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MULTIDIMENSIONAL SOCIAL EVALUATION: AN APPLICATION TO THE MEASUREMENT OF HUMAN DEVELOPMENT

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This paper deals with the axiomatic derivation of social evaluation indices in a multidimensional context. The resulting evaluation formula is the geometric mean of the egalitarian equivalent values of the different characteristics under consideration. We provide an application to the measurement of human development and compare the results obtained with those corresponding to the standard (additive) index.

1. Introduction

This paper deals with the construction of social evaluation indices in a multidimensional context. The key purpose is to evaluate the performance of a society as a function of the individual achievements in several dimensions. We follow a non-welfaristic approach, in the sense that our evaluation index is defined directly on the space of *social states* (matrices whose entries represent the agents' achievements in the relevant dimensions), rather than on the joint utility space. More specifically, we aim at contributing to the discussion of the measurement of human development, along the lines laid down by the United Nations (a specially relevant case of this family of evaluation problems).

The Human Development Index (HDI) is an indicator of this type, proposed by the United Nations in order to assess the well-being of a society (United Nations Development Programme 2006–08). Based on Amartya Sen's idea of *functionings and capabilities* (see Sen, 1985), it consists of the arithmetic mean of the partial indices that approach the achievements of the society in three basic dimensions: health, education, and material well-being. Achievements in health are associated with the variable *life expectancy at birth*. Achievements in education

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are approximated by a mixture of two variables, literacy rate and a combined gross enrollment rate (with weights of two-thirds and one-third, respectively). Finally, the achievements in material well-being are measured through the log of the standard per capita GDP. There are also some companion indices that provide measures of some additional aspects (gender and deprivation).

The HDI is simple and intuitive, refers to very relevant aspects of the socio-economic performance, and uses data that are available in most countries. Those features allow for widespread international comparisons that are accessible to non-specialists. That probably explains its popularity and the relevance given by the media to the yearly publication of each new wave of data.

The HDI is certainly a step ahead in the conventional description of the economic performance of a society, as it goes beyond the mere comparison of per capita GDP values. Yet, it is subject to some criticisms, concerning the number and nature of the selected functionings, the choice of the variables that measure those functionings, the lack of theoretical justification of the aggregation formula, or the absence of distributional considerations, among others. The reader is referred to the works of Osberg (1985), Anand and Sen (1994a, 1994b), Philipson and Soares (2001), Osberg and Sharpe (2002), Pinilla and Goerlich (2003), Becker *et al.* (2005), Grimm *et al.* (2008), Hicks (1997), Chakravarty (2003), and Foster *et al.* (2005), for a critical appraisal and some alternative formulations. The last three contributions are those closer to the analysis in this paper.

Chakravarty (2003) follows an axiomatic approach and provides a generalization of the HDI in two steps. First, he gives necessary and sufficient conditions for a characteristic to be measured as a function of its normalized value alone (assuming a compact range, this normalization consists of subtracting the min of the variable and dividing by its range). Then, he provides additional conditions that allow aggregation of those normalized values in terms of an additive formula (the arithmetic mean). The extension of the HDI refers to the possibility of having non-constant rates of substitution between characteristics.

Following a constructive approach, Hicks (1997) and Foster *et al.* (2005) present alternative versions of the HDI that incorporate distributive aspects in the evaluation formula. Hicks (1997) suggests deflating the normalized value in each dimension by a factor involving the Gini index of the corresponding dimension and then aggregating those transformed partial indices using the arithmetic mean. Foster *et al.* (2005) point out that this measure does not satisfy "subgroup consistency" and propose a different measure compatible with such a property. This new measure (actually a family of them) is based on the notion of a *generalized mean*, that may be associated with different specifications of the discount factor, substituting the Gini index by one of the Atkinson's family. The same generalized mean is used to aggregate those partial indices. The introduction of distributive concerns can be interpreted, both in Hicks (1997) and Foster *et al.* (2005), as substituting the original values in each dimension by their corresponding *egalitarian equivalent* ones, suitably defined.

Our focus here is somewhere in the intersection of those contributions. On the one hand, we would like to have a distribution sensitive index for the evaluation of human development. On the other hand, we are concerned with the precise justification of the aggregation formula and, in particular, on the suitability of the arithmetic mean. Indeed, the additive structure of the HDI entails a very particular trade-off between the characteristics, with or without distributive considerations, as it implies assuming full substitutability between all of them (linear indifference curves). That amounts to saying, for instance, that no matter how bad the health state is, it can be compensated with further education or additional income, *at a constant rate*, which is not very natural.

As in Chakravarty (2003), we follow here an axiomatic approach and provide a family of indices in which the marginal rates of substitution between characteristics are not constant. As in Hicks (1997) and Foster *et al.* (2005), our indices are distribution sensitive. More precisely, we provide an axiomatic characterization of an index that consists of the geometric mean of the egalitarian equivalent values of the characteristics. The distributive concern is incorporated in the specification of the egalitarian equivalent values, and admits the Gini index, the (normalized) Theil index, and the Atkinson's family of inequality measures, among others. In the extreme case when distributive aspects do not matter, we obtain a multiplicative version of the standard HDI (i.e. the geometric mean of the mean values).

In order to illustrate the relevance of this approach we consider an empirical application that consists of a comparative study of different formulations of the human development index. We compare the standard human development index with two alternative versions. The first one is similar to the standard index, but without using logs to measure the income. The second alternative is a multiplicative version of the previous one. The results show that choosing the additive or the multiplicative formula matters in two ways. On the one hand, the positions in the raking of human development may change significantly, especially for the medium developed countries. On the other hand, the aggregation procedure substantially affects the distribution of the indicators (in particular the standard HDI shows a much lower dispersion than its multiplicative version).

The rest of the paper is organized as follows. Section 2 presents the formal model and the key results. Section 3 contains the empirical application.

2. The Model and the Results

Let a society consist of n individuals, $N = \{1, 2, \ldots, n\}$, and suppose we want to assess its global performance as a function of the achievements of its members with respect to a set $K = \{1, 2, \ldots, k\}$ of characteristics. The HDI is a case in point, where the characteristics under consideration refer to health, education, and income. A **social state** is a matrix Y with n rows (one for each individual) and k columns (one for each characteristic). The element y_{ij} of matrix Y describes the value of the variable j for individual i. We assume that the values of each of these characteristics vary in the interval [0,1]. This amounts to saying that all variables have been previously normalized in order to make them comparable, independently on the units in which they are originally measured. This normalization procedure can always be done whenever the original values of the variables, z_{ij}, vary in a compact interval $[z_j^0, z_j^*]$, for some non-negative scalars $z_j^0, z_j^* \in \mathbb{R}_+$, with $z_j^0 < z_j^*$ for all $j \in K$. In that case we simply define:

$$y_{ij} = \frac{z_{ij} - z_j^0}{z_j^* - z_j^0},$$

and get the desired normalization. Note that the choice of the upper and lower bounds may not be innocuous, even though we shall not discuss here this subject.[1]

The space of admissible social state matrices is, therefore, $\Omega = [0,1]^{nk}$, that includes the extreme cases Y^0 and Y^*, corresponding to those matrices made out of zeroes and ones, respectively. We denote by $\mathbf{y}_j$ (bold letter) the j-th column of matrix Y that describes the distribution of the j-th characteristic in the population. The vectors $\mathbf{0}_n(j)$, $\mathbf{1}_n(j)$ describe the j-th column of matrices Y^0 and Y^*, respectively. Y_{-j} is an $n \times (k - 1)$ matrix obtained from Y by deleting its j-th column. We can therefore write $Y = (Y_{-j}, \mathbf{y}_j)$, assuming that $\mathbf{y}_j$ actually occupies the j-th position in the array of columns.

Consider now the following:

Definition: *A **Social Evaluation Index** is a continuous single-valued mapping $I : \Omega \to \mathbb{R}$ that provides a numerical evaluation of social states.*

A social evaluation index is a continuous function that maps social states into real numbers, in the understanding that higher values of this index correspond to better social states.

Now we consider some properties that introduce value judgments on the evaluation formulae.

The first property, *monotonicity*, establishes that the index increases when all agents in the society strictly improve their achievements.[2]

Monotonicity. For each $Y, Y' \in \Omega$, if $Y >> Y'$ then $I(Y) > I(Y')$.

The second property, *symmetry with respect to the characteristics*, establishes that all characteristics are equally important (recall that all variables have already been normalized). That is, a permutation of the characteristics does not affect the social evaluation. Formally:

Symmetry with respect to the characteristics. For each $Y \in \Omega$, if $\pi_C(Y)$ denotes a permutation of the columns of Y, then $I(\pi_C(Y)) = I(Y)$.

The third property, *normalization*, fixes the scale of the index. It establishes that when the matrix is uniform (i.e. all entries are identical), the index takes on the same value. Formally:

Normalization. Let $Y(\alpha) = \alpha[\mathbf{1}_n(1), \ldots, \mathbf{1}_n(k)]$, for some $\alpha \in [0,1]$. Then, $I(Y(\alpha)) = \alpha$.

It is worth noting that the combination of monotonicity and normalization has strong implications. On the one hand, it implies that the range of I is the interval [0,1], i.e. for all $Y \in \Omega$, $I(Y) \in [0,1]$ with $I_{\min} = I(Y^0) = 0$, $I_{\max} = I(Y^*) = 1$.

[1]See Chakravarty (2003, Th. 1) for an axiomatic deduction of this type of reference variable.
[2]The notation $Y >> Y'$ means that $y_{ij} > y'_{ij}$ for all $i \in N$ and all $j \in K$.

On the other hand, it introduces a cardinality feature in the social evaluation index (it actually implies a unique representation of the index).[3]

The following properties, minimal lower boundedness and separability, are borrowed from the theory of multi-attribute decision-making developed in Bossert and Peters (2000). *Minimal lower boundedness* states that there is no trade-off between characteristics when all members of the society are at their worst level in one of them. In other words, if for some characteristic $j \in K$ we have $y_{ij} = 0$, for all $i \in N$, then the social evaluation index I takes on its minimum value. Therefore, for a characteristic to provide a positive contribution to the social performance, at least one individual in society should be above that minimum level.[4] Formally:

Minimal lower boundedness. For all $Y, Y' \in \Omega$, all $j \in K$, $I(Y) \geq I\left(Y'_{-j}, \mathbf{0}_n(j)\right)$.

Clearly, this property, together with monotonicity and normalization, implies that $I(Y_{-j}, \mathbf{0}_n(j)) = 0$, for all $j \in K$, all $Y \in \Omega$.

Separability is a property closely related to the preferential independence axiom in utility theory (Keeney and Raiffa, 1976, ch. 3). It establishes that if social state Y is considered at least as good as social state Y', when there is a common value of a characteristic (both have an identical column $\mathbf{y}_j$), then this relation holds for all common values of this column. For this property to be compatible with minimal lower boundedness, it is only required on those matrices with strictly positive entries. Formally:

Separability. For each $Y, Y' \in \Omega$ with $Y, Y' >> Y^0$, and each $j \in K$,

$$I\left(Y_{-j}, \mathbf{y}_j\right) \geq I\left(Y'_{-j}, \mathbf{y}_j\right) \Rightarrow I\left(Y_{-j}, \mathbf{y}'_j\right) \geq I\left(Y'_{-j}, \mathbf{y}'_j\right).$$

Given a social evaluation index *1* satisfying the aforementioned properties, and a social state matrix Y, we denote by $\xi\left(Y_{-j}, \mathbf{y}_j\right) \in \mathbb{R}$ the **egalitarian equivalent value** associated with the distribution of the j-th characteristic in Y. That is, $\xi(Y_{-j}, \mathbf{y}_j)$ is implicitly defined by the following equation:

$$I(Y) = I\left(Y_{-j}, \mathbf{1}_n(j)\xi\left(Y_{-j}, \mathbf{y}_j\right)\right).$$

When $\xi(Y_{-j}, \mathbf{y}_j)$ is independent on Y_{-j} (that is, $\xi\left(Y_{-j}, \mathbf{y}_j\right) = \xi\left(Y'_{-j}, \mathbf{y}_j\right)$), for all admissible Y'_{-j}, we shall simply write: $\xi(\mathbf{y}_j)$.[5]

The next result is obtained:

Theorem: *A social evaluation index satisfies monotonicity, symmetry with respect to the characteristics, normalization, minimal lower boundedness, and separability, if and only if it takes the form*

[3]This relation is formally stated in a lemma in the Appendix.

[4]Note that the strength of this requirement partly depends on how we define the minimum value of the characteristic under consideration. For instance, if $y_{ij} = 0$ describes an absolute minimal value (e.g. income below subsistence), then this property is automatically fulfilled. When $y_{ij} = 0$ represents some conventional reference level, then the requirement implicitly says that this level is the minimum admissible in order to allow other characteristics to be taken into account.

[5]We show (see step 3 of the proof of Theorem 1) that, under our assumptions, the egalitarian equivalent value is well defined and actually independent of Y_{-j}.

$$I(Y) = \prod_{j \in K} (\xi(\mathbf{y}_j))^{1/k},$$

where $\xi(\mathbf{y}_j)$ is the egalitarian equivalent value of $\mathbf{y}_j$. Moreover, all these properties are logically independent.

(The proof is given in the Appendix)

This theorem says that assuming the five properties above amounts to measuring social states as the geometric mean of the egalitarian equivalent values of the corresponding characteristics.

The theorem identifies a family of multiplicative indices, $\mathbf{I}_M$, rather than a single one. To achieve a fully closed formula we have to make precise the content of the egalitarian equivalent values, $\xi(.)$. This is a notion that expresses our concern for equity in the distribution. Following the standard approach in the theory of economic inequality, under standard conditions,[6] we can define

$$\xi(\mathbf{y}_j) = \mu(\mathbf{y}_j)[1 - f(\mathbf{y}_j)],$$

where $\mu(\mathbf{y}_j)$ is the mean of the j-th characteristic and $f(.)$ is an index of inequality. We can then modulate our concern for equality via the specification of the inequality index used to define the egalitarian equivalent values.

Note that not all of the usual indices are suitable for this purpose. This is so because each $\xi(\mathbf{y}_j)$ must be increasing in $\mathbf{y}_j$ and take on values in the interval [0,1]. That requires $f(\mathbf{y}_j)$ to range also in that interval. Natural candidates to be considered are, therefore, the Gini index, the normalized (first) index of Theil and the Atkinson's family of inequality indices.

Depending on the chosen inequality measure, the social evaluation index will exhibit more precise properties (e.g. population replication, subgroup consistency, etc.). Indeed, making use of the theory of inequality measurement one can fix the inequality index out of the properties one is willing to get (see, for instance, Foster *et al.*, 2005).

Observe that the inequality measure in this formulation applies to the normalized values $y_{ij} = (z_{ij} - z_j^0)/(z_j^* - z_j^0)$, with $z_{ij}, \in [z_j^0, z_j^*]$, for some non-negative scalars $z_j^0, z_j^* \in \mathbb{R}_+$. If $z_j^0 = 0$ for all $j \in K$, then any relative inequality index will measure, precisely, the distribution of the original values.

A special case is that in which we are not concerned with equality. This is precisely the case of the standard human development index. The next property, *distributional neutrality*, introduces this idea. It says that the average value of a characteristic is sufficient information to calculate the index. Formally:

Distributional neutrality in the *j*-th characteristic. For each $Y \in \Omega$, $I(Y) = I(Y_{-j}, \mathbf{1}_n(j)\mu(\mathbf{y}_j))$.

Trivially, when I is distributionally neutral in the j-th characteristic, $\xi(\mathbf{y}_j) = \mu(\mathbf{y}_j)$. Furthermore, as our indices satisfy *symmetry with respect to the*

[6]That basically requires the index to be anonymous (names do not matter) and quasi-concave (redistribution is good).

characteristics, whenever $I \in \mathbf{I}_M$ is distributionally neutral w.r.t. some characteristic, it will also be neutral w.r.t all of them.[7] Therefore, the following straightforward result is obtained:

Corollary: *An index $I \in \mathbf{I}_M$ satisfies distributional neutrality in some characteristic j, if and only if it takes the form:*

$$I(Y) = \prod_{j \in K} [\mu(\mathbf{y}_j)]^{1/k}.$$

This result says that assuming distributional neutrality amounts to taking the geometric mean of the mean values of each characteristic as the right indicator. The formula obtained in the Corollary is one of the limit cases in the family of indices analyzed in Foster *et al.* (2005).

Note that, even though the index in the Corollary discards the information concerning the inequality in the distribution of each characteristic, it penalizes countries with uneven mean achievements in the different dimensions, reflecting the view that there is some complementarity between characteristics rather than full substitutability. Also observe that this index satisfies the principle of population replication (i.e. the index does not change when we replicate the population). Therefore, we can apply this index to societies with different population size and compare the performance of those societies with respect to the same set of variables.

3. A Case Study

We devote this section to illustrating the differences between the human development index proposed by the United Nations and our multiplicative index. We follow the method of the Human Development Report 2006, using its data for health, education, and income corresponding to 177 countries. Even though our formulation allows for the introduction of distributive considerations, we shall develop our analysis here in terms of the formula in the Corollary. The reason is twofold: on the one hand, to facilitate the comparison on the effect of modifying the aggregation criterion; on the other hand, due to the fact that the variables related to health and education are directly average constructs, without much reference to individual values, so that their dispersion in the population cannot be clearly established (see below).

The health variable is measured through the life expectancy at birth,

$$m(\mathbf{h}) = \frac{\text{life expectancy at birth} - 25}{85 - 25}.$$

[7]By the same token, if we introduce distributive considerations with respect to a particular dimension in terms of a given inequality measure, the same inequality index should be applied to all dimensions.

The education variable is measured through a combination of the adult literacy rate and the gross enrollment ratio, with weights of two-thirds and one-third, respectively,

$$m(\mathbf{e}) = \frac{\left(\frac{2}{3}\text{adult literacy rate} + \frac{1}{3}\text{gross enrollment ratio}\right) - 0}{100 - 0}$$
$$= \frac{2}{3}\frac{\text{adult literacy rate} - 0}{100 - 0} + \frac{1}{3}\frac{\text{gross enrollment ratio} - 0}{100 - 0}.$$

The income variable is measured through the log of the GDP per capita

$$m(\mathbf{y}) = \frac{\log(\text{GDP per capita}) - \log 100}{\log 40{,}000 - \log 100}.$$

Alternatively, we also consider the partial indicator $m'(\mathbf{y})$ where we dispense with the logs:

$$m'(\mathbf{y}) = \frac{\text{GDP per capita} - 100}{40{,}000 - 100}.$$

With these four partial indicators we can define a collection of human development indices, the first of which corresponds to the United Nations original proposal; the last one is the index characterized in the Corollary. The other index represents an intermediate step that is useful to clarify the nature of the differences between the United Nations HDI and our proposal. That is:

UN Human Development Index. $\text{UNHDI} = \frac{1}{3}(m(\mathbf{h}) + m(\mathbf{e}) + m(\mathbf{y}))$.

The second index also has an additive structure, but the income indicator is taken without logs.

Additive Human Development Index. $\text{AHDI} = \frac{1}{3}(m(\mathbf{h}) + m(\mathbf{e}) + m'(\mathbf{y}))$.

The third index is the multiplicative version of the previous one (see the Corollary).

Multiplicative Human Development Index. $\text{MHDI} = (m(\mathbf{h}) \cdot m(\mathbf{e}) \cdot m'(\mathbf{y}))^{\frac{1}{3}}$.

We plot on the horizontal axis of Figure 1 the 177 countries in the UN reports, arranged in decreasing order according to the UNHDI. The vertical axis shows the values of these three indices. Peaks correspond to changes in the order for the indices with respect to UNHDI. This figure illustrates well that these indices are not ordinally equivalent.[8] Indeed, some countries jump substantially in the ranking (Colombia, for instance, loses 11 positions). The difference in the ranking between UNHDI and AHDI is due to the re-scaling effect derived from

[8]Note that this happens not only when we move from an additive formulation to a multiplicative one, but also when we use the income variable without logs.

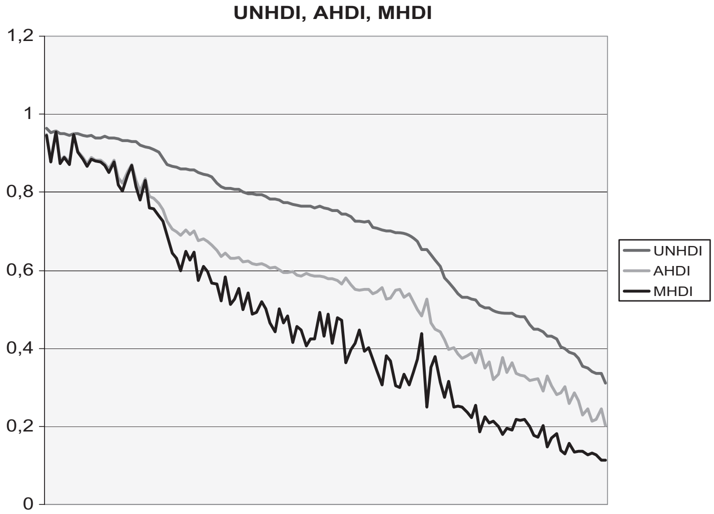

Figure 1. Absolute Values of the Different Indices (2006)

taking out the logs in the income measure, because that change alters the weight with which this variable enters the index. The difference between the AHDI and the MHDI ranking derives from the effect of the differences in the partial indices for each country: for a constant sum, the more equal the partial indices the higher the value of its product (this also explains the fact that the additive indices dominate the multiplicative one).

The cardinal information that these indices provide is also substantially different. A first approximation comes from the analysis of their dispersion, approximated by some free scale measure. The *coefficient of variation* is a standard measure of this type. As one would expect, the dispersion is larger for the MHDI than for the UNHDI. The coefficient of variation of the UNHDI is 0.26, whereas the value for the MHDI is 0.53 (more than twice that of the UNHDI). This suggests that the UNHDI is somehow hiding the differences between countries. On the other hand, the coefficient of variation for the AHDI is 0.35, pointing out that the difference in the dispersion is mainly due to the aggregation procedure and not to the use of logs.

Figure 2 presents the distribution of these indices, normalizing to one the highest value achieved by a country with each index. Note that here the same point in the horizontal axis may correspond to different countries, depending on the chosen index. The graphic shows a rather different picture of the distribution of the development levels, depending on the indicator. For those countries with higher values of the UNHDI the observed difference is mainly due to the measurement in logs of the income levels (which is anyway relevant for the whole distribution). The choice of an additive or a multiplicative formula becomes much more important for those countries in the mid range of the distribution and it is also substantial for those with lower values. A way of illustrating these differences is by noting that the least developed countries represent 32 percent

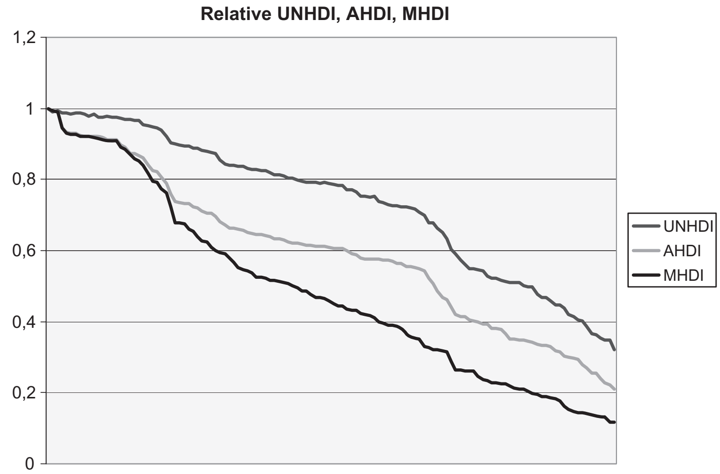

Figure 2. Relative Values of the Different Indices (2006)

of the most developed ones, according to the UNHDI, whereas this figure is 11 percent according to the MHDI. In terms of the correlations, the three indices are quite related. The correlations between the AHDI and UNHDI, and the AHDI and MHDI are 0.98 in both cases; while the correlation between the UNHDI and the MHDI is 0.94.

Table 1 provides the data with which these figures have been constructed, for a representative sample of 50 countries with different development levels. The table gives the values of the partial indices measuring the achievements in health, education, and material well-being, and the resulting human development indices, as well as the changes in the ranking of the MHDI with respect to the UNHDI. A positive number in this last column indicates that the country has gained positions in the MHDI with respect to the UNHDI.

In summary, this application shows that the choice of the aggregation formula for the partial indices matters as it affects both the ranking of the countries and the dispersion of their values. The standard HDI reduces the observed inequality between countries both due to the use of logs in the measurement of income and to the additive aggregation procedure. The multiplicative formula proposed here presents some advantages. First, it is theoretically well justified. Second, it does not assume constant rates of substitution between characteristics. And third, it allows for distributive considerations. The discussion above suggests that dealing properly with the distributive aspects calls for a modification of the way in which the United Nations measures the achievements in health and education.[9] This topic is left for future research.

[9]See, however, the original approach in Grimm *et al.* (2008), that allows for the introduction of distributive aspects recurring to the computation of the index by income groups.

TABLE 1
Human Development Indices for a Sample of Countries (2006)

	Means				HDI			Ranking Changes
	$m(\mathbf{h})$	$m(\mathbf{e})$	$m(\mathbf{y})$	$m'(\mathbf{y})$	UNHDI	AHDI	MHDI	
High human development								
Ireland	0.88	0.99	1.00	1.00	0.957	0.957	0.955	–2
United States	0.88	0.97	1.00	1.00	0.950	0.950	0.949	–3
Canada	0.92	0.97	0.96	0.79	0.950	0.892	0.889	1
Belgium	0.90	0.98	0.96	0.79	0.947	0.889	0.885	–1
Netherlands	0.89	0.99	0.96	0.79	0.947	0.889	0.885	–1
Austria	0.90	0.96	0.96	0.79	0.940	0.882	0.879	–5
United Kingdom	0.89	0.97	0.96	0.79	0.940	0.882	0.879	–5
Denmark	0.87	0.99	0.96	0.79	0.940	0.882	0.878	–5
Sweden	0.92	0.98	0.95	0.74	0.950	0.880	0.874	6
Japan	0.95	0.94	0.95	0.74	0.947	0.877	0.871	4
France	0.91	0.97	0.95	0.74	0.943	0.873	0.868	4
Finland	0.89	0.99	0.95	0.74	0.943	0.873	0.867	4
Italy	0.92	0.96	0.94	0.70	0.940	0.859	0.851	1
Germany	0.90	0.96	0.94	0.70	0.933	0.852	0.845	–1
Spain	0.91	0.98	0.92	0.62	0.937	0.836	0.820	3
Greece	0.89	0.97	0.90	0.55	0.920	0.803	0.779	2
Portugal	0.87	0.96	0.88	0.49	0.903	0.772	0.740	0
Medium human development								
Brazil	0.76	0.88	0.74	0.21	0.793	0.616	0.519	–4
Thailand	0.75	0.86	0.73	0.20	0.780	0.602	0.502	–8
Colombia	0.79	0.86	0.72	0.18	0.790	0.612	0.501	–2
Turkey	0.73	0.81	0.73	0.20	0.757	0.579	0.488	–18
Iran, Islamic Rep. of	0.76	0.75	0.72	0.18	0.743	0.565	0.472	–17
China	0.78	0.84	0.68	0.14	0.767	0.588	0.456	7
Peru	0.75	0.87	0.67	0.14	0.763	0.585	0.446	8
South Africa	0.37	0.80	0.79	0.28	0.653	0.484	0.437	–28
Paraguay	0.77	0.86	0.65	0.12	0.760	0.584	0.431	5
Philippines	0.76	0.89	0.64	0.11	0.763	0.588	0.425	12
Ecuador	0.82	0.86	0.61	0.09	0.763	0.591	0.405	17
Egypt	0.75	0.73	0.62	0.10	0.700	0.527	0.380	–3
Namibia	0.37	0.79	0.72	0.18	0.627	0.448	0.378	–16
Morocco	0.75	0.54	0.63	0.11	0.640	0.466	0.351	–8
Honduras	0.72	0.77	0.56	0.07	0.683	0.520	0.337	1
Bolivia	0.66	0.87	0.55	0.07	0.693	0.532	0.334	5
India	0.64	0.61	0.58	0.08	0.610	0.443	0.313	–4
Mauritania	0.47	0.49	0.49	0.04	0.483	0.335	0.218	–9
Guinea	0.48	0.34	0.51	0.05	0.443	0.290	0.202	–11
Senegal	0.52	0.39	0.47	0.04	0.460	0.316	0.200	–5
Low human development								
Kenya	0.37	0.69	0.41	0.03	0.490	0.362	0.190	5
Cote d'Ivoire	0.35	0.46	0.46	0.04	0.423	0.282	0.181	–6
Rwanda	0.32	0.61	0.42	0.03	0.450	0.320	0.177	2
Nigeria	0.31	0.63	0.41	0.03	0.450	0.322	0.173	3
Benin	0.49	0.40	0.40	0.03	0.430	0.305	0.170	1
Tanzania, U. Rep. of	0.35	0.62	0.32	0.01	0.430	0.328	0.147	3
Zambia	0.21	0.63	0.37	0.02	0.403	0.287	0.139	2
Ethiopia	0.38	0.40	0.34	0.02	0.373	0.266	0.136	–1
Central African Republic	0.24	0.42	0.40	0.03	0.353	0.228	0.136	–2
Burundi	0.32	0.52	0.32	0.01	0.387	0.285	0.134	2
Burkina Faso	0.38	0.23	0.41	0.03	0.340	0.212	0.133	–2
Malawi	0.25	0.64	0.31	0.01	0.400	0.301	0.129	7
Niger	0.33	0.26	0.34	0.02	0.310	0.202	0.113	1

Appendix

Lemma. An index that satisfies monotonicity and normalization together is cardinal.

Proof. Let I be a monotonic and normalized index. Now, consider a continuous and increasing function $G:\mathbb{R}\to\mathbb{R}$, and take the transformation of a social evaluation index I, $(G\circ I)$. One may wonder whether $(G\circ I)$ is also a suitable index to evaluate social states, and fulfils those properties. Since $(G\circ I)$ must satisfy normalization, $(G\circ I)(Y(\alpha))=\alpha$. On the other hand,

$$(G\circ I)(Y(\alpha))=G(I(Y(\alpha)))=G(\alpha).$$

Therefore, $G(\alpha)=\alpha$, and G is the identity function. This implies not only that I is cardinal but also that this representation is unique, as $I(Y^0)=0$, $I(Y^*)=1$, and there is no degree of freedom left.

Proof of the Theorem

It is not difficult to check that such indices satisfy the properties. We show here the converse. Let I be an index that satisfies all the five properties.

Step 1. Multiplicative structure of the index. In this step we show that an index that satisfies monotonicity, normalization, minimal lower boundedness, and separability is multiplicative. Indeed, by *monotonicity* and *normalization* we can write:

$$0=I\left(Y^0\right)\le I(Y)\le I(Y^*)=1,\quad \text{for all } Y\in\Omega.$$

Take an arbitrary characteristic, $j\in K$. By *separability*, and *cardinality* (obtained in application of the lemma), since $\mathbf{y}_j$ is independent of Y_{-j}, there exist real valued mappings u,v such that:[10]

$$I(Y)=u\left(\mathbf{y}_j\right)+v\left(\mathbf{y}_j\right)I\left(Y_{-j},\mathbf{1}_n(j)\right).$$

By letting $Y=\left(Y_{-j}^0,\mathbf{y}_j\right)$, *minimal lower boundedness* implies $u(\mathbf{y}_j)=0$, for all $\mathbf{y}_j$. Now, letting $Y=(Y_{-j}^*,\mathbf{y}_j)$, we get $v\left(\mathbf{y}_j\right)=I(Y_{-j}^*,\mathbf{y}_j)$ and, consequently,

$$I(Y)=I(Y_{-j}^*,\mathbf{y}_j)\cdot I\left(Y_{-j},\mathbf{1}_n(j)\right).$$

Define now the function $\widehat{I}\left(Y_{-j}\right):=I\left(Y_{-j},\mathbf{1}_n(j)\right)$, that satisfies *separability* and *minimal lower boundedness*, and apply to $\widehat{I}$ the argument above for some characteristic $h\neq j$. That is, by *separability* and *cardinality*, there are two functions $\widehat{u}$ and $\widehat{v}$ such that

$$\widehat{I}\left(Y_{-j}\right)=\widehat{u}\left(\mathbf{y}_h\right)+\widehat{v}\left(\mathbf{y}_h\right)\widehat{I}\left(Y_{-jh},\mathbf{1}_n(h)\right).$$

[10]See Keeney and Raiffa (1976, ch. 5, 6). Note that, in principle, both u and v may depend on the characteristic j considered. Yet we shall not make specific this feature in order to save notation.

By letting $Y_{-j} = (Y_{-jh}, \mathbf{y}_h)$, $Y^*_{-j} = (Y^*_{-jh}, \mathbf{y}_h)$, we get

$$\hat{I}(Y_{-j}) = \hat{I}(Y^*_{-jh}, \mathbf{y}_h) \cdot \hat{I}(Y_{-jh}, \mathbf{1}_n(h)).$$

Since, by definition, $\hat{I}(Y_{-j}) = I(Y_{-j}, \mathbf{1}_n(j))$, we have that

$$I(Y_{-j}, \mathbf{1}_n(j)) = I(Y^*_{-h}, \mathbf{y}_h) \cdot I(Y_{-jh}, \mathbf{1}_n(j), \mathbf{1}_n(h)).$$

Replacing $I(Y_{-j}, \mathbf{1}_n(j))$ in $I(Y)$,

$$\begin{aligned} I(Y) &= I(Y^*_{-j}, \mathbf{y}_j) I(Y_{-j}, \mathbf{1}_n(j)) \\ &= I(Y^*_{-j}, \mathbf{y}_j) \cdot I(Y^*_{-h}, \mathbf{y}_h) \cdot I(Y_{-jh}, \mathbf{1}_n(j), \mathbf{1}_n(h)). \end{aligned}$$

By repeating this procedure for all characteristics, we arrive at:

$$I(Y) = I(Y^*_{-1}, \mathbf{y}_1) \cdot I(Y^*_{-2}, \mathbf{y}_2) \cdot \ldots \cdot I(Y^*_{-k}, \mathbf{y}_k).$$

Therefore, if we define

$$F_j(\mathbf{y}_j) = I(Y^*_{-j}, \mathbf{y}_j),$$

we get:

$$I(Y) = \prod_{j \in K} F_j(\mathbf{y}_j),$$

with $F_j(\mathbf{0}_n) = 0$ and $F_j(\mathbf{1}_n) = 1$.

Step 2. Adding symmetry. By *symmetry with respect to the characteristics*, $F_j(.) = F_h(.) = F(.)$, for all $j,h \in K$. Therefore, we can write:

$$I(Y) = \prod_{j \in K} F(\mathbf{y}_j).$$

Step 3. Existence of the egalitarian equivalent value. We now see that the egalitarian equivalent value, as it is defined in Section 2, exists, and it does not depend on Y_{-j}. As a preliminary stage, we show that if $(Y^*_{-j}, \mathbf{y}_j) \in \Omega$ then there exists $\xi(\mathbf{y}_j) \in [0,1]$ such that $I(Y^*_{-j}, \mathbf{y}_j) = I(Y^*_{-j}, \mathbf{1}_n \xi(\mathbf{y}_j))$. To do so, define a function $g: [0,1] \to \mathbb{R}$ as follows: $g(\xi) = I(Y^*_{-j}, \mathbf{1}_n \xi)$. Such a g is a continuous function, with $g(0) = I(Y^*_{-j}, \mathbf{0}_n) = 0$, and $g(1) = I(Y^*_{-j}, \mathbf{1}_n) = 1$. Since $0 \le I(Y^*_{-j}, \mathbf{y}_j) \le 1$, the mean value theorem guarantees that there exists some $\xi(\mathbf{y}_j)$ such that $I(Y^*_{-j}, \mathbf{y}_j) = g(\xi(\mathbf{y}_j)) = I(Y^*_{-j}, \mathbf{1}_n \xi(\mathbf{y}_j))$.

Now we show that the egalitarian equivalent value exists for any Y_{-j}. Indeed, let $\xi(\mathbf{y}_j) \in [0,1]$ be such that $I(Y^*_{-j}, \mathbf{y}_j) = I(Y^*_{-j}, \mathbf{1}_n \xi(\mathbf{y}_j))$. We prove that such a value also satisfies $I(Y_{-j}, \mathbf{y}_j) = I(Y_{-j}, \mathbf{1}_n \xi(\mathbf{y}_j))$. By definition,

$$F(\mathbf{y}_j) = I(Y^*_{-j}, \mathbf{y}_j) = I(Y^*_{-j}, \mathbf{1}_n \xi(\mathbf{y}_j)) = F(\mathbf{1}_n \xi(\mathbf{y}_j)).$$

Then,

$$\begin{aligned} I(Y_{-j}, \mathbf{y}_j) &= \prod_{h \neq j} F(\mathbf{y}_h) \cdot F(\mathbf{y}_j) \\ &= \prod_{h \neq j} F(\mathbf{y}_h) \cdot F(\mathbf{1}_n \xi(\mathbf{y}_j)) \\ &= I(Y_{-j}, \mathbf{1}_n \xi(\mathbf{y}_j)). \end{aligned}$$

Step 4. Closing the formula. The previous result indicates that the egalitarian equivalent value $\xi(\mathbf{y}_j)$ obtained in previous step does not depend on the distribution of the other characteristics. Making use of *normalization* once more, and the definition of egalitarian equivalent value above, we can write:

$$\begin{aligned} \xi(\mathbf{y}_j) &= I[\mathbf{1}_n \xi(\mathbf{y}_j), \ldots, \mathbf{1}_n \xi(\mathbf{y}_j)] \\ &= [F(\mathbf{1}_n \xi(\mathbf{y}_j))]^k. \end{aligned}$$

Therefore,

$$F(\mathbf{y}_j) = [\xi(\mathbf{y}_j)]^{1/k},$$

for all $j \in K$, and the result follows. Notice, furthermore, that whenever $\xi(\mathbf{y}_j) \neq 0$, we can also guarantee that $\xi(\mathbf{y}_j)$ is unique.

Step 5. Separation of the properties. In order to see that all properties in Theorem 1 are independent, we provide examples of indices satisfying all but one property.

$$I_1(Y) = \min_{i,j} y_{ij}.$$

$$I_2(Y) = HDI(Y) = \frac{1}{k} \sum_{j=1 \ldots k} \mu(y_j).$$

$$I_3(Y) = \prod_{j=1}^{k} \mu(\mathbf{y}_j)^{\alpha_j}$$, with $\alpha_j > 0$ for all j, $\Sigma_j \alpha_j = 1$, and for some l,m, $\alpha_l \neq \alpha_m$.

$$I_4(Y) = \prod_{j=1}^{k} \mu(\mathbf{y}_j).$$

$$I_5(Y) = \prod_{j=1}^{k} \left[\mu(\mathbf{y}_j)^{1/k} - \text{range}(\mathbf{y}_j)^k \right].$$

In the following table we summarize the properties satisfied by these indices:

	Mon	Sym	Norm	MLB	Sep
I_1	Yes	Yes	Yes	Yes	No
I_2	Yes	Yes	Yes	No	Yes
I_3	Yes	No	Yes	Yes	Yes
I_4	Yes	Yes	No	Yes	Yes
I_5	No	Yes	Yes	Yes	Yes

References

Anand, S. and A. Sen, "Human Development Index: Methodology and Measurement," *Human Development Report Office Occasional Paper 12*, UNDP, 1994a.

———, "The Income Component of the Human Development Index," *Journal of Human Development*, 1, 83–107, 1994b.

Chakravarty, S. R., "A Generalized Human Development Index," *Review of Development Economics*, 7, 99–114, 2003.

Becker, G. S., T. J. Philipson, and R. R. Soares, "The Quantity and Quality of Life and the Evolution of World Inequality," *The American Economic Review*, 95, 277–91, 2005.

Bossert, W. and H. Peters, "Multi-Attribute Decision-Making in Individual and Social Choice," *Mathematical Social Sciences*, 40, 327–39, 2000.

Foster, J. E., L. F. López-Calva, and M. Székely, "Measuring the Distribution of Human Development: Methodology and Application to Mexico," *Journal of Human Development and Capabilities*, 6, 5–25, 2005.

Grimm, M., K. Harttgen, S. Klasen, and M. Misseklhorn, "A Human Development Index by Income Groups," *World Development*, 36, 2527–46, 2008.

Hicks, D. A., "The Inequality-Adjusted Human Development Index: A Constructive Proposal," *World Development*, 25, 1283–98, 1997.

Keeney, R. L. and H. Raiffa, *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*, John Wiley, New York, 1976.

Osberg, L., "The Measurement of Economic Welfare," in D. Laidler (coordinator), *Approaches to Economic Well-Being*, University of Toronto Press, Toronto, 1985.

Osberg, L. and A. Sharpe, "An Index of Economic Well-Being for Selected OECD Countries," *Review of Income and Wealth*, 48, 291–316, 2002.

Philipson, T. and R. Soares, "Human Capital, Longevity, and Economic Growth: A Quantitative Assessment of Full Income Measures," Mimeo, University of Chicago, 2001.

Pinilla, R. and F. J. Goerlich, "Renta per capita y potencial de calidad de vida (QLP) en España (1981–1999)," Working Paper WP-EC 2003-02, Ivie, 2003.

Sen, A., *Commodities and Capabilities*, North-Holland, Amsterdam, 1985.

United Nations Development Programme, *Human Development Report*, New York, United Nations, 2006–08.