Review of Income and Wealth
Series 56, Number 1, March 2010

POLARIZATION ORDERINGS OF INCOME DISTRIBUTIONS

BY SATYA R. CHAKRAVARTY*

Indian Statistical Institute, Kolkata

AND

CONCHITA D'AMBROSIO

Università di Milano-Bicocca, DIW Berlin and Econpubblica, Università Bocconi

This paper considers an intermediate notion of polarization which is defined as a convex mix of relative and absolute concepts of polarization. While absolute polarization indices remain unchanged under equal absolute augmentation in all incomes, relative indices do not change under equiproportionate variations in all incomes. We then identify the class of intermediate polarization indices whose orderings of alternative income distributions agree with the rankings generated by intermediate polarization curves. The ranking relation developed is implemented by a simple graphical device. Finally, a numerical illustration of the results developed in the paper is provided using data from Southern European countries.

1. INTRODUCTION

An index of polarization is a measure of the extent of decline of the middle class. A well-off middle class is important for every society because a flourishing middle class makes significant contributions to economic growth as well as to social stability (see Pressman, 2006, for a discussion on the importance of the middle class). By definition, a middle class is in the central position between the poor and the wealthy. A person with low income may not be able to become rich, but may have an expectation of achieving the status enjoyed by a middle class person. Therefore, such a person is likely to work hard to fulfill his expectation and unlikely to revolt against the society. In contrast, a highly polarized society may give rise to social conflicts and tensions (see Esteban and Ray, 1999). This may be regarded as one of the major reasons for studying polarization.

The issue of polarization has recently received wide attention in the Economics literature. See, for example, Esteban and Ray (1991, 1994, 1999), Duclos *et al.* (2004), Wolfson (1994, 1997), Wang and Tsui (2000), Chakravarty and Majumder

Note: We thank two anonymous referees, J. Deutsch, J. Silber, K.Y. Tsui, and participants at the seminar organized by Università di Roma, La Sapienza, and at the "Income Polarization: Measurement, Determinants and Implications" Research Workshop of the Israel Science Foundation for useful comments and suggestions. Chakravarty thanks Università Bocconi, Milano, for financial support and hospitality. D'Ambrosio thanks MIUR (Prin 2007) for financial support.

*Correspondence to: Satya R. Chakravarty, Economic Research Unit, Indian Statistical Institute, 203, R.T. Road, Kolkata 700 108, W.B., India (satya@isical.ac.in).

(2001), D'Ambrosio (2001), and Chakravarty *et al.* (2007).[1] There are two approaches to the measurement of polarization: the Wolfson approach and the Esteban and Ray approach. According to the first approach, polarization is shrinkage of the middle class; on the other hand, the Esteban and Ray approach regards polarization as clustering around local means of the distribution, wherever these local means are located on the income scale. Of the two approaches, since the former is a situation of bipolarization, the latter is more general than the former.

This paper follows the Wolfson approach to measuring polarization. The two characteristics that are considered as being intrinsic to the Wolfson notion of polarization are non-decreasing spread and non-decreasing bipolarity. According to non-decreasing spread, a movement of incomes from the middle position to the tails of the income distribution makes the distribution at least as polarized as before. In other words, as the distribution becomes more spread out from the middle position, polarization does not diminish. On the other hand, non-decreasing bipolarity requires that a clustering of incomes below or above the median leads to a distribution at least as polarized as before. Equivalently, a reduction of gaps between any two incomes, above or below the median, does not lessen polarization. Thus, polarization involves both an inequality-like constituent, the non-decreasing spread criterion, which does not decrease both inequality and polarization, and an equality-like constituent, the clustering or bunching principle, which does not lower polarization, while not augmenting any inequality measure that fulfills the Pigou–Dalton transfers principle, a requirement under which inequality is non-decreasing for a transfer of income from a rich to a poor. Thus, polarization and inequality are two different concepts, although there is a nice complementarity between them. (See the references cited above for further discussion.)

It is evident that a particular index of polarization will generate a complete ranking of alternative distributions of income. However, using more than one index, we may get different rankings of the distributions. Given the diversity of numerical indices, it is, therefore, reasonable to identify the class of indices that yields a similar ordering of different distributions. Two popular views of polarization are relative and absolute. According to the relative concept, an income distribution measure remains unaltered if all the incomes are changed equiproportionally. On the other hand, an absolute measure remains invariant under equal absolute addition to all incomes. Bossert and Pfingsten (1990) considered a general notion of inequality invariance that depends on a value judgment parameter, μ. The parameter can be varied such that the relative and absolute views of inequality are incorporated as extreme cases and hence the general notion can be called *intermediate invariance*. More precisely, intermediate invariance is a convex mix of relative and absolute concepts. That is, a measure does not change if income changes consist of an equal absolute amount and an amount proportional to the original income. Amiel *et al.* (2007), in investigating whether people's perceptions

[1]Some of these studies and several other studies have examined the extent of polarization in different countries. For instance, Morris *et al.* (1994), Wolfson (1997), Gradin (2000), Chakravarty and Majumder (2001), Zhang and Kanbur (2001), Chakravarty *et al.* (2007), and Esteban *et al.* (2007), looked at the extent of polarization in the U.S., Canada, Spain, India, China, and five OECD countries (the U.S., the U.K., Canada, Germany, and Sweden), respectively, over different periods.

of income polarization are consistent with the key axioms, found very little support for relative polarization and more, but still not full, support for an absolute concept of polarization. They write that "Clearly it may also make sense to consider alternatively an 'intermediate' position between scale-independence and translation independence" (Amiel *et al.*, 2007, p. 4). This intermediate approach is still to be introduced in the polarization literature and the aim of this paper is to fill this gap.

In this paper we address the problem of ranking income distributions using intermediate polarization indices. The ranking relation is implemented by a simple graphical device, which we call the *intermediate polarization curve*. The curve shows for any cumulative population proportion the shortfall (excess) of its normalized income from (over) the corresponding income that it would enjoy under the distribution where everybody has the median income. We show that of two income distributions x and y, if x is not intermediate polarization inferior to y, that is, the intermediate polarization curve of x lies nowhere below that of y, then x is regarded as at least as polarized as y by all intermediate, symmetric, population replication invariant polarization indices that fulfill non-decreasing spread and non-decreasing bipolarity. Furthermore, the converse is also true. The population sizes and the medians of the distributions concerned need not be the same for this general result to hold.

We then illustrate the results developed in this paper using the eight waves of the European Community Household Panel (ECHP) data which covers the period 1994–2001. We focus on Southern European countries, namely Greece, Italy, Portugal, and Spain. It is explicitly demonstrated that (a) inequality and polarization are two different concepts, (b) relative and absolute indices of polarization reflect two different notions of polarization, and (c) many ambiguous relative polarization comparisons may become unambiguous in the absolute case. However, in some special cases the reverse could also be true.

The paper is organized as follows. Desiderata for an index of polarization are presented rigorously in Section 2. Section 3 defines the intermediate polarization curve formally, discusses the ordering associated with non-intersecting intermediate polarization curves, and isolates the class of polarization indices that agrees with this ordering. Section 4 presents the numerical results. Section 5 concludes.

2. Postulates for an Index of Polarization

The purpose of this section is to present the postulates for an index of polarization rigorously. For a population of size n, a typical income distribution is a vector $x = (x_1, x_2, \ldots x_n)$, where x_i is the income of person i. Each x_i is assumed to be drawn from $[a, \infty)$, a non-degenerate interval in the non-negative part of the real line $\mathbb{R}$. The set of income distributions for this population is $\mathbb{D}^n$, the n-fold Cartesian product of $[a, \infty)$. The set of all possible income distributions is $\mathbb{D} = \cup_{n\in\mathbb{N}} \mathbb{D}^n$, where $\mathbb{N}$ is the set of positive integers. For the sake of simplicity and convenience, the lower bound of the interval $[a, \infty)$ has been taken to be non-negative, which in turn implies non-negativity of all incomes. Extension of our results to the situation where some incomes are negative may be an interesting investigation.

For any function $f: \mathbb{D} \to \mathbb{R}$, we write f^n for the restriction of f on $\mathbb{D}^n$. All income distributions are assumed to be illfare ranked, that is, for all $n \in \mathbb{N}$, $x \in \mathbb{D}^n$, $x_1 \leq x_2 \leq \ldots \leq x_n$. For any $n \in \mathbb{N}$, $x \in \mathbb{D}^n$, the mean and median of x are denoted respectively by $\lambda(x)$ and $m(x)$. If n is odd, $m(x)$ is the $\left(\frac{n+1}{2}\right)^{th}$ observation in x. But if n is even, the arithmetic mean of the $\left(\frac{n}{2}\right)^{th}$ and the $\left(\frac{n}{2}+1\right)^{th}$ observations in x is taken as the median.

Let $\bar{n} = \frac{n+1}{2}$. We write x_- and x_+ for the subvectors of x that include x_i for $i < \bar{n}$ and $i > \bar{n}$, respectively. Thus, for any $n \in \mathbb{N}$, $x \in \mathbb{D}^n$, $x = (x_-, x_+)$ if n is even and $x = (x_-, m(x), x_+)$ if n is odd. For all $n \in \mathbb{N}$, $x, y \in \mathbb{D}^n$, $x \geq y$ means that $y_i \leq x_i$ for all $i = 1, 2, \ldots n$. The n-coordinated vector of ones is denoted by 1^n, $n \in \mathbb{N}$.

Some more preliminaries are necessary for the purpose at hand. For all $n \in \mathbb{N}$, $x, y \in \mathbb{D}^n$, x is said to be obtained from y by a simple increment if $x_j > y_j$ for some j and $x_i = y_i$ for all $i \neq j$ and we write xCy to represent this. Given that income distributions are non-decreasingly ordered, the transformation C allows only rank preserving increments. For $x, y \in \mathbb{D}^n$, where $n \in \mathbb{N}$ is arbitrary, we say x is obtained from y by a progressive transfer, if there is a pair (i, j), such that $x_i - y_i = y_j - x_j > 0$, $x_j > x_i$ and $y_k = x_k$ for all $k \neq i, j$. That is, x and y are identical except for a positive transfer of income from the rich person j to the poor person i. This is equivalent to the statement that x can be obtained from y through a progressive transfer and we write xTy to indicate this. Note that only rank preserving transfers are allowed under the operation T.

A polarization index I is a real valued function defined on $\mathbb{D}$, that is, $I: \mathbb{D} \to \mathbb{R}^1$. For all $n \in \mathbb{N}$, $x \in \mathbb{D}^n$, the functional value $I^n(x)$ indicates the level of polarization associated with the distribution x. A polarization index $I: \mathbb{D} \to \mathbb{R}^1$ should satisfy the following postulates.

Non-decreasing Spread (NS): For all $n \in \mathbb{N}$, if $x, y \in \mathbb{D}^n$, where $m(x) = m(y)$, are related through anyone of the following cases:

(a) $x_+ = y_+$, $y_- C x_-$, (b) $x_- = y_-$, $x_+ C y_+$, (c) $y_- C x_-$, $x_+ C y_+$, then $I^n(x) \geq I^n(y)$.

Non-decreasing Bipolarity (NB): For all $n \in \mathbb{N}$, if $x, y \in \mathbb{D}^n$, where $m(x) = m(y)$, are related through anyone of the following cases:

(a) $x_- T y_-$, $x_+ = y_+$, (b) $x_+ T y_+$, $x_- = y_-$, (c) $x_- T y_-$, $x_+ T y_+$, then $I^n(x) \geq I^n(y)$.

Symmetry (SM): For all $n \in \mathbb{N}$, $x \in \mathbb{D}^n$, $I^n(x) = I^n(Px)$, where P is any $n \times n$ permutation matrix.[2]

[2]An $n \times n$ matrix with entries 0 and 1 is called a permutation matrix if each of its rows and columns sums to one.

Principle of Population (PP): For all $n \in \mathbb{N}$, $x \in \mathbb{D}^n$, $I^n(x) = I^{kn}(y)$, where y is any k-fold replication of x, that is $y = (x^1, x^2, \ldots x^k)$ with each $x^i = x$.

Normalization (NM): For all $n \in \mathbb{N}$, $I^n(c1^n) = 0$, where $c > 0$ is any scalar.

Continuity (CN): For all $n \in \mathbb{N}$, I^n is a continuous function on $\mathbb{D}^n$.

The postulates **NS** and **NB** were considered among others by Wolfson (1994, 1997), Wang and Tsui (2000) and Chakravarty and Majumder (2001). **NS** is a monotonicity condition. Since rank-preserving increments (reductions) in incomes above (below) the median widens the distribution, polarization should not go down. That is, greater distancing between the groups below and above the median, should not make the distribution less polarized. **NB** is a bunching or a clustering principle. Because a rank-preserving egalitarian transfer between two individuals on the same side of the median brings the individuals closer to each other, polarization should be non-diminishing. As an egalitarian transfer demands non-increasingness of inequality, **NB** explicitly establishes that inequality and polarization are two non-identical concepts. (It may be worthwhile to mention that our formulation of **NS** and **NB** is slightly different from that of Wang and Tsui (2000) in the sense that while the Wang–Tsui formulation puts the monotonic and redistributive changes in a compact form, we subdivide them explicitly into three components for the sake of simplicity.) **SM** means that polarization remains unchanged if we take a reordering of incomes. Thus, any characteristic other than income, e.g. names of the individuals, is not relevant to the measurement of polarization. One implication of **SM** is that a polarization index can be defined directly on ordered income distributions (as we have done). According to **PP**, if a population is replicated several times, the levels of polarization of the replicated and the original distributions are the same. In view of **PP**, we can regard polarization as an average concept. **PP** enables us to compare polarization across populations. Clearly, the median income remains unaltered under replications of the population. Postulate **NM** is a cardinality property of the polarization index. It says that for a perfectly equal income distribution, level of polarization is zero. **CN** means that a polarization index will not take sudden jumps for small income changes. Thus, a polarization index will not be oversensitive to minor observational errors in incomes. Clearly, given the difficulties in measuring incomes accurately, it is reasonable to require a polarization index to vary continuously with incomes. It may be mentioned that while we do not need **CN** and **NM** for proving our theorems, they are regarded as desirable postulates of a polarization index because of their intuitive appeal.

3. The Intermediate Polarization Ordering

We begin this section by defining the absolute and the (Wolfson) relative polarization curves. The absolute polarization curve (**APC**) of any income distribution shows for any population proportion, how far the total income enjoyed by that proportion, expressed as a fraction of the population size, is from the corresponding income that it would receive under the hypothetical situation where everybody enjoys the median income. For any $x \in \mathbb{D}^n$, the **APC** ordinate corresponding to the population proportion $\frac{k}{n}$, $(1 \le k \le \bar{n})$, is:

(1)
$$AP\left(x;\frac{k}{n}\right)=\frac{1}{n}\sum_{k\le i\le \bar{n}}(m(x)-x_i)$$

and corresponding to the population proportion $\frac{k}{n}$, ($\bar{n}\le k\le n$), the ordinate is:

(2)
$$AP\left(x;\frac{k}{n}\right)=\frac{1}{n}\sum_{\bar{n}\le i\le k}(x_i-m(x)).$$

Note that the ordinate at $\frac{\bar{n}}{n}$ involves the income level $x_{\bar{n}}=m(x)$. Now if n is odd, $x_{\bar{n}}$ is one of the incomes in the distribution x. However, for even n, although $x_{\bar{n}}$ is not in x, we define the ordinate at $\frac{\bar{n}}{n}$, since in polarization measurement, the median is the reference income level (see Chakravarty *et al.*, 2007). For any income distribution, $x\in\mathbb{D}^n$, AP (x; p) when divided by the median, that is, $\frac{AP(x,p)}{m(x)}$, becomes the (Wolfson) relative polarization curve (**RPC**) (see Wolfson, 1994, 1997).

Note that the **RPC** is a relative measure, that is, it is homogeneous of degree zero in incomes. In contrast, the **APC** is an absolute measure—it is translation invariant, that is, invariant under equal augmentation in all incomes. The relative and absolute concepts drop out as two polar cases of an intermediate or compromise condition, which requires that a weighted average of income variations along the relative and absolute scales should not change polarization. More precisely, the distributions x and $(x + c(\mu x + (1-\mu)1^n))$ should possess the same level of polarization, where $0\le\mu\le 1$ is a parameter that reflects the evaluator's value judgment on polarization equivalence since it states which distributions are considered as polarization equivalent to a given distribution x, and c is scalar such that $\left(x+c\left(\mu x+(1-\mu)1^n\right)\right)\in\mathbb{D}^n$. Note that here we express the vector 1^n in income units so that $\mu x + (1-\mu)1^n$ becomes well defined. The following simple generalization of the **RPC** and **APC** fulfils the intermediate condition:

(3)
$$\textbf{IPC}\left(x;\frac{k}{n},\mu\right)=\frac{1}{n}\sum_{k\le i\le \bar{n}}\frac{m(x)-x_i}{\mu m(x)+1-\mu},\quad 1\le k\le \bar{n},$$

and

(4)
$$\textbf{IPC}\left(x;\frac{k}{n},\mu\right)=\frac{1}{n}\sum_{\bar{n}\le i\le k}\frac{x_i-m(x)}{\mu m(x)+1-\mu},\quad \bar{n}\le k\le n.$$

Given the value judgment parameter μ, $0\le\mu\le 1$, (3) and (4) define the intermediate polarization curve (**IPC**) associated with the income distribution $x\in\mathbb{D}^n$. For a typical income distribution $x\in\mathbb{D}^n$ and for any $0\le\mu\le 1$, up to $\frac{\bar{n}}{n}$, the midpoint of the horizontal axis the **IPC** is decreasing, at $\frac{\bar{n}}{n}$ the **IPC** coincides with the horizontal axis and then it increases monotonically. If x is an equal distribution, then its **IPC** becomes the horizontal axis itself. For a distribution x, where x_+ is unequal but $(x_-, m(x))$ is equal, the **IPC** runs along the horizontal axis up to its mid point $\frac{\bar{n}}{n}$, and starting from $\frac{\bar{n}}{n}$, the curve rises gradually. Similarly if x is such that x_- is unequal but $(m(x), x_+)$ is equal, then the **IPC** is decreasing up to the mid-point of the horizontal axis, after which it runs through this axis.

For $\mu = 0$, $\mathbf{IPC}\left(x; \frac{k}{n}, \mu\right)$ becomes the **APC** considered in (1) and (2). On the other hand, the **RPC** drops out as a particular case of $\mathbf{IPC}\left(x; \frac{k}{n}, \mu\right)$ if $\mu = 1$. The closer is μ to unity (zero), the greater is the concern for relative (absolute) polarization. Since $\mathbf{IPC}\left(x; \frac{k}{n}, \mu\right)$ is a generalization of **APC** and **RPC**, from now on we will deal with it.

Given $x \in \mathbb{D}^n$, the area under the **APC** associated with x is

$$Q^n(x) = 2\left[\lambda(x_+) - \lambda(x_-) - A_G^n(x)\right], \tag{5}$$

where $A_G^n(x) = \lambda(x) - \frac{\sum_{i=1}^{n}(2(n-i)+1)x_i}{n^2}$ is the absolute Gini coefficient of the income distribution x (see Chakravarty *et al.*, 2007; Chakravarty, 2009). This in turn shows that the area under the **IPC** of x becomes

$$F_\mu^n(x) = \frac{2\left[\lambda(x_+) - \lambda(x_-) - A_G^n(x)\right]}{m(x) + 1 - \mu}. \tag{6}$$

Note that F_μ^n satisfies the intermediate invariance condition. In contrast, Q^n is an absolute index. Thus, while Q^n involves the absolute Gini coefficient as a component, the corresponding component of its intermediate sister F_μ^n is the intermediate Gini index (see Bossert and Pfingsten, 1990). F_μ^n coincides with Q^n if $\mu = 0$. On the other hand, if $\mu = 1$, F_μ^n becomes the Wolfson (relative) index of polarization (see Wolfson, 1994, 1997). It may be important to note that these indices satisfy all the postulates outlined in Section 2.

Wang and Tsui (2000) suggested a polarization index which is linear in incomes with a given rank order of incomes. Their index is given by $P_{Wt}^n(x) = \frac{\sum_{i=1}^{n} a_i^n x_i}{m(x)}$, where the sequence $\{a_i^n\}$ fulfils some restrictions for satisfaction of **NS** and **NB** (see proposition 3 of Wang and Tsui, 2000). The Wolfson index drops out as a special case of this index for a particular choice of a_i^n. Since our index in (6) is a rescaled version of the Wolfson index with the scaling factor $\frac{m(x)}{\mu m(x)+1-\mu}$, it is also a particular member of the Wang–Tsui index. However, it should be noted that while F_{μ}^{n} is population replication invariant, $P_{Wt}^{n}(x)$, in its general form, is not so. Consequently, while F_{μ}^{n} is suitable for cross-population comparison, $P_{Wt}^{n}(x)$, in its general form, is not.

An alternative intermediate index of polarization that fulfills all the postulates can be the following:

$$C_{\mu}^{n}(x)=\frac{\left(n^{-1}\sum_{i=1}^{n}|x_i-m(x)|^{r}\right)^{\frac{1}{r}}}{\mu m(x)+1-\mu},\quad 0<r\le 1. \tag{7}$$

C_{μ}^{n} is an increasing function of the parameter r. For $r=1$, C_{μ}^{n} becomes the intermediate mean deviation about the median so that in the extreme cases $\mu=0$ and 1 it coincides respectively with the absolute and relative mean deviations about the median. A rank preserving progressive transfer of income between two individuals on the same side of the median increases C_{μ}^{n} by a larger amount the lower is the value of r. For any $r<1$, the increase in C_{μ}^{n} because of a rank preserving progressive transfer on the same side of the median will be higher the lower are the incomes of the donor and the recipients.

The criterion that we wish to use here for ranking alternative distributions of income by intermediate polarization indices relies on the **IPC**. Given any two income distributions $x, y\in\mathbb{D}^n$, x is said to dominate y with respect to intermediate polarization ($x\succeq_{IP} y$, for short) if the **IPC** of x lies nowhere outside that of y. Formally, $x\succeq_{IP} y$ means that

$$\mathbf{IPC}\left(x;\frac{k}{n},\mu\right)\ge\mathbf{IPC}\left(y;\frac{k}{n},\mu\right) \tag{8}$$

for all $1\le k\le n$. Note that the relation $\succeq_{IP}$ is transitive, that is, for any $x, y, z\in\mathbb{D}^n$, if $x\succeq_{IP} y$ and $y\succeq_{IP} z$ hold, then $x\succeq_{IP} z$ holds. Since for any $x\in\mathbb{D}^n$, x is as polarized as itself, $\succeq_{IP}$ satisfies reflexivity also. However, it is not complete, that is, there may exist $x, y\in\mathbb{D}^n$, such that neither $x\succeq_{IP} y$ nor $y\succeq_{IP} x$ holds. Clearly, such a situation arises if the **IPC**s of x and y intersect. Thus, $\succeq_{IP}$ is a quasiordering—it is reflexive, transitive but not complete.

The following result gives an implication of the intermediate polarization dominance relation $\succeq_{IP}$ for income distributions over differing population sizes and arbitrary medians.

Theorem 1. *Let $x\in\mathbb{D}^l$, $y\in\mathbb{D}^n$ be arbitrary. Then the following statements are equivalent:*
(a) $x\succeq_{IP} y$.

(*b*) $I^l(x) \geq I^n(y)$ *for all intermediate polarization indices* $I : \mathbb{D} \to \mathbb{R}$ *that satisfy* **NS**, **NB**, **SM**, *and* **PP**.

Proof: See Appendix.

Theorem 1 indicates that an unambiguous ranking of income distributions by all population replication invariant, symmetric polarization indices satisfying **NS** and **NB** can be obtained if and only if their **IPC**s do not intersect. But if the two curves intersect, we can get two different indices with these properties that will rank the distributions in opposite directions. Clearly, for any two distributions x and y with the same median if, for all μ, $x \succeq_{IP} y$ holds, then the **APC** and **RPC** of y are nowhere inside the respective curve of x ($x \succeq_{AP} y$, and $x \succeq_{RP} y$ for short). But in general, the three orderings are likely to be different. In fact, in some cases, ambiguous comparison in terms of $\succeq_{RP}$ may be unambiguous by $\succeq_{AP}$. For instance, suppose that of two distributions x and y, where $m(x) > m(y)$, the **RPC** of x intersects that of y from above at a point below the mid-point of the horizontal axis. That is, below the point of intersection, the **RPC** of x lies everywhere above but after that it lies on and below that of y. Because $m(x) > m(y)$, multiplication of these curves by the corresponding normalizing factors may generate an upward shift in the resulting curve of x to the right of the point of intersection such that $x \succeq_{AP} y$ holds. Thus, higher median may be sufficient for pushing the lower curve upward to ensure absolute dominance unambiguously.

If we assume that the population size is fixed, say n, then the following result drops out as a Corollary to Theorem 1.

Corollary 1. *Let* $x, y \in \mathbb{D}^n$ *be arbitrary. Then the following statements are equivalent:*

(*a*) $x \succeq_{IP} y$.

(*b*) $I^n(x) \geq I^n(y)$ *for all intermediate polarization indices* $I^n : \mathbb{D}^n \to \mathbb{R}$ *that satisfy* **NS**, **NB**, *and* **SM**.

Corollary 1 shows that an unambiguous ranking of income distributions over a given population size by intermediate polarization indices can be achieved through the pairwise comparisons of the **IPC**s of the distributions. We can also focus our attention on the fixed median arbitrary population case. In this case, the domain of definition of polarization indices is $\mathbb{D}_m = \{x \in \mathbb{D} | m(x) = m\}$. For all indices that are consistent on $\mathbb{D}_m$ with $\succeq_{IP}$ the required postulates are **NS**, **NB**, **SM**, and **PP**.

Finally, it should be clear that if both mean income and median are fixed, indices that agree with the ordering $\succeq_{IP}$ should satisfy **NB**, **SM**, and **PP**. We can also have situations where mean is fixed, medians are different, and population size is equal/unequal. For consistency with $\succeq_{IP}$, while in the former case the intermediate indices should meet **NB** and **SM**, in the latter case they are required to satisfy **PP** in addition to **NB** and **SM**. But in the practical situations, where mean, median and population size are likely to vary, Theorem 1 is the result that can be applied for polarization comparison.

4. An Empirical Illustration

The purpose of this section is to illustrate the polarization orderings, using the European Community Household Panel (ECHP) data. We base our analysis on all the waves available under ECHP, which cover the period 1994–200l. The surveys are conducted at a European national level. The ECHP is an ambitious effort at collecting information on the living standards of the households of the **EU** member states using common definitions, information collection methods, and editing procedures. It contains detailed information on incomes, socioeconomic characteristics, housing amenities, consumer durables, social relations, employment conditions, health status, subjective evaluation of well-being, etc. The unit of our analysis is the individual. The calculation uses required sample weights. Household incomes were equivalized using the square-root equivalence scale. All incomes were then expressed in purchasing power standards (PPS) using purchasing power parities provided by Eurostat in the ECHP Country file.

According to Pressman (2006, p. 7), "four factors have been advanced as explanations for the declining middle class: (1) depolarization only when we follow the absolute criterion and compare wave 2 with wave 1, with the former being more polarized. Similar results hold for Spain where only the absolute criterion gives a clear verdict between waves 2 geographic factors, (2) structural or microeconomic factors such as the loss of middleclass manufacturing jobs and the decline of labor unions, (3) macroeconomic factors such as unemployment resulting from the business cycle, and (4) changes in public policy." The only relevant factor in Pressman (2006) is public policy interacting with the type of welfare state implemented. These findings led us to restrict our attention to the South of Europe since these countries share similarities in terms of the welfare state and the active role played there by families. We focus on Greece, Italy, Spain, and Portugal and analyze what has happened to their middle class during the period 1994–200l.

We allow for the possibility of different value judgments in polarization dominance and compute polarization curves for the relative ($\mu = 1$), absolute ($\mu = 0$), and intermediate ($\mu = 0.5$) views. The advantage of considering polarization orderings is at least twofold: we avoid the arbitrariness caused by choosing the middle class—whenever dispersion around the median increases, the middle class, however defined, becomes worse off; and the choice of the polarization index is irrelevant—when dominance exists, all indices, consistent with the ordering, will rank the distributions in the same way.

To demonstrate that polarization and inequality are different phenomena, we also rank the countries using the Lorenz and absolute Lorenz criteria.

Median incomes are reported in Table 1. In all the waves, while Italy is the country with the highest median, Portugal emerged as the country with the lowest median. Medians increased in all the countries over the years analyzed but not at the same rate. At the two extremes are Italy, with the lowest average growth rate, and Spain, which witnessed the highest rate of increase.

Results on polarization dominance within each country in the eight waves for any two consecutive years are reported in Table 2. For any country $W_i > W_{i+1}$ means that the i-th wave polarization dominates the $(i + 1)$-th wave, where $i = 1$,

TABLE 1

Median Household Equivalent Incomes per Person (in PPS)

Median	Greece	Italy	Portugal	Spain
Wave 1	7,200.25	9,856.09	6,775.18	8,572.39
Wave 2	7,696.66	10,445.18	7,319.13	8,730.28
Wave 3	7,970.40	10,462.85	7,489.80	8,890.14
Wave 4	8,277.12	10,945.79	7,892.19	9,511.34
Wave 5	8,937.34	11,451.80	8,186.39	9,985.21
Wave 6	9,167.64	12,142.44	8,653.21	10,806.21
Wave 7	10,087.24	13,298.33	9,100.61	12,001.87
Wave 8	10,376.20	13,439.07	9,840.16	12,920.21

Source: Authors' calculations from ECHP.

TABLE 2

Polarization Dominance within Each Country in the Eight Waves

	Greece	Italy	Portugal	Spain
Absolute	$W_4 > W_3$	$W_2 > W_1$	$W_2 > W_1$, $W_4 > W_3$	$W_2 > W_1$, $W_7 > W_6$, $W_8 > W_7$
Intermediate	$W_4 > W_3$, $W_5 > W_6$		$W_2 > W_3$, $W_5 > W_6$	
Relative	$W_4 > W_3$, $W_5 > W_6$		$W_2 > W_3$, $W_5 > W_6$	

Source: Authors' calculations from ECHP.

2, . . . , 7. The absolute polarization criterion shows less ambiguous comparisons in all the countries except in Greece. In the latter, wave 4 is more polarized than wave 3 according to all the three criteria, while wave 5 is more polarized than wave 6 only for the intermediate and relative cases. In Italy we observe a univocal increase in polarization only when we follow the absolute criterion and compare wave 2 with wave 1, with the former being more polarized. Similar results hold for Spain where only the absolute criterion gives a clear verdict between waves 2 and 1, 7 and 6, and 8 and 7, in each case the former being more polarized than the latter. In Portugal all the three criteria were able to rank two waves with the results coinciding for the intermediate and relative cases. We observe an increase in relative and intermediate polarizations from wave 2 to 3 and from wave 5 to wave 6. The absolute polarization ordering, on the other hand, reported wave 2 more polarized than wave 1, and wave 4 more polarized than wave 3. New interesting insights emerge from the polarization curves plotted in Figures 1–3. In all the countries there is less overlap between the absolute polarization curves as compared to the other two curves, particularly, for Spain followed by Greece. An asymmetry in distances from the median exists in all cases, with distances from incomes higher than the median being more pronounced than those from incomes on the other side of the median. This observation is a consequence of the longer right tail of the curves. In the absolute framework, wave 3 appears to be highly polarized and the curvature of all the curves increased on an average over time. The absolute polarization curves of Italy behave quite differently from those of the other countries. Here the right tail is quite stable over time, and, as in the case of Spain, it moved less than the left tail. For the intermediate and relative curves the

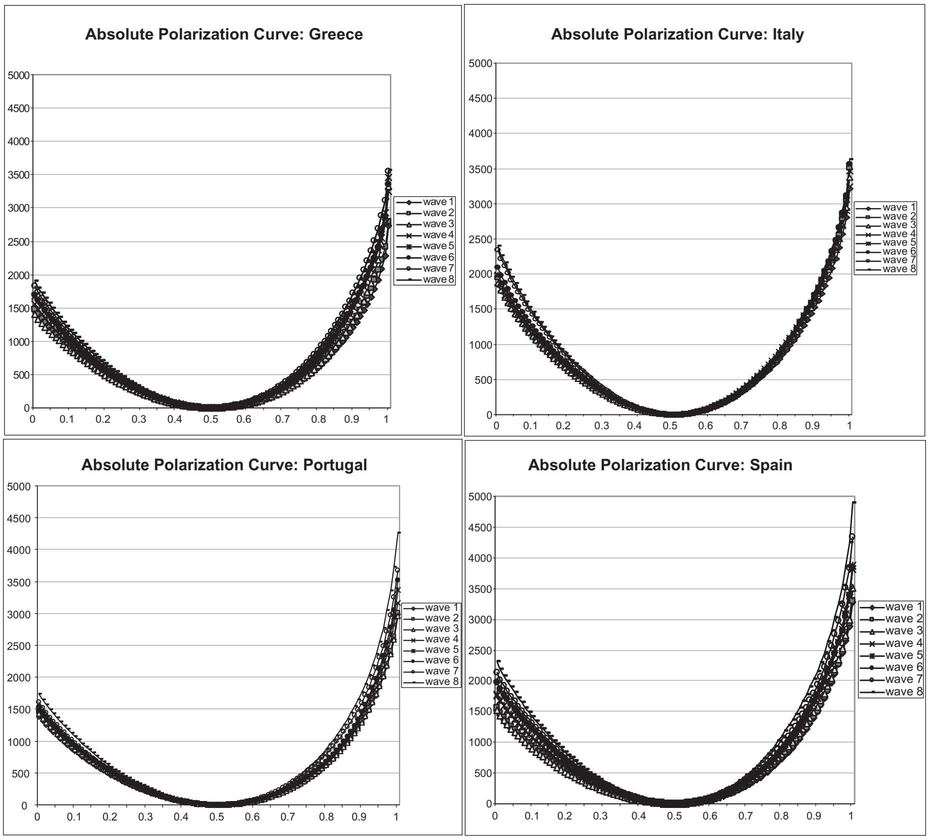

Figure 1. Absolute Polarization Curves

Source: Authors' calculations from ECHP.

opposite happened: relative distances of higher incomes from the median changed more over time. These moves flattened the curves over time.

In Tables 3–5 we report polarization dominance between countries using the three criteria. In these tables (and in Tables 6–7) for any two countries $C_1 >^i C_2$ by which we mean that C_1 polarization (respectively inequality) dominates C_2 in wave *i*. In Figure 4 we plot the polarization curves of the four countries in wave 1. We omit the plot for the other waves because of the high level of resemblance with wave 1. As in the country specific analysis, the absolute criterion (results contained in Table 3) produces less ambiguous cases. Italy is the most polarized country on average, followed by Spain. Looking at the curves we conclude that this result is a consequence of higher distances of lower incomes from the median. Since in the intermediate and relative cases the distances are expressed in relative terms, these differences in distances of lower incomes decrease and the left tail of the polarization curve of Italy overlaps with the ones of the other countries. According to the intermediate and relative notions of polarization, Italy is the least polarized country on average while Portugal is the country showing the highest level of polarization. These results are very much different from those obtained for the

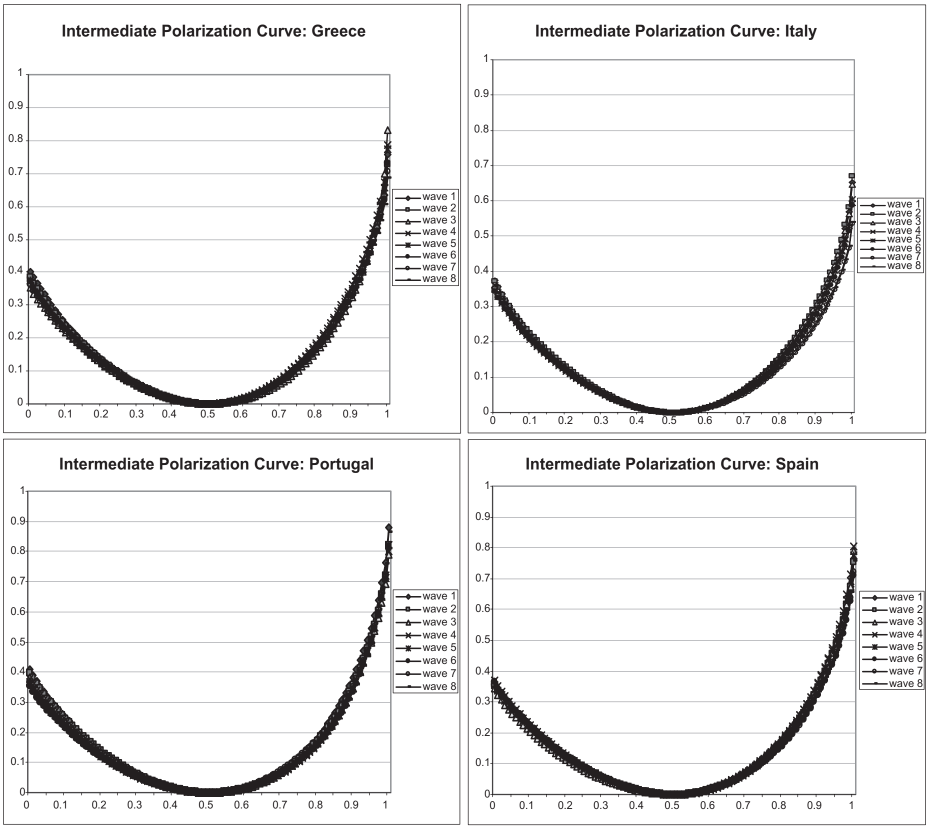

Figure 2. Intermediate Polarization Curves

Source: Authors' calculations from ECHP.

absolute polarization criterion. As shown in Figure 4, these differences follow from differences between relative distances of incomes higher than the median from the median itself.

In Tables 6 and 7 we report inequality dominance according to the absolute Lorenz criterion and the Lorenz criterion. The Lorenz (absolute Lorenz) curve of any $x \in \mathbb{D}^n$ will be the plot of $\dfrac{\sum_{j=1}^{i} x_j}{n\lambda(x)} \left(\dfrac{\sum_{j=1}^{i}\left(x_j - \lambda(x)\right)}{n}\right)$ against $\dfrac{i}{n}$, where $i = 1, 2, \ldots n$. Unanimous ranking of income distributions by all relative (absolute) inequality indices can be obtained through pairwise comparison of Lorenz (absolute Lorenz) curves of the distributions. More precisely, for any two income distributions $x \in \mathbb{D}^m$, $y \in \mathbb{D}^n$ if x Lorenz (absolute Lorenz) dominates y, then y is regarded as at least as unequal as x by all relative (absolute) inequality indices that satisfy the Pigou–Dalton transfers principle, symmetry, and population replication invariance, where Lorenz (absolute Lorenz) dominance is defined in the same way as polarization dominance. Furthermore, the converse is also true. From the

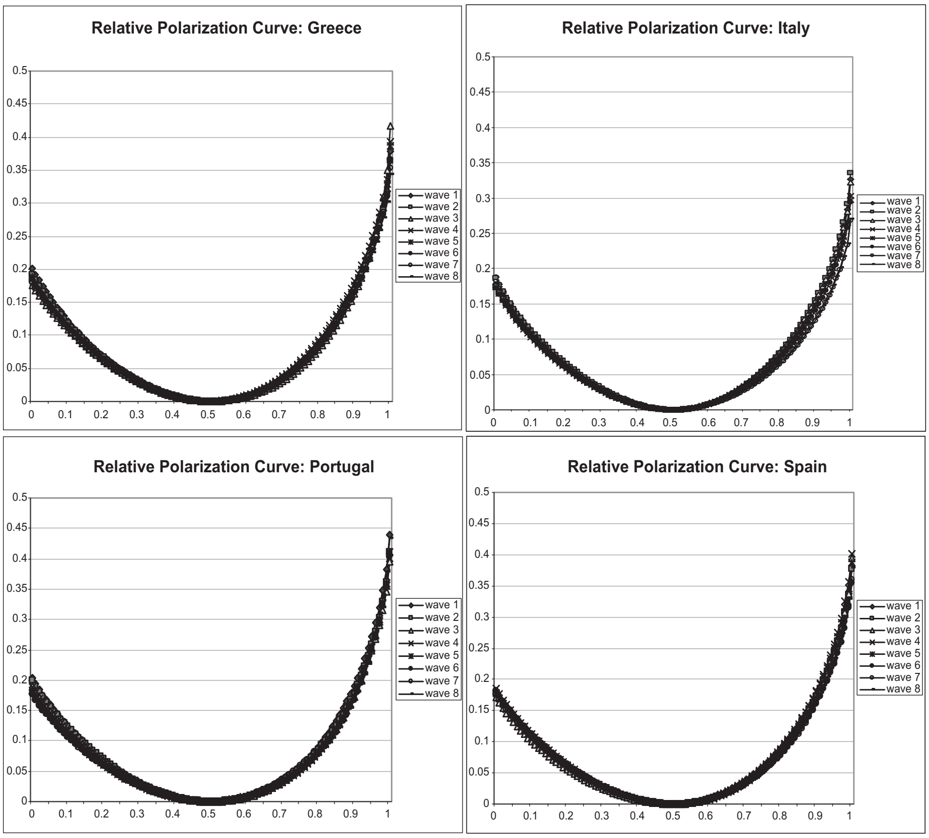

Figure 3. Relative Polarization Curves

Source: Authors' calculations from ECHP.

TABLE 3

Inter-Country Dominance by the Absolute Polarization Criterion in the Eight Waves

Absolute Polarization	Greece	Italy	Portugal	Spain
Greece			$>^{7}$	
Italy	$>^{1}, >^{2}, >^{3}, >^{4}, >^{5}, >^{6}, >^{8}$		$>^{1}, >^{2}, >^{3}, >^{4}, >^{5}, >^{6}, >^{7}$	$>^{2}, >^{3}$
Portugal				
Spain	$>^{1}, >^{2}, >^{6}, >^{7}, >^{8}$		$>^{1}, >^{2}, >^{4}, >^{5}, >^{6}, >^{7}, >^{8}$	

Source: Authors' calculations from ECHP.

rankings, it is evident that polarization and inequality are quite different phenomena. Italy, on an average, is the most unequal country, followed by Spain.

5. Conclusions

Intermediate polarization indices depend on a convex mixture of absolute and relative income differentials. That is, they satisfy a general notion of polarization

TABLE 4

Inter-Country Dominance by the Intermediate Polarization Criterion in the Eight Waves

Intermediate Polarization	Greece	Italy	Portugal	Spain
Greece		$>^1, >^5, >^6, >^8$		
Italy				
Portugal	$>^1, >^2, >^3$	$>^1$		$>^1$
Spain		$>^5, >^6$		

Source: Authors' calculations from ECHP.

TABLE 5

Inter-Country Dominance by the Relative Polarization Criterion in the Eight Waves

Relative Polarization	Greece	Italy	Portugal	Spain
Greece		$>^1, >^2, >^4, >^5, >^6, >^8$		
Italy				
Portugal	$>^1, >^2, >^3$	$>^1, >^2$		$>^1$
Spain		$>^4, >^5, >^6$		

Source: Authors' calculations from ECHP.

invariance which contains the popular notions of absolute and relative invariances as special cases. In this paper, we have isolated the class of all polarization indices of intermediate type, whose ordering of two income distributions agree with that generated by their non-intersecting intermediate polarization curves. A numerical illustration of our results was provided using ECHP data for the period 1994–2001.

Appendix

Proof of Theorem 1: In proving the theorem, we will assume that l and n are odd. A similar proof will run for the case when thay take even values.

Proof: **(a)** $\Rightarrow$ **(b)**: Let z and t be n- and l-fold replications of x and y respectively so that $z, t \in \mathbb{D}^{nl}$. Since **IPC** is population replication invariant, **IPC**s of x and z, and **IPC**s of y and t will be the same. Therefore, $x \succeq_{IP} y$ gives $z \succeq_{IP} t$. Further, $m(x) = m(z)$ and $m(y) = m(t)$. Suppose that $m(x) > m(y)$. Define $u = t + \frac{(m(z) - m(t))}{\mu m(t) + 1 - \mu}(\mu t + (1-\mu)1^n)$. Since **IPC** satisfies the intermediate invariance condition, $IP\left(t; \frac{k}{nl}, \mu\right) = IP\left(u; \frac{k}{nl}, \mu\right)$. Therefore, $z \succeq_{IP} t$ is same as $z \succeq_{IP} u$. Note also that $m(u) = m(z)$. Now by $z \succeq_{IP} u$, we have,

$$\sum_{i=\overline{nl}+1}^{k}(z_i - m(z)) \geq \sum_{i=\overline{nl}+1}^{k}(u_i - m(u)), \quad \overline{nl} + 1 \leq k \leq n,$$

which in view of $m(u) = m(z)$ implies that

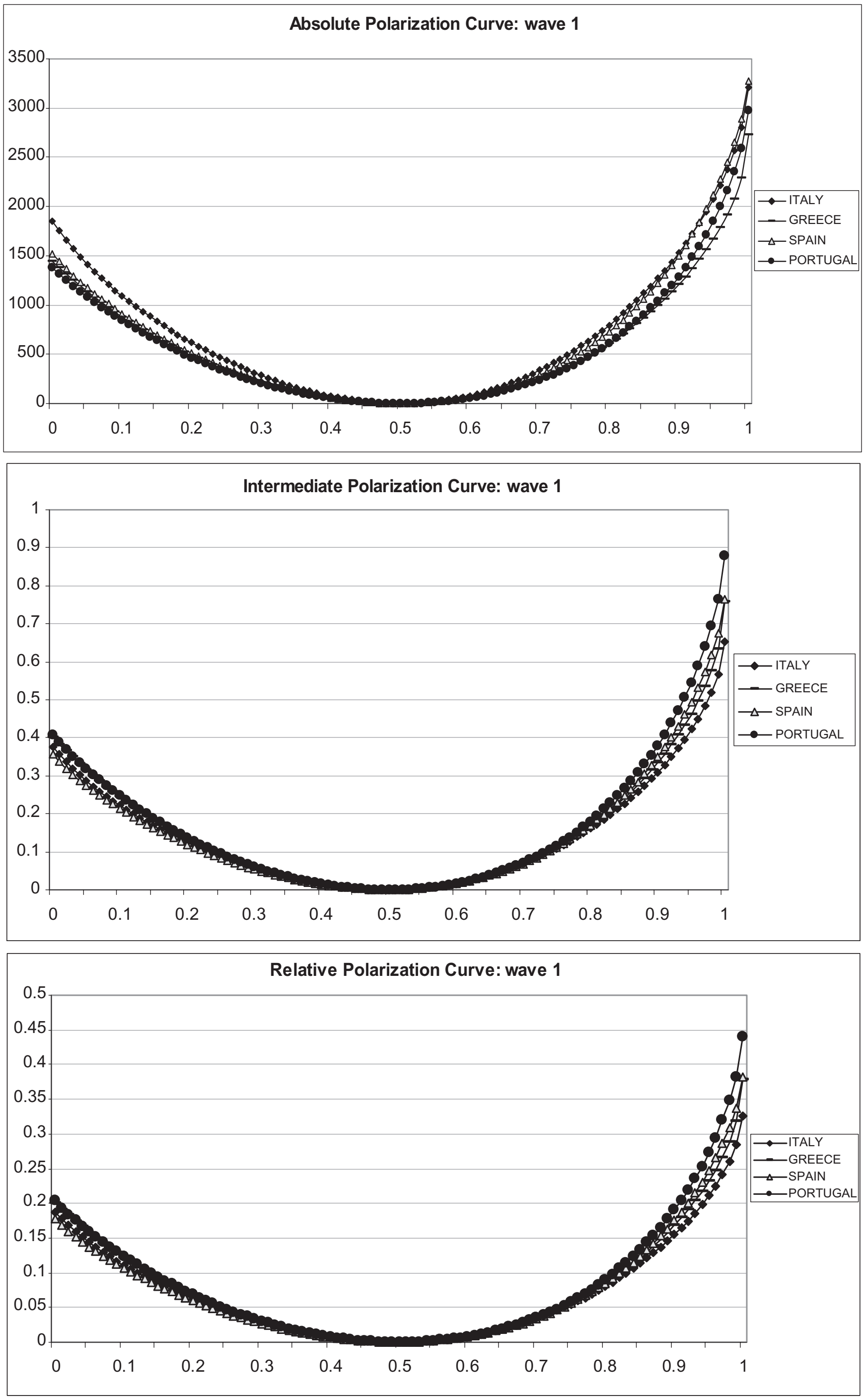

Figure 4. Polarization Curves Between Countries in Wave 1

Source: Authors' calculations from ECHP.

TABLE 6

Inter-Country Dominance by the Absolute Lorenz Criterion in the Eight Waves

Absolute Lorenz	Greece	Italy	Portugal	Spain
Greece		$>^1$	$>^2, >^5, >^8$	$>^1, >^8$
Italy	$>^2, >^3, >^7, >^8$		$>^2, >^3, >^8$	$>^2, >^4, >^8$
Portugal		$>^1$		$>^1, >^2$
Spain	$>^2, >^3, >^4, >^5, >^6, >^7$		$>^3, >^4, >^5, >^6, >^7$	

Source: Authors' calculations from ECHP.

TABLE 7

Inter-Country Dominance by the Lorenz Criterion in the Eight Waves

Lorenz	Greece	Italy	Portugal	Spain
Greece				
Italy	$>^6, >^7, >^8$		$>^1, >^5, >^6, >^7, >^8$	$>^4, >^5, >^6, >^7, >^8$
Portugal				
Spain	$>^1, >^5, >^6$		$>^1, >^2, >^5, >^6, >^7$	

Source: Authors' calculations from ECHP.

(9) $$\sum_{i=\overline{nl}+1}^{k} z_i \geq \sum_{i=\overline{nl}+1}^{k} u_i,$$

where $\overline{nl}+1 \leq k \leq n$. (9) means that z_+ is weakly majorized by u_+ (Marshall and Olkin, 1979, p. 10).

From $z \succeq_{IP} u$ we can establish an analogous relationship between z_- and u_-. Using arguments similar to that employed in Chakravarty *et al.* (2007) it can be shown that the overall distribution z can be derived from the corresponding distribution u through some spread augmenting movements away from the median and/or some equalizing transfers on the same side of the median.

Since I^n satisfies **NS** and **NB**, we have $I^{nl}(z) \geq I^{nl}(u)$. Note that I^{nl} is symmetric because it has been defined on ordered distributions. As I^{nl} is an intermediate index, $I^{nl}(u) = I^{nl}(t)$. Hence, $I^{nl}(z) \geq I^{nl}(t)$. By **PP**, we have $I^{nl}(z) = I^{l}(x)$ and $I^{nl}(t) = I^{n}(y)$. Hence $I^{l}(x) \geq I^{n}(y)$.

If $m(x) \leq m(y)$, we define $u = t + \frac{(m(t)-m(z))}{\mu m(t)+1-\mu}(\mu t + (1-\mu)1^n)$. Then $IP(t; p, \mu) = IP(x; p, \mu)$, so that $z \succeq_{IP} t$ is the same as $z \succeq_{IP} u$. Furthermore, t and u have the same median $m(y)$. The rest of the proof is analogous to the steps employed earlier and hence omitted.

(b) ⇒ (a): Consider the distributions z, t defined above. Consider also the polarization index $I_k^l(x) = \sum_{i=1}^{k} \frac{|m(x)-x_i|}{(\mu m(x)+1-\mu)l}$, where $1 \leq k \leq l$. This index satisfies **NB**, **NS**, **SM**, **PP**, and intermediate polarization invariance. Thus, we have $I_k^{nl}(z) \geq I_k^{nl}(t)$ for $1 \leq k \leq nl$, which in turn implies $IP\left(z; \frac{k}{nl}, \mu\right) \geq IP\left(t; \frac{k}{nl}, \mu\right)$ for all k, $1 \leq k \leq nl$, that is, $z \succeq_{IP} t$. Since **IPC** is population replication invariant, $z \succeq_{IP} t$ is same as $x \succeq_{IP} y$. This completes the proof of the theorem.

References

Amiel, Y., F. A. Cowell, and X. Ramos, "On the Measurement of Polarisation: A Questionnaire Study," DARP Working Paper 90, 2007 (dowloadable from http://sticerd.lse.ac.uk/dps/darp/darp90.pdf).

Bossert, W. and A. Pfingsten, "Intermediate Inequality: Concepts, Indices and Welfare Implication," *Mathematical Social Sciences*, 19, 117–34, 1990.

Chakravarty, S. R., *Inequality, Polarization and Poverty*, Springer Verlag, New York, 2009.

Chakravarty, S. R. and A. Majumder, "Inequality, Polarization and Welfare: Theory and Applications," *Australian Economic Papers*, 40, 1–13, 2001.

Chakravarty, S. R., A. Majumder, and S. Roy, "A Treatment of Absolute Indices of Polarization," *Japanese Economic Review*, 58, 273–92, 2007.

D'Ambrosio, C., "Household Characteristics and the Distribution of Income in Italy: An Application of Social Distance Measures," *Review of Income and Wealth*, 47, 43–64, 2001.

Duclos, J-Y., J. M. Esteban, and D. Ray, "Polarization: Concepts, Measurement, Estimation," *Econometrica*, 72, 1737–72, 2004.

Esteban, J. M. and D. Ray, "On the Measurement of Polarization," Boston University Institute for Economic Development Working Paper 18, February 1991.

———, "On the Measurement of Polarization," *Econometrica*, 62, 819–51, 1994.

———, "Conflict and Distribution," *Journal of Economic Theory*, 87, 379–415, 1999.

Esteban, J. M., C. Gradin, and D. Ray, "Extensions of a Measure of Polarization, with an Application to Income Distribution of Five OECD Countries," *Journal of Economic Inequality*, 5, 1–19, 2007.

Gradin, C., "Polarization by Sub-population in Spain, 1973–91," *Review of Income and Wealth*, 46, 457–74, 2000.

Marshall, A. W. and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

Morris, M., A. Bernhart, and M. S. Handcock, "Economic Inequality: New Methods for New Trends," *American Sociological Review*, 59, 205–19, 1994.

Pressman, S., "The Decline of the Middle Class: An International Perspective," LIS Working Paper No. 280, March 2006.

Wang, Y. Q. and K. Y. Tsui, "Polarization Orderings and New Classes of Polarization Indices," *Journal of Public Economic Theory*, 2, 349–63, 2000.

Wolfson, M. C., "When Inequalities Diverge," *American Economic Review, Papers and Proceedings*, 84, 353–8, 1994.

———, "Divergent Inequalities: Theory and Empirical Results," *Review of Income and Wealth*, 43, 401–21, 1997.

Zhang, X. and R. Kanbur, "What Differences Do Polarization Measures Make? An Application to China," *The Journal of Development Studies*, 37, 85–98, 2001.