev
Dif (Std Err)	3.3936 (0.4667)	4.5203 (0.4949)	5.4616 (0.5126)	6.2386 (0.4694)
BIPOL(Std Err)	0.5029 (0.0783)	0.6685 (0.1017)	0.8031 (0.1084)	0.9213 (0.1098)
$\chi^2(9)$ $(1-F(x))$	9.4422 (0.3975)	29.061 (0.0006)	6.1940 (0.7203)	2.5353 (0.9799)

TABLE A2

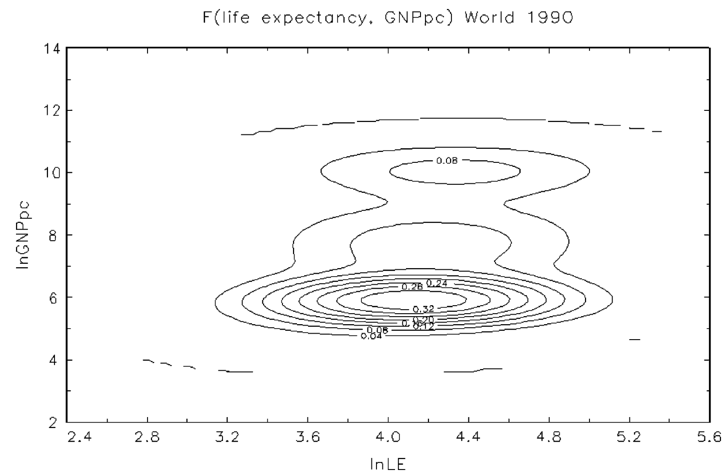
Results: Second Set of Experiments

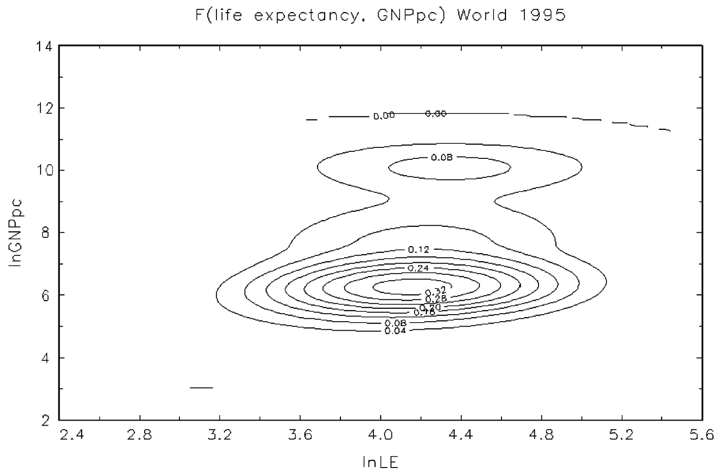
	N = 100	N = 150	N = 200
Dif (Std Err)	3.4940 (0.6195)	3.3936 (0.4667)	3.3062 (0.4096)
BIPOL(Std Err)	0.5112 (0.1014)	0.5029 (0.0783)	0.4867 (0.0720)
$\chi^2(9)$ $(1 - F(x))$	5.0662 (0.8285)	9.4422 (0.3975)	4.0154 (0.9104)

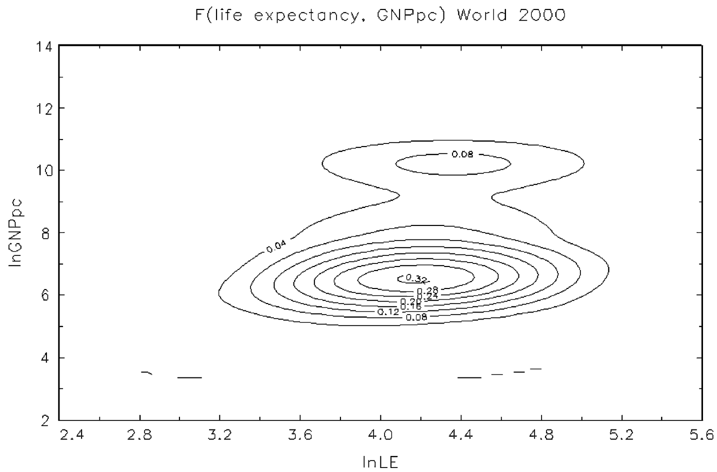
Appendix 2: Nations Included in the Sample

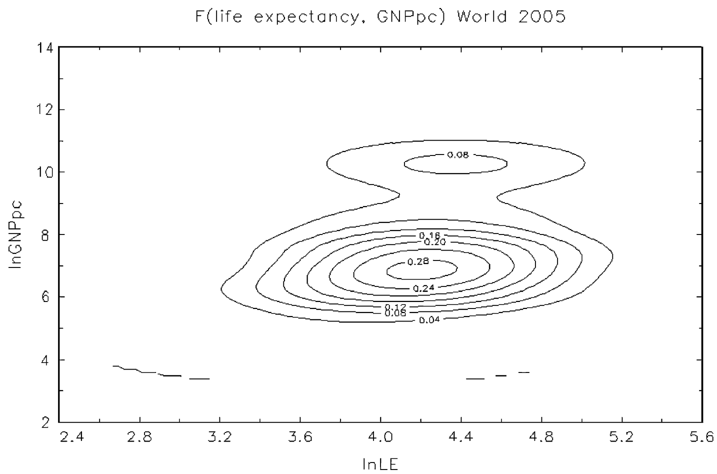
Albania, Algeria, Angola, Argentina, Armenia, Australia, Austria, Azerbaijan, Bahrain, Bangladesh, Belarus, Belgium, Belize, Benin, Bhutan, Bolivia, Botswana, Brazil, Bulgaria, Burkina Faso, Burundi, Cameroon, Canada, Cape Verde, Central African Republic, Chad, Chile, China, Colombia, Comoros, Congo, Dem. Rep., Congo, Rep., Costa Rica, Cote d'Ivoire, Croatia, Czech Republic, Denmark, Djibouti, Dominican Republic, Ecuador, Egypt, El Salvador, Estonia, Ethiopia, Finland, France, Gabon, Gambia, Georgia, Germany, Ghana, Greece, Guatemala, Guinea, Guinea-Bissau, Guyana, Haiti, Honduras, Hong Kong (China), Hungary, Iceland, India, Indonesia, Iran, Ireland, Israel, Italy, Jamaica, Japan, Jordan, Kazakhstan, Kenya, Korea, Kyrgyz Republic, Lao PDR, Latvia, Lebanon, Lesotho, Liberia, Lithuania, Luxembourg, Macao (China), Madagascar, Malawi, Malaysia, Mali, Malta, Mauritania, Mauritius, Mexico, Micronesia (Fed. Sts.), Moldova, Mongolia, Morocco, Namibia, Nepal, Netherlands, New Zealand, Nicaragua, Niger, Nigeria, Norway, Pakistan, Panama, Paraguay, Peru, Philippines, Poland, Portugal, Russian Federation, Rwanda, Samoa, Sao Tome and Principe, Saudi Arabia, Senegal, Sierra Leone, Singapore, Slovak Republic, Slovenia, Solomon Islands, South Africa, Spain, Sri Lanka, St. Vincent and the Grenadines, Sudan, Suriname, Swaziland, Sweden, Switzerland, Syria, Tajikistan, Tanzania, Thailand, Togo, Tonga, Trinidad and Tobago, Tunisia, Turkey, Uganda, Ukraine, United Arab Emirates, United Kingdom, United States, Uruguay, Uzbekistan, Vanuatu, Venezuela, Vietnam, Yemen, Zambia, Zimbabwe.

Appendix 3: Contour Plots




F(life expectancy, GNPpc) World 1995
lnGNPpc
lnLE
2
4
6
8
10
12
14
2.4
2.8
3.2
3.6
4.0
4.4
4.8
5.2
5.6


F(life expectancy, GNPpc) World 2000
lnGNPpc
lnLE
2
4
6
8
10
12
14
2.4
2.8
3.2
3.6
4.0
4.4
4.8
5.2
5.6


F(life expectancy, GNPpc) World 2005
lnGNPpc
lnLE
2
4
6
8
10
12
14
2.4
2.8
3.2
3.6
4.0
4.4
4.8
5.2
5.6

References

Alesina, A. and E. Spolare, "On the Number and Size of Nations," *Quarterly Journal of Economics*, 113, 1027–56, 1997.

Anderson, G. J., "The Empirical Assessment of Multidimensional Welfare, Inequality and Poverty: Sample Weighted Generalizations of the Kolmorogorov-Smirnov Two Sample Test for Stochastic Dominance," *Journal of Economic Inequality*, 6, 73–87, 2008a.

———, "Indices and Tests for Alienation Based Upon Gini type and Distributional Overlap Measures," in Gianni Betti and Achilli Lemmi (eds), *Advances in Income Inequality and Concentration Measures*, Routledge, London and New York, 70–85, 2008b.

Anderson, G. J., I. Crawford, and A. Leicester, "Efficiency Analysis and the Lower Convex Hull Approach," in N. Kakwani and J. Silber (eds), *Quantitative Approaches to Multidimensional Poverty Measurement*, Palgrave Macmillan, Basingstoke, 176–91, 2008.

Anderson G. J., Y. Ge, and T. W. Leo, "Distributional Overlap: Simple, Multivariate, Parametric and Non-Parametric Tests for Alienation, Convergence and General Distributional Difference Issues," *Econometric Reviews*, 2009a.

Anderson, G. J., O. Linton, and J. Wang, "Nonparametric Estimation of a Polarization Measure," Mimeo, University of Toronto, 2009b.

Anderson, G. J., O. Linton, and T. W. Leo, "Examining Convergence Hypotheses: Overlapping Distributions and the Polarization Trapezoid," Mimeo, Economics Department, University of Toronto, 2009c.

Duclos, J-Y., J. Esteban, and D. Ray, "Polarization: Concepts, Measurement and Estimation," *Econometrica*, 72, 1737–72, 2004.

Duclos, J-Y., D. Sahn, and S. D. Younger, "Robust Multidimensional Poverty Comparisons," *The Economic Journal*, 116, 943–68, 2006.

Esteban, J. and D. Ray, "On the Measurement of Polarization," *Econometrica*, 62, 819–52, 1994.

Esteban, J. and D. Ray, "A Comparison of Polarization Measures," UFAE and IAE Working Papers, Unitat de Fonaments de l'Anàlisi Econòmica (UAB) and Institut d'Anàlisi Econòmica (CSIC), 2007.

Esteban, J., C. Gradin, and D Ray, "Extensions of a Measure of Polarization with an Application to the Income Distribution of Five OECD Countries," Mimeo, Instituto de Analisis Economico.

Foster, J. and M. Wolfson, "Polarization and the Decline of the Middle Class in Canada and the USA," Mimeo, Vanderbilt University, 1992.

Gigliarano, C. and K. Mosler, "Constructing Indices of Multivariate Polarization," *Journal of Economic Inequality*, 2009.

Grusky, D. B. and R. Kanbur, *Poverty and Inequality. Studies in Social Inequality*, Stanford University Press, 2006.

Hartigan, J. A. and P. M. Hartigan, "The Dip Test of Unimodality," *Annals of Statistics*, 13, 70–84, 1985.

Kakwani, N. and J. Silber, *Quantitative Approaches to Multidimensional Poverty Measurement*, Palgrave Macmillan, 2008.

Kiefer, J., "On Large Deviations of the Empiric D.F. of Vector Chance Variables and a Law of the Iterated Logarithm," *Pacific Journal of Mathematics*, 11, 649–60, 1961.

Kiefer, J. and J. Wolfowitz, "On the Deviations of the Empiric Distribution Function of Vector Chance Variables," *Transactions of the American Mathematical Society*, 87, 173–86, 1958.

Pagan, A. and A. Ullah, *Nonparametric Econometrics Themes in Modern Econometrics*, Cambridge University Press, 1999.

Pearson, K., "On a Criterion that a Given System of Deviations from the Probable in the Case of a Correlated System of Variables is Such that it Can Reasonably Be Supposed to Have Arisen from Random Sampling," *Philosophical Magazine*, 50, 157–75, 1900.

Sen, A. K., "Poor Relatively Speaking," *Oxford Economic Papers*, 35, 153–69, 1983.

Silverman, B. W., *Density Estimation for Statistics and Data Analysis*, Chapman & Hall, 1986.

Townsend, P., "A Sociological Approach to the Measurement of Poverty—A Rejoinder to Professor Amartya Sen," *Oxford Economic Papers*, 37, 659–68, 1985.

Wang, Y. Q. and K. Y. Tsui, "Polarization Orderings and New Classes of Polarization Indices," *Journal of Public Economic Theory*, 2, 349–63, 2000.

Wolfson, M. C., "When Inequalities Diverge," *American Economic Review, Papers and Proceedings*, 94, 353–8, 1994.

Zhang, X. and R. Kanbur, "What Difference Do Polarization Measures Make? An Application to China," *Journal of Development Studies*, 37, 85–98, 2001.

Zheng, B., "Aggregate Poverty Measures," *Journal of Economic Surveys*, 11, 123–62, 1997.