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UNIT CONSISTENCY AND BIPOLARIZATION OF INCOME DISTRIBUTIONS

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Many polarization measures proposed in the literature assume some invariance condition. Clearly, each invariance condition imposes a specific value judgment on polarization measurement. In inequality and poverty measurement, B. Zheng suggests rejecting these invariance conditions as axioms, and proposes replacing them with the unit-consistency axiom. This property demands that the inequality or poverty rankings, rather than their cardinal values, are not altered when income is measured in different monetary units. Following Zheng's proposal we explore the consequences of the unit-consistency axiom in the bipolarization field. We introduce a new family of Krtscha-type intermediate bipolarization indices, and also propose and characterize a class of intermediate polarization orderings which are unit-consistent. Finally, a short empirical application using data from Spain is also provided to illustrate how the bipolarization orderings proposed may be used in practice.

1. Introduction

In recent years, the concern about the extent to which a society is polarized has attracted a great deal of attention, since polarization has been proved to be closely linked to the generation of tension and social unrest. This interest has led several authors to try to define measures to capture the phenomenon. Foster and Wolfson (1992), Esteban and Ray (1994), and Wolfson (1994, 1997) conceptualized the notion of polarization and developed corresponding measures. Following these seminal papers a number of studies have been devoted to proposing different polarization indices and introducing criteria to order distributions in terms of polarization. In most of these proposals some invariance condition, be that scale, translation, or intermediate, is assumed.[1]

Zheng (2007a, 2007b, 2007c) highlights the limitations of any of these invariance conditions since they all impose value judgments on the measurement. On the

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[1]For instance: Davis and Huston (1992), Jenkins (1995), Milanovic (2000), Wang and Tsui (2000), Chakravarty and Majumder (2001), Kanbur and Zhang (2001), Duclos *et al.* (2004), Duclos and Échevin (2005), Lasso de la Vega and Urrutia (2006), Chakravarty *et al.* (2007), Deutsch *et al.* (2007), and Chakravarty and D'Ambrosio (2010).

other hand, it is clear that, in empirical applications, the units in which income is measured should not affect the ranking results. In other words, it makes no sense that income distribution comparisons vary when the units in which income is measured change. Zheng introduces a new ordinal principle, the unit-consistency axiom, which requires the inequality or poverty rankings, rather than the inequality or poverty levels, not to be affected by the units in which incomes are expressed.[2] Obviously, all relative indices satisfy this principle. In addition, Zheng (2007c) explores the implications of the unit-consistency axiom for inequality orderings. He proposes an appealing and comprehensive notion of intermediateness for inequality orderings through the Lorenz curve and characterizes the intermediate orderings which are unit-consistent. He also shows that the Krtscha-type dominance (Krtscha, 1994) is the only Lorenz ordering which is both intermediate and unit-consistent. These contributions are the background for our study.

More precisely, in Section 2, this paper proposes a straightforward extension of the unit-consistency axiom to the bipolarization indices. We also analyze the implications of this property for the polarization measures proposed in the literature. Then a new family of Krtscha-type intermediate bipolarization indices, which is unit-consistent, is proposed.

Section 3 is devoted to the polarization orderings. Since the polarization comparisons may be sensitive to the choice of polarization measure, a standard procedure to avoid any conflict is to demand unanimous agreement among classes of polarization measures. In the inequality field, the Lorenz curve is generally used to test whether one distribution is unambiguously more unequal than another providing that the Pigou–Dalton transfer principle is accepted. The polarization curve proposed by Foster and Wolfson (1992) plays a role similar to that of the Lorenz curve and has already been used as a tool for ordering distributions in terms of bipolarization (for instance, Chakravarty *et al.*, 2007; Chakravarty and D'Ambrosio, 2010). Taking these papers as a reference, and following Zheng (2007c), we propose and characterize a class of intermediate Foster–Wolfson orderings, and show that the only unit-consistent members are those related to the Krtscha-type intermediate notion.

Section 4 provides an empirical application based on data from the Spanish Household Budget Surveys (HBS) for 1973/74, 1980/81, and 1990/91, and the more recent continuous HBS for 1998 and 2003. In this illustration we show how our approach can be used to analyze polarization across regions and over time.

The paper ends with some concluding remarks. Most of the proofs of our paper follow both Zheng (2007c) and Chakravarty *et al.* (2007).

2. Unit-Consistent Bipolarization Indices

2.1. *Basic Notions about Bipolarization Indices*

We consider a population of $n \geq 2$ individuals. Individual *i*'s income is denoted by $x_i \in \mathbb{R}_{++} = (0, \infty)$, $i = 1, \ldots, n$. An income distribution is represented

[2]Diez *et al.* (2008) have generalized some of Zheng's results to a multidimensional setting.

by a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$. We let $D = \bigcup_{n=1}^{\infty} \mathbb{R}^n_{++}$ represent the set of all finite dimensional income distributions and denote the mean and the median of any $\mathbf{x} \in D$ by $\bar{\mathbf{x}}$ and $m(\mathbf{x})$ respectively.

In this paper we assume that a bipolarization measure is a function $P: D \to \mathbb{R}$ which fulfils the following four basic properties:

(i) *Continuity*: $P(\mathbf{x})$ is a continuous function of $\mathbf{x}$.

(ii) *Symmetry*: $P(\mathbf{x}) = P(\mathbf{x}\Pi)$ for any $\mathbf{x} \in D$ and where Π is any $n \times n$ permutation matrix.

(iii) *Replication Invariance*: $P(\mathbf{y}) = P(\mathbf{x})$ if $\mathbf{y}$ is obtained from $\mathbf{x}$ by a replication.

(iv) *Normalization*: $P(\mathbf{x}) = 0$ if all the incomes of the distribution $\mathbf{x}$ are identical.

The symmetry axiom allows us, without loss of generality, to assume that all income distributions are ranked, that is, for all $n \in \mathbb{N}$ and for all $\mathbf{x} \in \mathbb{R}^n_{++}$, $x_1 \leq x_2 \leq \ldots \leq x_n$. In this paper, for the sake of simplicity and whenever it is not confusing, we use the same notation for an income distribution and its ranked permutation. Let $\bar{n} = (n+1)/2$. We denote by $\mathbf{x}_+$ and $\mathbf{x}_-$ the sub-vectors of $\mathbf{x}$ which contains x_i for $i > \bar{n}$ and x_i for $i < \bar{n}$ respectively. Hence, for any $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^n_{++}$, $\mathbf{x} = (\mathbf{x}_-, \mathbf{x}_+)$ if n is even and $\mathbf{x} = (\mathbf{x}_-, m(\mathbf{x}), \mathbf{x}_+)$ if n is odd. Moreover, $\bar{\mathbf{x}}_+$ and $\bar{\mathbf{x}}_-$ stand for the mean of $\mathbf{x}_+$ and $\mathbf{x}_-$ respectively.

The replication invariance principle permits comparisons of distributions of different sizes.

None of these properties are sufficient to guarantee that the function P is able to capture the essence of polarization. A polarization measure is usually assumed to have two basic properties. First higher within-group homogeneity is bound to increase polarization. Second, heterogeneity across groups contributes to an increase in polarization. In the case of bipolarization measures these principles are usually known as *Non-Decreasing Spread* and *Non-Decreasing Bipolarity* axioms respectively, and are formulated in the following way:

(v) *Non-Decreasing Spread:* $P(\mathbf{x}) \leq P(\mathbf{y})$ whenever $m(\mathbf{x}) = m(\mathbf{y})$, and $\mathbf{y}$ is derived from $\mathbf{x}$ by a rank-preserving increase of the income of an individual above the median, and/or by a rank-preserving reduction of the income of an individual below the median.

(vi) *Non-Decreasing Bipolarity:* $P(\mathbf{x}) \leq P(\mathbf{y})$ whenever $m(\mathbf{x}) = m(\mathbf{y})$, and $\mathbf{y}$ is derived from $\mathbf{x}$ by a rank-preserving progressive transfer from a richer person to a poorer one, both on the same side of the median.

It should be stressed that these axioms only allow distributions with the same median to be compared. However, assuming some invariance condition, this restriction may be overcome. In fact, to derive polarization indices, invariance properties are often invoked. A *relative* polarization index remains unchanged with proportional changes in all incomes, whereas an *absolute* polarization index remains unchanged if the same amount of money is added to all individual incomes. These two notions correspond to what in the field of inequality are known as the rightist and leftist points of view (See Kolm, 1977). There are also intermediate concepts of invariance which can be applied to the measurement of polarization. Chakravarty and D'Ambrosio (2010) adopted the notion of intermediateness proposed by Pfingsten (1986), which requires that any combination of an equal proportional increase in all incomes, and an equal amount increase in all incomes, should not change the polarization level. That is, **x** and **z** are intermediate polarization invariant if and only if:

$$\frac{x_i - m(\mathbf{x})}{\mu m(\mathbf{x}) + 1 - \mu} = \frac{z_i - m(\mathbf{z})}{\mu m(\mathbf{z}) + 1 - \mu}$$

for all $i = 1, \ldots, n$, and where the value judgment parameter μ, $0 \leq \mu \leq 1$, indicates the degree of intermediateness. The two extreme cases, that is, for $\mu = 0$ and $\mu = 1$, correspond to the absolute and relative cases respectively. This transformation can also be expressed as:

$$z_i = (1 + \lambda\mu)x_i + \lambda(1 - \mu)$$

where λ is any real constant such that $(1 + \lambda\mu)x_i + \lambda(1 - \mu) > 0$.

Assuming this notion of intermediateness, Chakravarty and D'Ambrosio (2010) introduce two new classes of bipolarization intermediate measures. First, they propose the following family:

(1) $$F_\mu(\mathbf{x}) = \frac{2(\bar{\mathbf{x}}_+ - \bar{\mathbf{x}}_- - A_G(\mathbf{x}))}{\mu m(\mathbf{x}) + 1 - \mu}$$

where $A_G(\mathbf{x})$ is the absolute Gini coefficient of income distribution **x**. It is worth noting that if $\mu = 0$, expression (1) becomes the absolute polarization index proposed by Chakravarty *et al.* (2007), whereas if $\mu = 1$, F_1 coincides with the Wolfson relative index of polarization (Wolfson, 1994).

Chakravarty and D'Ambrosio (2010) also propose the following two-parameter family:

(2) $$C_\mu(\mathbf{x}) = \frac{\left(n^{-1} \sum_{1 \leq i \leq n} |x_i - m(\mathbf{x})|^r\right)^{\frac{1}{r}}}{\mu m(\mathbf{x}) + 1 - \mu}, \quad \text{with } 0 < r \leq 1.$$

Other intermediate invariance conditions introduced in the literature may also be applied to the polarization field. For instance, we propose incorporating the intermediate inequality notion proposed by Krtscha (1994), and characterized and generalized by Yoshida (2005), as follows: distributions **x** and **z** are Krtscha-type intermediate polarization invariant if and only if:

$$\frac{x_i - m(\mathbf{x})}{(m(\mathbf{x}))^\lambda} = \frac{z_i - m(\mathbf{z})}{(m(\mathbf{z}))^\lambda}$$

for all $i = 1, \ldots, n$ and for some constant λ, $0 \leq \lambda \leq 1$, which can be regarded as the degree of intermediateness. The two polar cases coincide with the absolute, if $\lambda = 0$, and the relative, for $\lambda = 1$, notions. Equivalently, this invariance property states that **x** and **z** are invariant according to this intermediate condition if and only if **x** is transformed to **z** through:

$$z_i = \gamma^{\lambda} x_i + (\gamma - \gamma^{\lambda}) m(\mathbf{x})$$

for some constant $\gamma > 1$.

Similarly, it is not difficult to prove that the expressions below can be interpreted as classes of Krtscha-type intermediate bipolarization indices:

$$P_{\lambda}(\mathbf{x}) = \frac{2(\bar{\mathbf{x}}_{+} - \bar{\mathbf{x}}_{-} - A_G(\mathbf{x}))}{(m(\mathbf{x}))^{\lambda}} \tag{3}$$

where $A_G(\mathbf{x})$ is again the absolute Gini coefficient of income distribution $\mathbf{x}$; and

$$P_{\lambda}(\mathbf{x}) = \frac{\left(n^{-1} \sum_{1 \le i \le n} |x_i - m(\mathbf{x})|^r \right)^{\frac{1}{r}}}{(m(\mathbf{x}))^{\lambda}} \quad \text{with } 0 < r \le 1. \tag{4}$$

This Krtscha-type intermediate notion will play an important role in our paper.

Zoli (1998, 2003) generalized the intermediate notion of both Pfingsten- and Krtscha-type. We suggest applying Zoli's two-parameter generalization to the polarization field and state that $\mathbf{x}$ and $\mathbf{z}$ are Zoli-type intermediate polarization invariant if and only if:

$$\frac{x_i - m(\mathbf{x})}{(\mu m(\mathbf{x}) + 1 - \mu)^{\lambda}} = \frac{z_i - m(\mathbf{z})}{(\mu m(\mathbf{z}) + 1 - \mu)^{\lambda}} \tag{5}$$

for all $i = 1, \ldots, n$ and for some constants μ and λ with $\mu, \lambda \in [0,1]$. Equivalently we may state that: $\mathbf{x}$ and $\mathbf{z}$ are intermediate Zoli-type invariant if and only if $\mathbf{x}$ is transformed to $\mathbf{z}$ through:

$$z_i = \gamma^{\lambda} x_i + (\gamma - \gamma^{\lambda}) m(\mathbf{x}) + (1 - \mu)(\gamma - 1)/\mu$$

for some constant $\gamma > 1$.

Expression (5) becomes Pfingsten's condition when $\lambda = 1$, and the generalization of Krtscha's condition proposed by Yoshida (2005) when $\mu = 1$. It is not difficult to derive that the following class of intermediate polarization indices may be considered as a class of Zoli-type intermediate bipolarization indices:

$$F_{\lambda\mu}(\mathbf{x}) = \frac{2(\bar{\mathbf{x}}_{+} - \bar{\mathbf{x}}_{-} - A_G(\mathbf{x}))}{(\mu m(\mathbf{x}) + 1 - \mu)^{\lambda}}. \tag{6}$$

Other answers have been given in the literature to the question concerning the amount of income that has to be distributed among all the individuals without altering the level of inequality. Such alternative approaches could be applied to the field of polarization. Such a discussion is however beyond the aim of this paper.

2.2. *Unit-Consistency Axiom for Polarization Measures*

As already mentioned, Zheng (2007a, 2007b) analyzed in depth the implications of the invariance conditions usually assumed to define inequality and poverty measures. He proposed a new axiom of unit-consistency, which requires that the inequality or poverty rankings between two distributions should not be affected by the unit in which income is expressed. This axiom has a straightforward generalization to the polarization field:

(vii) *Unit-Consistency Axiom*: A polarization measure P is *unit-consistent* if for any two distributions $\mathbf{x}$ and $\mathbf{y} \in \mathrm{D}$, such that $P(\mathbf{x}) < P(\mathbf{y})$, then $P(\theta\mathbf{x}) < P(\theta\mathbf{y})$ for any $\theta > 0$.

It is clear that the scale invariance principle implies unit-consistency, and hence, every relative polarization measure is unit-consistent. Even, interestingly enough, the absolute index proposed by Chakravarty *et al.* (2007), which corresponds with equation (1) when $\mu = 0$, is unit-consistent. In fact:

$$Q(\theta\mathbf{x}) = 2\left(\overline{\theta\mathbf{x}}_+ - \overline{\theta\mathbf{x}}_- - A_G(\theta\mathbf{x})\right) = 2\theta(\bar{\mathbf{x}}_+ - \bar{\mathbf{x}}_- - A_G(\mathbf{x})) = \theta Q(\mathbf{x}).$$

The Krtscha-type bipolarization indices introduced in the previous section, equations (3) and (4), also fulfill the unit-consistency axiom:

$$P_\lambda(\theta\mathbf{x}) = \frac{\theta}{\theta^\lambda}\frac{2(\bar{\mathbf{x}}_+ - \bar{\mathbf{x}}_- - A_G(\mathbf{x}))}{(m(\mathbf{x}))^\lambda} = \frac{\theta}{\theta^\lambda}P_\lambda(\mathbf{x})$$

$$P_\lambda(\theta\mathbf{x}) = \frac{\theta}{\theta^\lambda}\frac{\left(n^{-1}\sum_{1\le i\le n}|x_i - m(\mathbf{x})|^r\right)^{\frac{1}{r}}}{(m(\mathbf{x}))^\lambda} = \frac{\theta}{\theta^\lambda}P_\lambda(\mathbf{x}).$$

However the two classes of intermediate bipolarization indices proposed by Chakravarty and D'Ambrosio (2010), equations (1) and (2), violate the unit-consistent axiom except for the two polar cases. Consider the two distributions $\mathbf{x} = (1,2,3,4,5)$ and $\mathbf{y} = (0.1,0.1,0.6,1,2)$. Computing the index according to equation (1) with $\mu = 0.01$ we get $F_{0.01}(\mathbf{x}) = 4.3137 > F_{0.01}(\mathbf{y}) = 1.0321$, whereas multiplying $\mathbf{x}$ and $\mathbf{y}$ by $\theta = 1000$ we find $F_{0.01}(\theta\mathbf{x}) = 141.9812 < F_{0.01}(\theta\mathbf{y}) = 147.06$. With respect to the indices in equation (2), for the same values of the parameters with $r = 1$, we obtain $C_{0.01}(\mathbf{x}) = 0.980 > C_{0.01}(\mathbf{y}) = 0.5622$ and $C_{0.01}(\theta\mathbf{x}) = 32.268 < C_{0.01}(\theta\mathbf{y}) = 80.114$. In general, examples can be found for the rest of the members of these two families.

3. Unit-Consistent Bipolarization Orderings

3.1. *Basic Notions about Bipolarization Orderings*

In order to establish unanimous bipolarization rankings of income distributions, generalizations of the curve proposed by Foster and Wolfson (1992) have been introduced in the literature. Foster and Wolfson (1992) define, for any ranked income distribution $\mathbf{x}$, a relative polarization curve, which we refer to as the FWC_R

curve, which shows to what extent the current distribution is different from the hypothetical situation in which everybody enjoys the median income. The ordinate corresponding to the k/n percentage of population is computed according to the following expression:[3]

$$FWC_R(\mathbf{x};k)=\begin{cases}\dfrac{1}{n}\displaystyle\sum_{k\le i\le \bar{n}}\dfrac{(m(\mathbf{x})-x_i)}{m(\mathbf{x})} & \text{if} \quad 1\le k\le \bar{n}\\ \dfrac{1}{n}\displaystyle\sum_{\bar{n}\le i\le k}\dfrac{(x_i-m(\mathbf{x}))}{m(\mathbf{x})} & \text{if} \quad \bar{n}\le k\le n\end{cases}$$

where $\bar{n}=(n+1)/2$. Consequently, the ordinate of the curve at $k=\bar{n}$ involves the median. If n is odd, the median coincides with one of the income values of the distribution **x**. In contrast, if n is even, the value does not belong to the distribution. Nevertheless, the ordinate at $\bar{n}$ is defined since the median is the reference level in the bi-polarization measurement (see for instance Chakravarty *et al.*, 2007). This remark will remain valid in all the definitions of polarization curves given below.

This FWC_R curve allows the introduction of the following dominance criterion: for any two distributions **x** and **y** ∈ D, **x** relative *FW* dominates **y**, and we write $\mathbf{x}\succsim_{FW_R}\mathbf{y}$, if and only if

$$FWC_R(\mathbf{x};p)\ge FWC_R(\mathbf{y};p)$$

for all $p \in [0,1]$ and the strict inequality holds at least once. This dominance criterion establishes an unambiguous partial ranking of income distributions among a class of polarization relative measures. In other words, all the indices in this family give the same ranking for any pair of distributions, as long as their FWC_R curves do not cross.

Chakravarty *et al.* (2007) derived the absolute polarization curve, the FWC_A curve, scaling up the FWC_R curve by the median, according to the following expression:

$$FWC_A(\mathbf{x};k)=\begin{cases}\dfrac{1}{n}\displaystyle\sum_{k\le i\le \bar{n}}(m(\mathbf{x})-x_i) & \text{if} \quad 1\le k\le \bar{n}\\ \dfrac{1}{n}\displaystyle\sum_{\bar{n}\le i\le k}(x_i-m(\mathbf{x})) & \text{if} \quad \bar{n}\le k\le n\end{cases}.$$

They then derived the corresponding dominance criterion: **x** absolute *FW* dominates **y**, and we denote $\mathbf{x}\succsim_{FW_A}\mathbf{y}$, if and only if

$$FWC_A(\mathbf{x};p)\ge FWC_A(\mathbf{y};p)$$

for all $p \in [0,1]$ and the strict inequality holds at least once. They also identify the class of bipolarization indices which agree on their ranking when the FWC_A curves of the distributions do not intersect.

[3]In fact, although the FWC_R curve is originally defined using the cumulative distribution function (Foster and Wolfson, 1992), for the sake of coherence with the rest of the definitions we choose this equivalent formulation of the curve proposed by Chakravarty *et al.* (2007).

Similarly to the Lorenz curve in the inequality framework, a family of intermediate polarization curves may be defined. If the dominance condition proposed by Zoli (1998, 2003), corresponding to the generalized intermediate notion of both Pfingsten- and Krtscha-type is adopted, the ordinates of the polarization curve are as follows:

$$(7) \qquad FWC_Z(\mathbf{x};\mu,\lambda,k) = \begin{cases} \frac{1}{n}\sum_{k\le i\le \bar{n}} \frac{(m(\mathbf{x})-x_i)}{(\mu m(\mathbf{x})+1-\mu)^{\lambda}} & \text{if} \quad 1\le k\le \bar{n} \\ \frac{1}{n}\sum_{\bar{n}\le i\le k} \frac{(x_i-m(\mathbf{x}))}{(\mu m(\mathbf{x})+1-\mu)^{\lambda}} & \text{if} \quad \bar{n}\le k\le n \end{cases}$$

for some constants μ and λ (μ, $\lambda \in (0,1)$).

One of the extreme cases, corresponding to the case where $\lambda = 1$, coincides with the Pfingsten-type curve proposed by Chakravarty and D'Ambrosio (2010); the other one, when $\mu = 1$, can be considered the Krtscha-type polarization curve. In addition, since the class of bipolarization indices in expression (6) corresponds to the area under the FWC_Z curve for the different values of the λ and μ parameters, the indices in expressions (1) and (3) also coincide with the areas under the respective curves.

The dominance criterion related to this curve can be formulated as follows: for any two distributions **x** and **y** $\in$ D, **x** Zoli(μ, λ)-type *FW* dominates **y**, and we denote $\mathbf{x} \succsim_{FW_Z} \mathbf{y}$, if and only if

$$FWC_Z(\mathbf{x};\mu,\lambda,p) \ge FWC_Z(\mathbf{y};\mu,\lambda,p)$$

for all $p \in [0,1]$ and the strict inequality holds at least once.

The particular case when $\mu = 1$ will be referred to as the Krtscha(λ)-type FW dominance. Specifically, we denote by $\mathbf{x} \succsim_{FW_K} \mathbf{y}$ if distributions **x** and **y** $\in$ D are ordered according to the Krtscha(λ)-type criterion.

Finally, Zheng (2007c) proposes the most comprehensive notion of intermediateness.[4] Capturing the essence of what "intermediateness" means in relation to extremes, and since the natural extremes for inequality orderings are the absolute and the relative Lorenz orderings, he stresses that "it is reasonable for the intermediate notion to be sensitive to and only to the changes in these bounds."

Using the relative and absolute *FW* polarization curves instead of the Lorenz ones, we apply to the polarization field Zheng's axiom of intermediateness and write:

(viii) *The Intermediateness Axiom*: For any **x**,**y** $\in$ D, if **x** both relative and absolute FW dominates **y**, then **x** should intermediate FW dominates **y**. Specifically, for all $p \in [0,1]$, if $FWC_R \ge FWC_R(\mathbf{y};\, p)$ and $FWC_A(\mathbf{x};\, p) \ge FWC_A(\mathbf{y};\, p)$, then $FWC_I(\mathbf{x};\, \mu, p) \ge FWC_I(\mathbf{y};\, \mu, p)$ for all $\mu \in (0, 1)$, where μ is the parameter of intermediateness and $FWC_I(\mathbf{x};\, \mu, p)$ denotes any intermediate *FW* curve. In addition $FWC_I(\mathbf{x};\, \mu, p) \ge FWC_I(\mathbf{y};\, \mu, p)$ if and only if at least one of the above inequalities is strict.

[4]For a better understanding of our application to the polarization field of Zheng's ideas, a thorough reading of Section 3 in Zheng (2007c) is highly recommended.

Following Zheng (2007c, p. 524), it is easy to prove that both Pfingsten- and Krtscha-type dominances satisfy this axiom.

The following proposition establishes the implication of the intermediateness axiom for a general intermediate polarization curve.

Proposition 3.1.1. *An intermediate polarization curve FWC_I satisfies the intermediateness axiom if and only if there exists a function G which is continuous and increasing in its first two arguments such that*

$$FWC_I(\mathbf{x}; \mu, p) = G[FWC_R(\mathbf{x}; p), FWC_A(\mathbf{x}; p), \mu] \tag{8}$$

for all $p \in [0,1]$.

Proof. The proof is straightforward following that of Proposition 3.1 in Zheng (2007c). *Q.E.D.*

Since intuitively an "intermediate" curve should lie between the relative and the absolute Lorenz curves, Zheng proposes a particular way to consider intermediate curves using the well-known *quasilinear-weighted-means.* In polarization terms this "intermediate" curve, which we refer to as a Zheng-type intermediate polarization curve, may be defined as follows:

Definition 3.1.1. *For any distribution* $\mathbf{x} \in D$, *its Zheng-type intermediate polarization curve, referred to as the* FWC_{Zh} *curve, is defined as a quasilinear-weighted-mean of the relative and the absolute FW curves according to the following expression:*

$$FWC_{Zh}(\mathbf{x}; p) = f\left[\mu f^{-1}(FWC_R(\mathbf{x}; p)) + (1-\mu) f^{-1}(FWC_A(\mathbf{x}; p))\right]$$

where $\mu \in (0,1)$ *is the parameter of intermediateness and f is some continuous and strictly monotonic function.*

This curve allows us to introduce the corresponding dominance criterion: for any two distributions **x** and $\mathbf{y} \in D$ we say that **x** Zheng-type FW dominates **y**, and we denote $\mathbf{x} \succsim_{FW_{Zh}} \mathbf{y}$, if and only if

$$FWC_{Zh}(\mathbf{x}; p) \geq FWC_{Zh}(\mathbf{y}; p)$$

for all $p \in [0,1]$ and the strict inequality holds at least once.

It is clear that the Zheng-type curve is a particular case of equation (8). In addition the quasilinear-weighted-means have been widely used both in the economics and in the mathematics literature, and by no means is their formulation as complicated as it seems at first sight. For instance, when $f(t) = t$, we find that the ordinates of the Zheng-type curve are just a weighted arithmetic mean of the ordinates of the relative and absolute FW curves. The Zheng-type curve becomes the Krtscha-type one for $f(t) = e^t$ and the Pfingsten-type curve for $f(t) = 1/t$.

Furthermore, we are going to show that the Zheng-type curves fulfill seven desirable properties for an intermediate polarization curve. Following Aczél

(1966), these properties allow us to characterize the Zheng-type curves as the only polarization curves which satisfy the intermediateness axiom.

To formally establish these properties, let us consider the function G in equation (8), and let us denote by u the FWC_R curve and by v the FWC_A curve. The μ-parameter may be interpreted as the degree of intermediateness, with μ the weight attached to the relative ordinates and $(1-\mu)$ to the absolute ones.

First, it seems reasonable to assume that the roles played by the relative and the absolute curves are symmetric, that is:

Assumption 1. Symmetry: $G(u, v; \mu) = G(v, u; 1 - \mu)$

Second, if the relative and absolute curves are the same, then the intermediate curve coincides with them:

Assumption 2. Reflexivity: $G(u, u; \mu) = u$ for all $0 \leq \mu \leq 1$.

Moreover, the two polar cases, that is, when $\mu = 0$ and $\mu = 1$, correspond to the absolute and the relative curves respectively, and for any other value, the intermediate curve lies between these extremes. This property is known as *internality* according to Aczél's designation:

Assumption 3. Internality: $G(u, v; 0) = v$, $G(u, v; 1) = u$ and for $0 \leq \mu \leq 1$ $min(u, v) \leq G(u, v; \mu) \leq max(u, v)$

Two monotonicity assumptions may be derived with respect to both the μ-parameter and the curves. The curve is increasing in the weight associated to the variable whose value is higher. This means that when the relative curve lies above the absolute one, if the level of intermediateness increases, then the intermediate curve ordinates also increase. The same happens when the absolute curve lies above the relative and the μ-parameter decreases.

Assumption 4. Increasing in the μ-parameter: if $u > v$ and $\mu_1 < \mu_2$ then $G(u, v; \mu_1) < G(u, v; \mu_2)$

In addition the curve is also increasing in both variables, which means that if either the relative or the absolute curve ordinates increase then, with the same level of intermediateness μ, the intermediate curve ordinates also increase.

Assumption 5. Increasing in the curve ordinates: if $u_1 < u_2$ then $G(u_1, v; \mu) < G(u_2, v; \mu)$

Moreover, by definition, the intermediate curve according to equation (8) is homogeneous of degree 0 in the μ-parameter:

Assumption 6. Homogeneity of degree 0 in the weights.

The seventh requirement concerns the consistency of the procedure followed in the construction of the curve. This property ensures that the construction of the intermediate curve can be carried out in several steps, without changes in the final result. To state this condition it is useful to rewrite $G(u, v; \mu) = \tilde{G}(u, v; r, s)$ with $\mu = \frac{r}{r+s}$. Then we get:

Assumption 7. Aggregativity: $\tilde{G}\left[\tilde{G}(u, v; r, s), w; r+s, t\right] = \tilde{G}\left[u, \tilde{G}(v, w; s, t); r, s+t\right]$ for all u, v, w and for all $r, s, t \in (0, \infty)$.

If the above seven requirements are considered as appealing conditions for a intermediate curve FWC_I, then the only possibility for FWC_I is to be a curve according to Definition 3.1.1, that is, a Zheng-type curve.

Proposition 3.1.2. *Symmetry, Internality, Reflexivity, Increasing in the μ-parameter, Increasing in the curve ordinates, Homogeneity of degree 0 in the weights, and Aggregativity on the function G (equation 8) are necessary and sufficient conditions for the intermediate polarization curve FWC_I to be Zheng-type polarization curve.*
Proof. It is straightforward from Aczél (1966, p. 242). *Q.E.D.*

3.2. *Unit-Consistency Axiom for Polarization Orderings*

For polarization orderings and dominance to make sense, when applying them to cross-sections or time series, they should not depend on the units in which income is measured. The unit-consistency axiom for polarization orderings may be introduced as follows:

(ix) *Unit-Consistency Axiom*: A polarization dominance criterion $\succsim_P$ is *unit-consistent* if for any two distributions **x** and **y** ∈ D such that $\mathbf{x} \succsim_P \mathbf{y}$ then $\theta\mathbf{x} \succsim_P \theta\mathbf{y}$ for any $\theta > 0$.

Given the conclusions drawn in the previous section as regards the polarization indices, we may state that the relative and absolute as well as the Krtscha-type FW dominance criteria are unit-consistent.

The class of Pfingsten-type polarization orderings proposed by Chakravarty and D'Ambrosio (2010) violates the unit-consistent axiom. Consider the same distributions as in the previous example, that is, $\mathbf{x} = (1, 2, 3, 4, 5)$ and $\mathbf{y} = (0.1, 0.1, 0.6, 1, 2)$. Consider the same values of the parameters, i.e. $\mu = 0.01$ and $\theta = 1000$. The ordinates of the intermediate polarization curve, computed according to equation (7) when $\lambda = 1$, are (0.6, 0.2, 0, 0.2, 0.6) for **x**; the corresponding ones for **y** are (0.2, 0.1, 0, 0, 0.1, 0.4). Thus $FWC_Z(\mathbf{x}; 0.01, 1, p) > FWC_Z(\mathbf{y}; 0.01, 1, p)$ for all $p \in [0, 1]$. In contrast, multiplying the two distributions by $\theta = 1000$ we find (19.4, 6.5, 0, 0, 6.5, 19.4) and (28.6, 14.3, 0, 0, 11.4, 51.5) respectively.

Therefore $FWC_Z(\theta\mathbf{x}; 0.01, 1, p) < FWC_Z(\theta\mathbf{y}; 0.01, 1, p)$

Following Zheng (2007c, p. 522), it is easy to prove that the Zoli-type orderings, apart from cases where the Krtscha-type condition holds, are in general not unit-consistent.

Finally, if we take into consideration the Zheng-type dominance criterion, we get the following result, which is similar to his Proposition 3.3 which he proved for inequality orderings:

Proposition 3.2.1. *The Zheng-type intermediate polarization orderings are unit-consistent if and only if* $f(t) = \frac{1}{\beta}e^{t/\alpha}$ *for some constants* $\alpha \neq 0$ *and* $\beta > 0$.

Proof. The proof is straightforward following that of proposition 3.3 in Zheng (2007c). *Q.E.D.*

Substituting $f(t) = \frac{1}{\beta}e^{t/\alpha}$ in Zheng-type polarization curves according to Definition 3.1.1 leads to Krtscha-type curves.

3.3. *The Krtscha-Type Unit-Consistent Bipolarization Orderings*

In this last section we seek the class of bipolarization indices which rank two given distributions according to the Krtscha-type *FW* dominance, the only intermediate orderings among the wide class of Zheng-type ones which are unit-consistent.

Proposition 3.3.1. *For any two distributions* $\mathbf{x}$ *and* $\mathbf{y} \in D$, $\mathbf{x}$ *Krtscha* λ*-type FW dominates* $\mathbf{y}$ *if and only if* $P(\mathbf{x}) \geq P(\mathbf{y})$ *for all* $P: D \to \mathbb{R}$ *Krtscha* λ*-type bipolarization indices which fulfill the non-decreasing spread and the non-decreasing bipolarization axioms.*

Proof. The proof is straightforward. Following that of similar theorems derived in Chakravarty *et al.* (2007), we define and use the indices $z_i = \left(\frac{m(\mathbf{x})}{m(\mathbf{y})}\right)^{\lambda} y_i + \left(\frac{m(\mathbf{x})}{m(\mathbf{y})} - \left(\frac{m(\mathbf{x})}{m(\mathbf{y})}\right)^{\lambda}\right) m(\mathbf{y})$, when $m(\mathbf{x}) > m(\mathbf{y})$ and $t_i = \left(\frac{m(\mathbf{y})}{m(\mathbf{x})}\right)^{\lambda} x_i + \left(\frac{m(\mathbf{y})}{m(\mathbf{x})} - \left(\frac{m(\mathbf{y})}{m(\mathbf{x})}\right)^{\lambda}\right) m(\mathbf{x})$, when $m(\mathbf{x}) \leq m(\mathbf{y})$, and taking into account the indices $P_{\lambda}^{k}(\mathbf{x}) = \frac{1}{n} \frac{\sum_{1 \leq i \leq k} |x_i - m(\mathbf{x})|}{(m(\mathbf{x}))^{\lambda}}$ with $1 \leq k \leq n$ for the sufficiency part. *Q.E.D.*

4. An Empirical Application

We provide an empirical illustration of the methodology developed in this paper which is based on Spanish data for the period 1973–2003, using per capita equivalent expenditure as a proxy variable for per capita income.[5] We use the Household Budget Surveys (HBS) for the years 1973/74, 1980/81, and 1990/91, with approximately 20,000 household observations, as well as the Continuous Household Budget Survey from 1998 and 2003, with approximately 10,000 household observations. Both surveys were carried out by the Spanish Statistical Institute (Instituto Nacional del Estadística, INE). All of them are representative at a regional level (NUTS 2 regions).[6] The unit of analysis is the individual, so even if the household is the basic statistical unit, person weights are applied to the calculations, along with the required sample weights. All incomes are expressed in pesetas of 2001, using the consumer price indices provided by INE for each region.

In Table 1 we show the median incomes of the Spanish regions. Extremadura is the region with the lowest median in the whole period, followed by Andalucía, Castilla la Mancha, and Murcia. By contrast, Navarra, País Vasco, Baleares,

[5]As the variable representative of the standard of living we use the equivalent expenditure defined as monetary expenditure, plus non-monetary expenditure arising from self-consumption, self-supply, free meals, in-kind salary, and imputed rents for house ownership

[6]For methodological information about the HBS, see INE (various years). The autonomous cities of Ceuta and Melilla are excluded, since they do not appear in the HBS of 1973/74. "Transfers to other Households and Institutions" are excluded from the definition of expenditure in the HBS of 1980/81 and 1990/91, since they are included neither in the current Continuous HBS nor in the HBS of 1973/74. Moreover, the HBS for 1990/91 includes a different valuation criterion for the non-monetary expenditures from the one used by INE (see Arévalo *et al.*,1998).

TABLE 1
Median Household Equivalent Incomes Per Capita in Pesetas, 2001

Median	1973/74	1980/81	1990/91	1998	2003
Andalucía	473,966	552,864	747,140	705,956	766,568
Aragón	635,042	722,622	870,665	914,391	941,664
Asturias	635,000	726,422	1,079,221	887,564	922,679
Baleares	706,943	754,443	1,137,705	935,077	1,047,897
Canarias	650,069	608,824	883,707	687,668	762,281
Cantabria	723,175	765,834	910,392	832,429	867,994
Castilla León	501,443	632,009	854,286	824,280	845,263
Castilla la Mancha	507,041	526,250	797,223	708,472	662,980
Cataluña	802,050	817,343	1,122,912	990,385	974,081
C.Valenciana	634,491	673,492	837,712	817,153	926,788
Extremadura	416,948	472,035	672,494	616,069	610,829
Galicia	531,656	628,826	838,180	746,445	787,511
Madrid	826,142	828,418	1,180,170	1,083,634	975,780
Murcia	522,119	645,351	810,624	709,191	776,234
Navarra	722,042	896,754	1,223,966	1,044,319	1,019,995
País Vasco	790,814	853,927	1,078,498	1,013,352	1,129,606
La Rioja	716,776	789,080	892,019	871,284	936,419

Madrid, and Cataluña appear in the group with the highest medians in most years. It can also be seen that medians increase in all the regions during the period, but with fluctuations over the years. The lowest increase, both in absolute and relative terms, is for Canarias, Cantabria, and Madrid. In contrast, the highest growth is for Castilla León.

In measuring polarization, we allow for the possibility of different concepts of invariance, from relative to absolute, through the intermediate views, but always assume that the use of different monetary units should lead to the same orderings. We plot for each case three polarization curves: the relative FWC_R, the absolute FWC_A, and one intermediate from the Krtscha-type family ($\lambda = 0.5$), denoted by FWC_K. Polarization dominances between regions are obtained using the three criteria mentioned, the relative $\succsim_{FW_R}$, the absolute $\succsim_{FW_A}$, and the intermediate denoted by $\succeq_{FW_K}$, all of them unit-consistent.

In Table 2, the rankings of the different regions for 2003 are presented. The meaning of R (respectively A and I) in the cell corresponding to row-region i and column-region j, is that the region i is dominated by the region j by the ordering $\succsim_{FW_R}$ (respectively $\succsim_{FW_A}$ and $\succeq_{FW_K}$). In Figure 1 we plot the corresponding polarization curves for every region for 2003.

From Table 2, it is clear that the income distributions of Cantabria and Aragón dominate a great number of the Spanish regions (12 and 11, respectively), regardless of the invariance point of view taken into consideration. Therefore, an unambiguous ranking is obtained by all symmetric bipolarization indices satisfying non-decreasing spread and non-decreasing bipolarity, that are either relative, absolute, or Krtscha 0.5-type intermediate. Consequently, Cantabria and Aragón can be considered as the most polarized of the Spanish regions.

Their curves, in Figure 1, show that the level of polarization of these two regions stems, mainly, from the great distances of the incomes above the median rather than the distances of the incomes below it. Although this asymmetry occurs in general for all the regions, for Cantabria and Aragón it is much more pronounced.

TABLE 2

Polarization Dominance in 2003

	Andalucía	Aragón	Asturias	Baleares	Canarias	Cantabria	Castilla León	Castilla la Mancha	Cataluña	C.Valenciana	Extremadura	Galicia	Madrid	Murcia	Navarra	País Vasco	La Rioja
Andalucía		A,I,R		A		A,I,R	A,I,R		A	A	R	A,I,R	A,I		A,I	A	A
Aragón																	
Asturias	R	A,I,R		I		A,I,R	A,I,R			A,I,R	R	A,I,R	A,I,R		A,I,R	A	
Baleares	R	A,I,R				A,I,R	A,I,R			I	R	I,R	I,R		I		
Canarias	A,I,R	A,I,R	A,I	A,I		A,I,R	A,I,R		A	A	R	A,I,R	A,I,R		A,I,R	A,I	A,I
Cantabria		A															
Castilla León		A,I				A											
Castilla la Mancha	A,I,R	A,I,R	A	A		A,I,R	A,I,R		A	A	A,I,R	A,I,R	A,I	A,I	A,I	A	A
Cataluña	R	A,I,R		A	R	A,I,R	A,I,R			R	R	A,I,R	A,I,R	R	A,I	A	I
C.Valenciana		A,I,R				A,I,R	A,I,R					I,R	I				
Extremadura	A,I	A,I				A,I,R	A,I		A			A	A	A	A	A	A
Galicia																	
Madrid		I,R				I,R	I,R					I,R					
Murcia		A,I,R				A,I,R	A,I,R		A		I,R	A	A		A	A	A
Navarra		A,I,R				A,I,R	R					R					
País Vasco		A,I,R				A,I,R	I,R				R	I,R	R	R	I,R		
La Rioja		A,I,R				A,I,R	I,R				R	R		R	A	A	

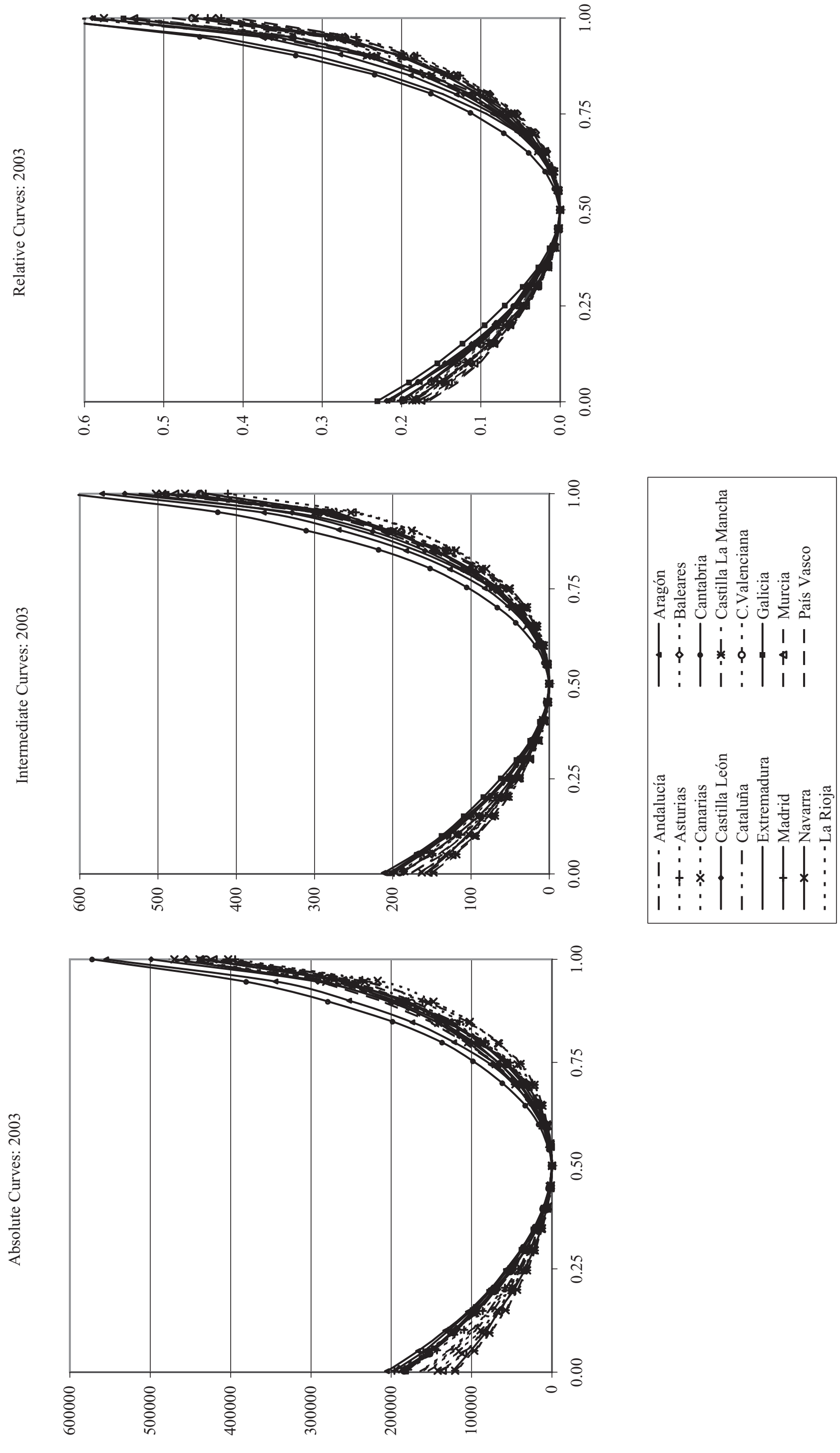

Figure 1. Polarization Curves in 2003

TABLE 3
Polarization Dominance Within Andalucía

	1973/74	1980/81	1990/91	1998	2003
1973/74		A,I	A	A,I	A,I,R
1980/81			A	A,I	A,I,R
1990/91	R	R			A,I,R
1998					
2003					

The group of the most polarized regions also includes Castilla León and Galicia. Canarias, Castilla la Mancha, Asturias, and Cataluña are the least polarized regions in this year. If an absolute approach to polarization is adopted, Castilla la Mancha is dominated by all regions but one, Canarias.

Interesting conclusions may be drawn from the analysis of Table 2. First, several pairs of regions are inconclusively ordered according to one polarization notion, but are unambiguously ranked under another. This is, for example, the case of Madrid and Aragón, Castilla la Mancha and Baleares, and País Vasco and Galicia. Second, the ordering of a pair of regions may depend on the polarization notion that is selected. The cases of the Balearic Islands and Andalucía, Cataluña and Andalucía, and País Vasco and Extremadura illustrate this issue. Finally, it can be mentioned that conclusions concerning the polarization of the País Vasco depend on the polarization notion taken into account. If polarization is relative, País Vasco is dominated by eight regions and does not dominate any, while if polarization is absolute the conclusions change, and País Vasco is dominated only by two regions and dominates eight.

Table 3 provides an illustration of binary comparisons of polarization over time (for the period 1973–2003) in the case of Andalucía, one of the regions with the lowest median in this period. Figure 2 gives the corresponding polarization curves. The same symbol interpretation as in Table 2 holds. It is clear that the distribution of 2003 dominates three of the previous decades, that is, those of 1973/74, 1980/81, and 1990/91. This dominance offers a robust result: polarization in 2003 can be considered to be higher than during the three previous decades, according to all relative, absolute, or Krtscha 0.5-type intermediate bipolarization indices. The sensitivity of the polarization results to the underlying invariance notion is evident when comparing the periods 1973/74 to 1990/91 and 1980/81 to 1990/91, since absolute polarization increases whereas relative polarization decreases.

Looking at the three polarization curves corresponding to all the invariance conditions (see Figure 2) it appears that the curves of 1998 and 2003 intersect. Therefore, for each invariance condition, either absolute, intermediate, or relative, there is an inconclusive result. In fact, different indices lead to different conclusions about the evolution of polarization in these years.

In Table 4 we report different Krtscha-type polarization indices (equation (4)) that are either absolute ($\lambda = 0$), intermediate ($\lambda = 0.5$), or relative ($\lambda = 1$), all of them unit-consistent. In each case the distributions of 1998 and 2003 are ranked in contradictory ways. For r equal to 1, polarization decreases, while for r equal to 0.2 and 0.5, polarization increases, regardless of the type of invariance selected.

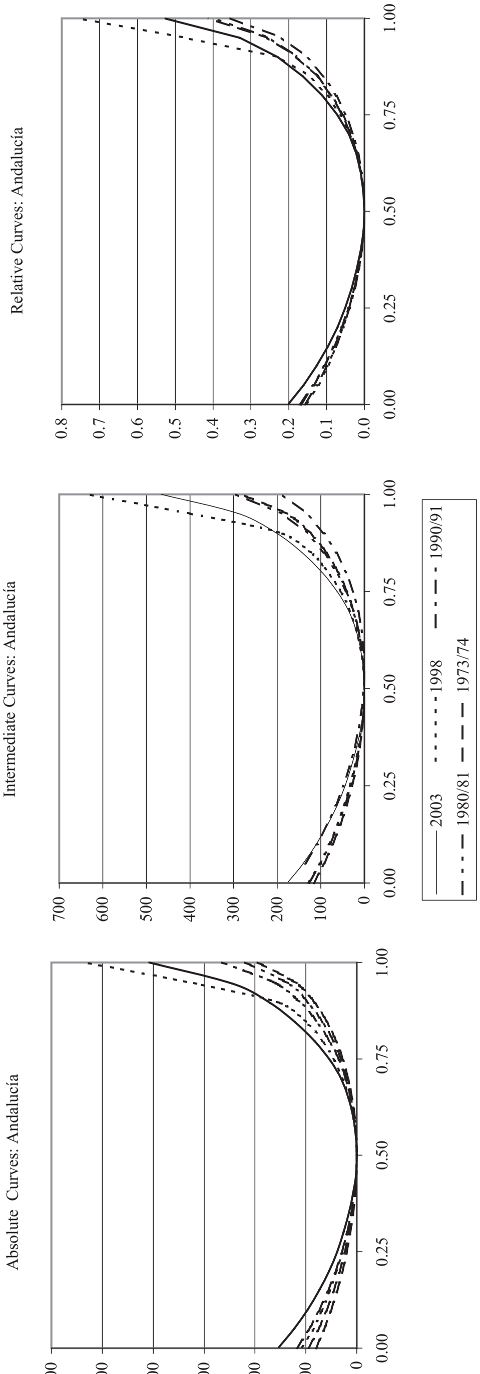

Figure 2. Polarization Curve Within Andalucía

TABLE 4
Krtscha-Type Bipolarization Indices for Andalucía

	1998			2003		
	r = 1	r = 0.5	r = 0.2	r = 1	r = 0.5	r = 0.2
$\lambda = 0$	594,376.4	319,362.3	158,939.4	560,578.1	424,374.7	352,903.5
$\lambda = 0.5$	707.4	380.1	189.2	640.3	484.7	403.1
$\lambda = 1$	0.8419	0.4524	0.2251	0.7313	0.5536	0.4604

5. Concluding Remarks

If it is true that a good measure of how much you like a proposal is how much you try to imitate it, it should be obvious that we really appreciate the unit-consistency axiom proposed by Zheng. As he stresses: "Unit consistency, in the sense that the use of different measuring units should not lead to contradictory conclusions, is important to all scientific studies. Recognizing this importance, it is surprising that until recently the issue has not been appropriately addressed in the inequality measuring literature" (Zheng, 2007c, p. 536). The previous statement applies naturally also to the case where inequality is replaced by polarization. The main contribution of this paper was to apply the unit-consistency axiom to the measurement of polarization, and characterize unit-consistent bipolarization dominance conditions and bipolarization measures.

We have proved that, apart from the relative measures, all the absolute indices proposed hitherto in the literature satisfy this axiom. If other value judgments are to be incorporated via intermediate measures, only the Krtscha-type ones we proposed should be adopted. With regard to bipolarization orderings, the characterization result provided in this paper establishes that only the Krtscha-type intermediate orderings are unit-consistent.

These results have two important consequences. On the one hand, if the unit-consistent axiom is invoked in empirical applications of polarization, the Krtscha-type invariance condition for some intermediate value is implicitly assumed and distributions with different medians are allowed to be compared. The second consequence is that, with other intermediateness notions, distribution polarization comparisons may vary with changes in the monetary units.

An empirical application using data from Spain shows how the theoretical results can be implemented in practice.

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