Review of Income and Wealth
Series 51, Number 1, March 2005

SEQUENTIAL STOCHASTIC DOMINANCE AND THE ROBUSTNESS OF POVERTY ORDERINGS

BY JEAN-YVES DUCLOS*

Université Laval

AND

PAUL MAKDISSI

Université de Sherbrooke

When comparing poverty across distributions, an analyst must select a poverty line to identify the poor, an equivalence scale to compare individuals from households of different compositions and sizes, and a poverty index to aggregate individual deprivation into an index of total poverty. A different choice of poverty line, poverty index or equivalence scale can of course reverse an initial poverty ordering. This paper develops easily-checked sequential stochastic dominance conditions that throw light on the robustness of poverty comparisons to these important measurement issues. These general conditions extend well-known results to any order of dominance, to the choice of individual versus family based aggregation, and to the estimation of "critical sets" of measurement assumptions. Our theoretical results are briefly illustrated using data for four countries drawn from the Luxembourg Income Study databases.

1. INTRODUCTION

The last decades have seen considerable developments of the methods that can be used to make comparisons of welfare distributions more robust to the choice of ethical indices. Earlier work focussed on inequality measurement (Kolm, 1969; Atkinson, 1970; Dasgupta *et al.*, 1973) and social welfare (e.g. Shorrocks, 1983), but the more recent literature has also pointed out that similar robustness is desirable for poverty measurement (Atkinson, 1987; Foster and Shorrocks, 1988a, 1988b; Zheng, 2000a, 2000b). This is especially important given the long list of available poverty indices and the uncertainty regarding the setting of poverty lines.

Less attention has been granted to robustness to the choice of equivalence scales to compare the resources of households of different compositions and sizes, although considerable uncertainty and debate also surround this choice and since a number of recent studies have emphasized the sensitivity of poverty profiles to it.[1]

Note: We thank G. Anderson, C. Elbers, G. Fields, M. Hoy, P. Lambert, J. Lanjouw, P. Lanjouw, S. Mongrain, D. Sahn, C. Téjédo, E. Thorbecke, S. Younger, as well as one anonymous referee and participants at various seminars for useful comments. This research was supported, in part, by the MIMAP/PEP network of the Canadian IDRC and by grants from SSHRC and FQRSC.

*Correspondence to: Jean-Yves Duclos, Département d'économique and CIRPÉE, Pavillon De Sève, Université Laval, Sainte-Foy, Québec, Canada, G1K 7P4 (jyves@ecn.ulaval.ca).

[1]Buhmann *et al.* (1987) show the important empirical impact of different equivalence scale parameters on poverty measurement. Coulter *et al.* (1992b) use the same type of parameterization and analyze its implication for the theoretical impact of equivalence scales on poverty measurement (see also Coulter *et al.*, 1992a). Banks and Johnson (1994), Jenkins and Cowell (1994) and Duclos and Mercader-Prats (1999) generalize the analysis for a class of parameterized equivalence scales extended to take household composition into account. These papers, in addition to those of Phipps (1991), Burkhauser *et al.* (1996), and De Vos and Zaidi (1997), also find that international comparisons of poverty and poverty profiles are strongly influenced by assumptions on household needs.

Two assumptions are often made on the sets of admissible equivalence scales. The first assumption says that the needs of a household increase with household size. The second assumes the existence of economies of scale and of public goods within the household, and thus says that household needs do not necessarily increase as fast as household size. Even if these two assumptions were not to be disputed, a multitude of equivalence scales would still respect them, including equivalence scales that are income dependent.[2] Subsequent poverty comparisons would then be determined by the scale used, and two alternative choices of equivalence scales could lead to clashing poverty orderings.

This sensitivity to the choice of equivalence scales is particularly important when poverty comparisons are intended to support recommendations for poverty-alleviating economic policies. The targeting of households on a contingency other than income (such as household type) is also often convenient for purposes of poverty alleviation, and this also requires confidence in the comparisons of the standards of living of individuals belonging to different household types.

This paper develops methods that enable robust poverty orderings of distributions for large classes of equivalence scales, poverty lines and poverty indices, as well as for alternative aggregation procedures. This will help ascertain the robustness of poverty comparisons to these important measurement tools. The methods rely on criteria of sequential stochastic dominance, which is useful when the target function is uni-dimensional with multiple inputs. These methods were first introduced in the literature by Atkinson and Bourguignon (1987) and Bourguignon (1989). They were adapted by Atkinson (1992) for poverty dominance of the first order, and this was subsequently extended by Jenkins and Lambert (1993) to enable comparisons of populations with different proportions of household types. Chambaz and Maurin (1998) have also used this framework for second-order poverty dominance.

Our results extend the previous stochastic dominance criteria to any order of dominance. This extension effectively allows analysts to choose any arbitrary order of dominance when seeking robust poverty orderings. The criteria are easy to check and have a straightforward graphical interpretation.

We also extend the previous dominance criteria to the choice of individual versus household based aggregation (previous work on sequential dominance has focussed on the latter type of aggregation). This is important since there are good ethical reasons for which to prefer individual as against household counts in comparing poverty across distributions. The choice also matters for establishing the robustness of poverty comparisons, particularly in the presence of significant demographic differences between two distributions (as we find in the illustration below).

To see how, consider the example shown in Table 1. Distributions A and B each have two households, of size 1 and size 2 respectively. The total incomes of each of these households are shown in the fourth column. If the poverty line for a single adult is 10, which of A and B has more poverty? If needs do not increase with size, total household income is the adult-equivalent income, and poverty is larger in A than in B, regardless of whether it is individuals or households that

[2]See Aaberge and Melby (1998) for an example of this.

TABLE 1

POVERTY COMPARISONS WITH VARIOUS EQUIVALENCE SCALES AND AGGREGATING APPROACHES

Distribution	Household Number	Household Size	Total Income	Per Capita Income	Intermediate Equivalent Income*
A	1	1	6	6	6
	2	2	24	12	16
B	1	1	12	12	12
	2	2	12	6	8
Aggregation over households			$P_A > P_B$	$P_A = P_B$	$P_A > P_B$
Aggregation over individuals			$P_A > P_B$	$P_A < P_B$	P_A ? P_B

Notes:
Poverty line for a single adult: 10.
Populations A and B have 2 households each.
*A household of size 2 has the needs of 1.5 single adults.

are counted. If there are no economies of scale in household size, *per capita* income is the adult-equivalent income; aggregating over households, poverty in *A* and *B* are the same, but poverty is larger in *B* if it is individuals that are counted instead. Using an intermediate "economies of scale" view (with the needs of a household of two persons being those of two single adults), we find that poverty is necessarily larger in *A* than in *B* for all depth-sensitive poverty indices when households are counted (the most severe case of poverty is found in *A*). When it is individuals that are counted, the poverty ranking is ambiguous, since it depends on a trade-off between the incidence of poverty (the individual headcount is larger in *B*) and the depth of poverty. Hence, comparing poverty in *A* and in *B* clearly depends on the choice of poverty index and equivalence scales, and on the chosen type of aggregation procedure.

Finally, an important contribution of the paper shows how sequential stochastic dominance can be used to identify "critical sets" of measurement assumptions. More precisely, we show how to estimate critical boundaries for the sets of equivalence scales and poverty lines over which a poverty comparison may be considered robust for a given order of dominance. These critical sets are, in our view, a better alternative to the traditional dominance testing procedure. The traditional procedure starts by fixing sets of measurement assumptions (a somewhat arbitrary upper bound for poverty lines, for instance) and then checks whether there is dominance under those given sets. Instead, we prefer to identify the sets of measurement assumptions under which dominance is obtained, and let the analyst determine whether his preferred measurement assumptions belong to those critical sets. Hence, instead of starting with assumptions on measurement and then searching for dominance, we identify the set of assumptions that ensure dominance.

Our approach also demonstrates clearly the trade-offs that arise in trying to infer robustness over sets of poverty indices, poverty lines and equivalence scales that are as large as possible. We find that the higher the order of dominance or the smaller the range of possible poverty lines for an equivalent adult, the wider the sets of equivalence scales over which robust poverty orderings can be inferred. Conversely, the smaller the sets of equivalence scales considered, the wider are the robust class of poverty indices and the robust range of poverty lines for single adults.

In addition to its methodological contributions, the paper also shows why and how many of the cross-country and temporal comparisons of poverty that are routinely found in the empirical literature are not robust to assumptions on the treatment of needs and on the units of aggregation. When robust comparisons *can* be made, they moreover may be valid only over limited ranges of measurement assumptions (as in the case of the comparison of Finland and Canada below). In such cases, we feel it is important to make explicit the limits of those ranges in drawing comparative poverty conclusions. Some of the tools introduced in this paper allow this to be done, in such a way that the *maximal* sets of measurement assumptions under which a distribution poverty-dominates another are then available.

The rest of the paper runs as follows. Section 2 develops sequential stochastic dominance conditions in the presence of heterogeneous household composition, aggregating either over households or over individuals. Methods to identify the complete sets of poverty indices, poverty lines and equivalence scales for which the orderings may be considered robust are described in Section 3. Section 4 applies our methods to data of four countries drawn from the Luxembourg Income Study. The last section concludes the paper and summarizes some of our main findings. All proofs appear in the Appendix.

2. Sequential Stochastic Dominance with Households or Individuals as Aggregating Units

It has usually been assumed in the sequential stochastic dominance literature that it is households that are counted when it comes to computing aggregate indices of poverty (e.g., Atkinson, 1992; Jenkins and Lambert, 1993; Chambaz and Maurin, 1998). Ultimately, however, it is the well-being of individuals that is important for normative purposes. Household formation matters, of course, but only in so far as it influences the standards of living of individuals. On ethical grounds, it is then ethically preferable to count individuals rather than households in comparing poverty. This section shows formally how doing so can change poverty orderings significantly.

For both approaches (the household-aggregating H approach and the individual-aggregating I approach), we consider classes Ξ_H^s and Ξ_I^s of additive poverty indices that obey some s-order ethical assumptions.[3] Households are heterogenous in size. For expositional and analytical simplicity, however, households of a given size are assumed to be homogenous in characteristics other that total income. We suppose that there are n different sizes of households.

We wish to determine if poverty decreases when we move from an initial income distribution A to an alternative income distribution B. Let $F_{Ak}(x)$ be the continuous distribution of the total income x of the households of k individuals in distribution A. Let $\Phi_{Ak}(y)$ be the distribution function of the *per capita* income of households of k individuals in distribution A. (If total household income is x, *per capita* income is x/k.) Throughout, the argument x and y will stand respec-

[3] Ok and Lambert (1999) have shown how second-order sequential dominance can be linked with non-additive social welfare indices, and thus presumably with non-additive poverty indices.

tively for the H and the I approaches. These distribution functions are defined on the interval $[0, a]$, where a is greater than the maximum total income and greater also than the maximum conceivable poverty line. Thus, we have $F_{Ak}(a) = 1$ and $\Phi_{Ak}(a) = 1$, for all k. Let θ_{Ak} and γ_{Ak} represent respectively subgroup k's share of households and individuals in population A, so that $F_A(x) = \Sigma_{k=1}^{n}\theta_{Ak}F_{Ak}(x)$ and $\Phi_A(y) = \Sigma_{k=1}^{n}\gamma_{Ak}\Phi_{Ak}(y)$.

Now let subgroup k's stochastic dominance curves be defined as $D_{Ak}^{1}(x) = F_{Ak}(x)$ and $D_{Ak}^{s}(x) = \int_0^x D_{Ak}^{(s-1)}(u)du$ for all integer $s \geq 2$, and similarly let $G_{Ak}^{1}(y) = \Phi_{Ak}(y)$ and $G_{Ak}^{s}(y) = \int_0^y G_{Ak}^{(s-1)}(u)du$ for all integers $s \geq 2$. $D_A^s(x)$, $F_{Bk}(x)$, θ_{Bk}, $F_B(x)$, $D_{Bk}^{s}(x)$, $D_B^s(x)$, $\Phi_{Bk}(y)$, γ_{Bk} and $G_{Bk}^{s}(y)$ are all defined analogously. It is well known that (using successive integrations)

$$D_{Ak}^{s}(x) = \frac{1}{(s-1)!}\int_0^x (x-u)^{s-1}dF_{Ak}(u) \tag{1}$$

and

$$G_{Ak}^{s}(x) = \frac{1}{(s-1)!}\int_0^x (x-u)^{s-1}d\Phi_{Ak}(u). \tag{2}$$

Expressions (1) and (2) have a clear link with the popular *FGT* indices (see Foster *et al.*, 1984).[4] As is well-known, these FGT indices are additively decomposable, and this naturally extends to the dominance curves $D_A^s(x)$ and $G_A^s(x)$. Denote by $C_{Al}^{s}(x)$ the contribution to total household poverty of the $n - l + 1$ neediest groups, $l = 1, \ldots, n$:

$$C_{Al}^{s}(x) = \sum_{k=l}^{n}\theta_{Ak}D_{Ak}^{s}(x), \tag{3}$$

and by $\Gamma_{Al}^{s}(y)$ the contribution to total individual poverty of the individuals belong to the l neediest types of individuals, $l = 1, \ldots, n$:

$$\Gamma_{Al}^{s}(y) = \sum_{k=1}^{l}\gamma_{Ak}G_{Ak}^{s}(y). \tag{4}$$

Since the poverty indices $P \in \Xi_H^s$ and $\Pi \in \Xi_I^s$ are assumed to be additive, we can write

$$P_A = \sum_{k=1}^{n}\theta_{Ak}\int_0^a p_k(x)dF_{Ak}(x) \tag{A1}$$

and

$$\Pi_A = \sum_{k=1}^{n}\gamma_{Ak}\int_0^a \pi_k(y)d\Phi_{Ak}(y). \tag{A4}$$

[4]The original formulation proposed by Foster *et al.* (1984) involves a normalization by z_k, which means that assumption (A3) later in the text may not be satisfied for these indices. Atkinson (1992) argues, however, that such normalization may be inappropriate in the context of heterogenous households. Davidson and Duclos (2000) also show that a normalization of poverty indices by different poverty lines can generate orderings that are more naturally interpretable in terms of relative inequality than in terms of poverty.

$p_k(x) \geq 0$ is the contribution of a household of size k and of total income x to aggregate household poverty; $\pi_k(y) \geq 0$ is the contribution to total individual poverty of an individual that is member of a k-type household. Sen (1981) argues that a poverty measure must satisfy the focus axiom, which says that poverty should be invariant to changes in the incomes of the non-poor. This imposes that $p_k(x) = 0 \ \forall x \geq z_k, \forall k$ and $\pi_k(y) = 0, \forall y \geq z_k, \forall k$, where z_k is k's poverty line (expressed in total or *per capita* income, according to the approach. P_B and Π_A are defined analogously. If poverty does not increase when we move from A to B, we have that

(5) $$\Delta P_{AB} = P_B - P_A \leq 0$$

and

(6) $$\Delta \Pi_{AB} = \Pi_B - \Pi_A \leq 0.$$

For s-order stochastic dominance, we also assume that

(A2) $$p_k(x) \in C^s$$

and

(A5) $$\pi_k(y) \in C^s,$$

where C^s is the set of functions which are (at least s-time) differentiable over $[0, a]$. These continuity assumptions imply that an infinitesimal increase in income does not induce a discrete variation in the functions $p_k(x)$ and $\pi_k(y)$. This rules out the popular (though discontinuous) poverty headcount.

In order to develop sequential stochastic dominance criteria of arbitrary order s, we need a final assumption for each of the H and I approaches.

2.1. *Household-aggregation Dominance*

For the H approach, we finally assume that:

(A3) $$(-1)^s p_n^{(s)}(x) \geq \ldots \geq (-1)^s p_2^{(s)}(x) \geq (-1)^s p_1^{(s)}(x) \geq 0.$$

For $s = 1$, assumption (A3) implies that an increase in household income x diminishes poverty, whatever the household type to which this increased income accrues. It also says that, for a given household income x, the potential for such poverty reduction is greater for households with more members.[5] (A3)'s normative assumptions are thus more stringent than the usual ones for first-order unidimensional dominance. In a sequential-stochastic-dominance framework, this weak version of the Pigou-Dalton principle is in fact equivalent to Sen's Weak Equity Axiom (see Sen, 1997, p. 18).

For $s = 2$, assumption (A3) says that an equalizing transfer of \$1 to a poor from a richer individual decreases poverty, and that this effect is stronger across households of larger sizes. The interpretation of (A3) for higher s can be made

[5]Assumption (A3) only orders derivatives across household types at a given level of total household income. In particular, it does not say whether a transfer from a richer household towards a poorer household may be desirable when the rich household is composed of more individuals than the poor household. See Ebert (1997) for more discussion of this issue.

using Fishburn and Willig (1984), where their general transfer principles give increasing weights to transfers occurring at the bottom of the distribution as s increases. Here, (A3) makes these principles normatively more important for larger households than for smaller ones. Hence, for each order s, we have the standard Fishburn and Willig normative interpretation of s-order unidimensional dominance (that is, the interpretation of $(-1)^s p_k^{(s)}(x) \geq 0$), joined with a weak version of the traditional normative interpretation of $s + 1$-order dominance (the interpretation of $(-1)^s p_k^{(s)}(x) \geq (-1)^s p_{k-1}^{(s)}(x)$ in a sequential context). Again, this normative interpretation can be seen as a generalization of Sen's Weak Equity Axiom. Together, assumptions (A1), (A2) and (A3) define the classes of poverty indices Ξ_H^s, $s = 1, 2, \ldots$.[6]

A consequence of assumptions (A2) and (A3) is that (in the manner of Jenkins and Lambert (1993) and Atkinson (1992)) poverty lines z_k across household types k can be ordered as follows:

$$z_1 \leq z_2 \leq z_3 \leq \ldots \leq z_n < a, \tag{7}$$

Denote by z_k^+ the maximum possible poverty line for a household of type k, with $z_k \leq z_k^+$, $\forall k = 1, \ldots, n$. (7) implies that

$$z_1^+ \leq z_2^+ \leq z_3^+ \leq \ldots \leq z_n^+ < a. \tag{8}$$

As is often done in the literature on poverty and equivalence scales, we can interpret the ratio of z_k over z_1 as the number of equivalent adults living in a household of type k. Denote by $m(k)$ the equivalence scale (relative to a household of a single adult) for such a household; we then have

$$m(k) = z_k / z_1, \quad \text{with } m(1) = 1. \tag{9}$$

Note, however, that this is only an *interpretation* of the ratio z_k/z_1, since (9) is not needed for the results below. By (7) and (9), it follows that

$$m(k) \leq m(k+1), \quad \text{for } k = 1, \ldots, n-1. \tag{10}$$

A common and usually undisputed restriction on equivalence scales is that $m(k)$ cannot exceed k, that is, with (7) and (9), that:[7]

$$m(k) \in [1, k]. \tag{11}$$

(8), (9) and (11) together suggest that z_k^+ may often sensibly (but does not need to) be set as follows:

$$z_k^+ = k z_1^+ \tag{12}$$

where z_1^+ is an agreed or pre-specified maximum poverty line for households of one person.

We are now ready to state a first result (for ease of exposition, the proofs of the propositions appear in the Appendix).

[6] We can show that assumptions (A2) and (A3) together with $p_k(x) = 0$ $\forall x \geq z_k$ yield $\Xi_H^s \subset \Xi_H^{s-1}$, for $s = 2, 3, \ldots$

[7] The assumptions made on $m(k)$ here do not imply the assumption of concavity in k sometimes found in the literature.

Proposition 1

$\Delta P_{AB} \leq 0$ *for all P satisfying (A1), (A2), (A3) and for all poverty lines* z_k, $k = 1, \ldots, n$, *such that* $z_k \leq z_k^+$, *if and only if*

$$\text{(DS)} \qquad C_{Al}^s(x) \geq C_{Bl}^s(x), \quad \forall_x \leq z_l^+, \quad \forall l.$$

Recall that $C_{Al}^s(x)$ is the contribution to total poverty in A of its $n - l + 1$ neediest groups, and similarly for $C_{Bl}^s(x)$. When $s = 1$, condition DS is identical to the necessary condition developed by Jenkins and Lambert (1993), and it is also similar to the one found in Atkinson (1992). Note here that we show in the appendix DS to be both necessary and sufficient for s-order sequential stochastic dominance, as was shown by Chambaz and Maurin (1998) for second-order dominance. Note also that condition DS orders A and B for a potentially very wide set of implicit equivalence scales, including scales that may be income dependent (in the manner of Aaberge and Melby, 1998). The only restrictions on equivalence scales imposed in our analysis flow from the ordering of poverty lines in (7), and from the ordering of the effects of (generalized) income transfers on poverty indices in (A3).

When graphed against x, condition DS for a given l shows the difference between A and B in the absolute contribution of the cumulative groups l to n to total household poverty, when the contribution is measured by the well-known additive decomposition of the FGT indices. Hence, to assess whether A has robustly more poverty than B, it is sufficient to determine whether the contribution of groups of larger households to overall household poverty is always larger in A than in B. For a robust ordering, this condition must be satisfied whatever the cumulative groups considered ($l = 1, 2, \ldots, n$) and for all $z_l \leq z_l^+$.

Note that the continuity assumption (A2) is important in validating the sole use of an s-order test in DS. In a framework with homogenous households, Zheng (1999) shows that if we allow discontinuity of derivatives at the poverty line and do not assume a minimum value for the ranges of poverty lines, the conditions for higher-order stochastic dominance will typically include those for lower-order dominance. This is also valid for sequential stochastic dominance. More generally, if there is some lower-order discontinuity of the functions $p_k(z)$ over some interval of z, then a lower-order test appears over that interval. Whether such discontinuities really do (or should) exist is ultimately a matter of subjective normative taste—this paper focusses on necessary and sufficient dominance conditions for the cases when it is assumed that they do not occur.[8]

2.2. *Individual-aggregation Dominance*

The final assumption for individual-based poverty indices is that

$$\text{(A6)} \qquad (-1)^s \pi_1^{(s)}(y) \geq (-1)^s \pi_2^{(s)}(y) \geq \ldots \geq (-1)^s \pi_n^{(s)}(y) \geq 0.$$

[8]As for unidimensional poverty dominance, "dual" or "p-" conditions for sequential poverty dominance could probably also be derived for second-order and third-order dominance, using Cumulative Poverty Gap or "TIP" curves (see Jenkins and Lambert, 1997). For third-order poverty dominance, in addition to verifying which TIP curve crosses from below, these dual conditions would probably involve comparing means and variances when TIP curves do cross. Sufficient dual conditions for third-order sequential *welfare* dominance are shown in Lambert and Ramos (2000) (theorem 2).

The normative interpretation of those assumptions is similar to the interpretation of (A3) in the H approach, as is the discussion of the properties of the poverty indices which obey (A6). Together, assumptions (A4), (A5) and (A6) define the classes Ξ_I^s of individual-based poverty indices.

Assumptions (A5) and (A6) imply that the individual poverty lines z_k can be ranked as follows:

$$a > z_1 \geq z_2 \geq z_3 \geq \ldots \geq z_n. \tag{13}$$

It also follows that

$$a > z_1^+ \geq z_2^+ \geq z_3^+ \geq \ldots \geq z_n^+. \tag{14}$$

where z_k^+ is the maximum possible poverty line for someone in a household of type k.

An implicit equivalence scale $m(k)$ (to transform the total income of a household of k individuals into an equivalent income for a one-person household) is now given by:

$$m(k) = k \cdot z_k / z_1, \quad \text{with } m(1) = 1. \tag{15}$$

Sensible bounds for $m(k)$ are again given by (11), and by (15) the maximum poverty line for individuals in household type k can typically be set to:

$$z_k^+ = z_1^+. \tag{16}$$

We can then state a second general result.

Proposition 2

$\Delta\Pi_{AB} \leq 0$ *for all poverty indices* Π *satisfying (A4), (A5) and (A6) and for all poverty lines* z_k*,* $k = 1, \ldots, n$*, such that* $z_k \leq z_k^+$*, if and only if*

$$\Gamma_{Al}^s(y) \geq \Gamma_{Bl}^s(y), \quad \forall y \leq z_l^+, \quad \forall l. \tag{DIS}$$

2.3. *Does the Choice of Aggregating Units Matter?*

Although the results of the I approach may at first sight appear similar to the H approach, they in fact differ significantly and, more importantly, they do not necessarily generate the same poverty orderings. Three reasons account for this.

First, approach I counts individuals rather that households and thus gives a higher statistical weight to members of larger households. Second, assumptions (A3) and (A6) on the rankings of the successive derivatives reverse the ordering of "needs." This can have immediate effects if, for example, in approach I dominance for individuals in households of one person initially fails, or alternatively if in the H approach dominance for individuals in households of n persons initially fails.

A numerical example illustrates this difference. Suppose that a distribution is made up of ten households. Five of these households are made of couples whereas the five others are composed of single people. In A, the incomes of the single people are {2, 2, 3, 4, 5} and the total incomes of the couples are {2, 2, 3, 4, 5}. In B, the incomes of the single people are given by {2, 2, 2, 4, 5} and those of the couples are {4, 4, 5, 5, 6}. Comparing A and B, we see that poverty increases for single

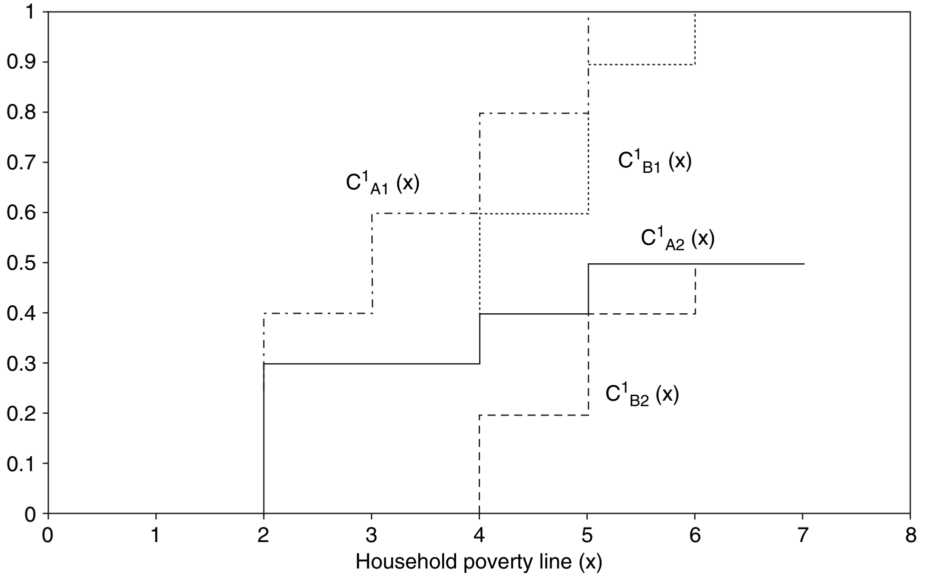

Figure 1. Sequential First-Order Dominance (Distribution B dominates Distribution A)

people and decreases for couples (for a poverty line of between 2 and 3 for instance). How, however, is overall poverty affected by the movement of *A* to *B*? Consider first the *H* approach for first-order dominance (condition D1 with $n = 2$). The first-order cumulative dominance curves, $C^1_{Al}(x)$ and $C^1_{Bl}(x)$, are shown in Figure 1. We note that *B* dominates *A*, since $C^s_{A2}(x) \geq C^s_{B2}(x)$ and $C^s_{A1}(x) \geq C^s_{B1}(x)$ for all x. Although at $x = 2$ poverty among single people is larger in *A* than in *B*, this is outweighted by the difference in poverty among couples (the needier group), that is,

(17) $$\frac{1}{2}D^1_{A1}(2) - \frac{1}{2}D^1_{B1}(2) = \frac{2}{10} - \frac{3}{10} = \frac{-1}{10}$$

is offset by

(18) $$\frac{1}{2}D^1_{A2}(2) - \frac{1}{2}D^1_{B2}(2) = \frac{2}{10} - \frac{0}{10} = \frac{2}{10}$$

and thus

(19) $$C^1_{A1}(2) - C^1_{B1}(2) = \frac{2}{10} + \frac{-1}{10} = \frac{1}{10} \geq 0.$$

We can thus infer robust poverty reduction when moving from *A* to *B*.

Consider now approach *I* to aggregating poverty. Condition DI1 says that it is necessary first to find dominance for single individuals. The increase in poverty for single persons (see (17)) can then not be compensated by the reduction in poverty for couples (see (18)). Within such a framework, we cannot infer robustness of overall poverty reduction when moving from *A* to *B*.

The third difference is that the income levels at which households and individuals are compared are not the same for approaches *H* and *I*. Approach *H* checks

$$(20) \qquad \sum_{k=l}^{n} [\theta_{Ak} D_{Ak}^{s}(x) - \theta_{Bk} D_{Bk}^{s}(x)] \geq 0,$$

by using total household income x whatever the type of households. Approach *I* checks

$$(21) \qquad \sum_{k=l}^{l} [\gamma_{Ak} G_{Ak}^{s}(y) - \gamma_{Bk} G_{Bk}^{s}(y)] \geq 0,$$

by using *per capita* income y regardless of household size. To see why this can matter, suppose distributions A and B are each made up of fifteen households. Five of these households are couples with a child, five others are couples without children, whereas the last five are single people. In A, each type of households has an identical distribution of income which is $\{2, 2, 3, 4, 5\}$. In B, singles and couples with a child have incomes $\{3, 3, 3, 4, 5\}$ while those of the couples without children are $\{2, 2, 2, 4, 5\}$. We thus note a poverty reduction for singles and couples with a child but a poverty increase for couples without children. How is overall poverty affected by the movement of A to B? On the one hand, with approach H, we note that at $x = 2$ the gain for couples with children,

$$(22) \qquad \frac{1}{3} D_{A3}^{1}(2) - \frac{1}{3} D_{B3}^{1}(2) = \frac{2}{15} - \frac{0}{15} = \frac{2}{15},$$

compensates for the loss for couples without children,

$$(23) \qquad \frac{1}{3} D_{A2}^{1}(2) - \frac{1}{3} D_{B2}^{1}(2) = \frac{2}{15} - \frac{3}{15} = \frac{-1}{15}.$$

Thus, we may infer dominance of B over A. On the other hand, with approach I, condition DI1 stipulates that the loss for individuals within couples without children must be compensated by the gain for the single people. Here, at $x = 2$ the gain of 1/15 for single people would indeed compensate for the loss of −1/15 for the group of couples without children if the comparisons were done at total household income x. With DIS, however, comparisons are rather made at common levels of *per capita* incomes, and the loss of 1/15 for couples without children is at a *per capita* income of \$1. Because there is no compensating gain at this *per capita* income for singles, we cannot conclude to a robust decrease in poverty with the *I* approach.

Hence, the conditions and results of the *I* and *H* approaches may clash with each other. It can also be argued that they should be used as complementary rather than as alternative conditions. We may not be certain, indeed, whether one approach is unambiguously preferable to the other. In such cases, both conditions should be tested simultaneously in establishing robust sequential poverty orderings. This will increase the robustness—but also decrease the completeness—of the resulting poverty orderings.

3. Assessing the Robustness of Poverty Orderings

When conditions DS or DIS are met, we may confidently assert that poverty in B is no greater than poverty in A under the conditions stated in the two propositions above. As indicated earlier, it may then be useful to consider z_1^+ as an upper bound for the poverty line of single persons, and to interpret the ratios of z_k^+/z_1^+ or kz_k^+/z_1^+ as indicative of the bounds of the ranges of equivalence scales over which the poverty robustness result holds.

An arguably better empirical strategy than testing dominance under an *a priori* set of somewhat arbitrary measurement assumptions is to estimate the maximum ranges of these assumptions under which there is dominance of one distribution over another. This is analogous to the "inverse optimum problem" of Ahmad and Stern (1984, 1991), which consists of estimating the preferences of a social planner that would lead to a given economic policy. This alternative strategy is particularly useful when the test of conditions DS or DIS has failed and we therefore cannot infer a robust poverty ordering over an initially specified set of poverty indices and poverty line upper bounds z_k^+.

In implementing this alternative empirical strategy, three alternative routes can be followed. The first route increases the order of stochastic dominance until a poverty ordering becomes robust over all of some pre-specified ranges of poverty lines. The second route infers (for a given s) critical bounds for restricted intervals of poverty lines for single persons while maintaining the ranges of the ratios of z_k^+/z_1^+ (and thus some given ranges of equivalence scales). Finally, the third route determines the critical ratios of z_k^+/z_1^+ up to which a poverty ordering is robust for a given poverty line upper bound z_1^+ for single people and for a pre-specified order of stochastic dominance s.

A reasonable empirical strategy then runs as follows.[9] In order to determine if poverty is unambiguously higher in A than in B, we can initially test for first-order sequential stochastic dominance using condition D1 or DI1. In that first step, we may set the z_k^+ as we wish (so long as (8), but, as mentioned above, sensible bounds are arguably given by equations (12) and (16)). If conditions D1 or DI1 does not lead to an unambiguous ranking, we conclude that it is not possible to obtain a poverty ordering of the two distributions which is robust to all members of Ξ_H^1 or Ξ_I^1 and over the pre-specified ranges of poverty lines determined by $z_1^+, z_2^+, z_3^+, \ldots$.

A second step then tests successively conditions D2, D3, . . . or DI1, DI2, . . . to determine if there is a smaller class Ξ_H^s or Ξ_I^s of poverty indices for which the ranking of these two distributions is robust. Alternatively (or simultaneously), we may choose not to restrict unduly the class of poverty indices for which the ranking of the two distributions will eventually be robust. Instead, we may prefer to restrict the interval of admissible poverty lines for single persons, while keeping constant the "reasonable" upper bounds for the equivalence scales (using restrictions (12) and (16)). Hence, for stochastic dominance of order s, we can search for a critical z_1^+ such that z_1^+ is the maximum value of ξ which obeys the condition

[9] As should become clear, this strategy can be followed either with the H or with the I aggregating approach.

$$(24) \qquad C^s_{Al}(x) \geq C^s_{Bl}(x), \quad \forall x \leq l\xi, \quad \forall l,$$

for the H aggregating approach, and the condition

$$(25) \qquad \Gamma^s_{Al}(y) \geq \Gamma^s_{Bl}(y), \quad \forall y \leq \xi, \quad \forall l,$$

for the I aggregating approach.

Finally, we may choose to restrict the intervals of the implicit admissible equivalence scales $m(k)$. This will limit the values of $z_2^+, z_3^+, \ldots$ without necessarily reducing z_1^+.[10] To follow this third route, we find estimates z_k^s, $k = 1, \ldots, n$, of the maximum poverty line z_k^+ for which a stochastic dominance condition of order s, DS or DIS, allows a robust ordinal ranking of poverty. With the H approach to aggregation, we find, in a first step, the values of $\hat{z}_l^s$, $l = 1, \ldots n$, such that $\hat{z}_l^s$ is the maximum value of ξ which respects the condition

$$(26) \qquad C^s_{Al}(x) \geq C^s_{Bl}(x), \quad \forall x \leq \xi.$$

This gives us a set of upper poverty line bounds $\hat{z}_1^s, \hat{z}_2^s, \ldots, \hat{z}_n^s$, that may or may not obey the assumptions made on the rankings of the z_k^+. To ensure that $z_k^s \leq z_{k+1}^s$ and that $m(k) \leq m(k+1)$, we proceed by iteration, first by defining $z_n^s = \hat{z}_n^s$, and then by setting the remaining z_k^s as follows:

$$(27) \qquad z_k^s = \min(\hat{z}_k, z_{k+1}^s), \quad \text{for } k = n-1, \ldots, 1.$$

Interpreting z_1^s as the robust upper bound for the poverty line of a single person, we may then use the estimated vector $z^s = (z_1^s, z_2^s, \ldots, z_n^s)$ to estimate the sets of equivalence scales for which a poverty ranking is robust at order s. This "critical" set of equivalence scales is given by $m(k) \in [1, z_k^s/z_1^s]$ with the additional conditions that $m(k) \leq m(k+1)$, for $k = 1, \ldots, n-1$. If we also wish (although we do not need) to ensure that $m(k) \leq k$, then we simply use instead the sets $m(k) \in [1, \min(k, z_k^s/z_1^s)]$. Finally, if a maximum bound z_1^+ for the range of poverty lines for single people were to be agreed a priori, and if it were the case that $z_1^+ < z_1^s$, the robust set of equivalence scales could be further extended, simply by using $[1, \min(k, z_k^s/z_1^+)]$ instead of $[1, \min(k, z_k^s/z_1^s)]$.

To follow instead the I approach to aggregation, we mostly proceed as above by searching for estimates of z_k^s, $k = 1, \ldots, n$, of the maximum poverty line z_k^+ for

[10]The method used here is analogous in spirit to that proposed by Lanjouw and Ravallion (1995). They wish to determine if large households (with n individuals) experience more poverty than smaller ones (for example, single people). By using the parametric equivalence scale $m(k) = k^\sigma$ proposed by Buhmann *et al.* (1987), they then estimate the maximum elasticity σ for which the classification is robust when one uses either a given poverty index or some of the stochastic dominance conditions stated in Atkinson (1987). Our approach is significantly different since it makes it possible to compare two distributions for which household types can vary within each distribution. Moreover, we impose smaller restrictions on the shape of the sets of allowable equivalence scales since these are not restricted to belong to a particular parametric class of equivalence scales.

In the same spirit, Bradbury (1997) determines the intervals of equivalence scales for which comparisons of poverty are robust. Although not using a parametric form for the sets of equivalence scales, his method imposes an assumption of concavity on the function $m(k)$ (an assumption that is not needed here). Bradbury uses the non-sequential dominance condition developed by Atkinson (1987) combined with a numerical algorithm to find the bounds of the intervals of admissible equivalence scales. The greater empirical simplicity of our approach comes from our use of the sequential stochastic dominance technique. See also Fleurbaey *et al.* (1998), who propose sequential second-order dominance tests that assume lower and higher bounds for equivalence scale intervals.

which a poverty ordering is robust. We find, in a first step, the values of $\hat{z}_l^s$, $l = 1, \ldots n$, such that $\hat{z}_l^s$ are the maximum values of ξ which respect the condition

$$\Gamma_{Al}^s(y) \geq \Gamma_{Bl}^s(y), \quad \forall y \leq \xi. \tag{28}$$

Again, this gives us a set of upper poverty line thresholds $\hat{z}_l^s$, $\hat{z}, \ldots, \hat{z}_n^s$, that may or may not obey the assumptions made on the rankings of the z_k^+ for the I approach and on the (sensible) ranges of equivalence scales. To ensure that $kz_k^s \leq (k+1)z_{k+1}^+$ and thus that $m(k) \leq m(k+1)$, we proceed by iteration, first by defining $z_n^s = \hat{z}_n^s$, and then by setting the remaining z_k^s as follows:

$$z_k^s = \min\left(\hat{z}_k^s, \frac{(k+1)}{k} z_{k+1}^s\right) \tag{29}$$

Just as before, we may use the vector $z^s = (z_1^s, z_2^s, \ldots, z_n^s)$ to determine the set of equivalence scales for which the ranking of poverty between two distribution is robust at dominance order s. This set is such that $m(k) \in [1, kz_k^s/z_1^s$ with the additional condition that $m(k) \leq m(k+1)$, for $k = 1, \ldots, n-1$. If we also wish to ensure that $m(k) \leq k$, then we may use instead the sets $m(k) \in [1, \min(k, kz_k^s/z_1^s)]$. Moreover, if we were to agree that the maximum poverty line for single people could not exceed z_1^+, with $z_1^+ < z_1^s$, then we could do as above and extend the robust set of equivalence scales by using $[1, \min(k, kz_k^s/z_1^+)]$ instead of $[1, \min(k, kz_k^s/z_1^s)]$.

For each of the two H and I approaches, if we feel the resulting set of robust equivalence scales is too limited at order s, we can proceed to higher order $s + 1$. If there were sequential dominance for some limited ranges of z_k (for all k) at an order s, then, for a given z_1^+, the sets of robust equivalence scales will necessarily become larger and larger as s increases (this follows from Lemma 1 of Davidson and Duclos, 2000).

4. Illustration

We now illustrate the previous methodological results using data on Canada, Finland, Italy and USA drawn from the Luxembourg Income Study (LIS) data sets. These countries were selected partly because 1991 data were available for them. We take household income to be disposable income (i.e. post-tax and transfer income) and we apply purchasing power parities drawn from the Penn World Tables[11] to convert national currencies into 1991 US dollars. We subdivide each population into six different types of households according to the number of people composing the households. We consider households of six or more individuals as part of the same category. All household observations are weighted by the LIS sample weights “hweight.” Finally, negative incomes are set to 0.

Table 2 shows average household incomes by household type for each country. American households of one, two and three people have the highest average

[11]We have used the price level over the GDP as PPP. Another possible choice would have been to choose the price level of consumption and this may have led to different results, given the well-known sensibility of international comparisons of living standards to such statistics. See Summers and Heston (1991) for the methodology underlying the computation of the parties we use, and http://www.nber.org/pwt56.html for access to the 1991 figures.

TABLE 2

Average Household Incomes by Household Type in Each Country (1991 US$)

	Canada	U.S.	Finland	Italy
1 person	14,806	15,406	11,158	11,387
2 persons	28,023	29,691	22,620	17,822
3 persons	33,659	33,993	28,673	24,021
4 persons	37,388	36,346	32,626	25,481
5 persons	40,020	36,385	33,675	26,404
6 persons or more	41,033	37,042	34,958	26,023

TABLE 3

Proportion of Each Household Type in Each Country

	Canada	U.S.	Finland	Italy
1 person	30.1%	29.1%	36.5%	16.1%
2 persons	27.9%	29.6%	30.4%	22.9%
3 persons	16.0%	16.7%	14.1%	24.3%
4 persons	16.3%	15.0%	13.0%	25.2%
5 persons	6.9%	6.1%	4.6%	8.0%
6 or more persons	2.8%	3.5%	1.4%	3.5%

TABLE 4

Critical Thresholds $\hat{z}_k^1$ Under which Each Country Dominates (i.e. has Less Poverty than) the U.S. when Aggregating Households (1991 US$)

	Canada	Finland	Italy
$\hat{z}_1^1$	42,674	7,242	9,496
$\hat{z}_2^1$	–	20,878	*X*
$\hat{z}_3^1$	55,156	29,739	*X*
$\hat{z}_4^1$	50,523	33,830	*X*
$\hat{z}_5^1$	84,428	–	*X*
$\hat{z}_6^1$	*X*	–	5,935

incomes while Canada has the highest average income for the larger households. Single people have the lowest average income in Finland. For the other types of households, it is in Italy where average income is lowest.

Table 3 shows the proportion of each household type in each country. The highest proportions of single people and households of two people are found in Finland. Italy, however, has a higher proportion of larger households.

In a first step, poverty in each one of these countries is compared with poverty in the U.S. Table 4 gives the first-order dominance thresholds $\hat{z}_k^1$ for which each country dominates (has less poverty than) the U.S. when poverty aggregation uses households. A horizontal bar (–) indicates that $\hat{z}_k^1$ tends towards infinity and an X indicates that the country is initially dominated by (i.e. has more poverty than) the U.S. when this household type is introduced. Table 4 shows that, although Canada clearly dominates the U.S. for households of more than one to five people, the U.S. initially dominates Canada for households of six people or more. Thus, we cannot

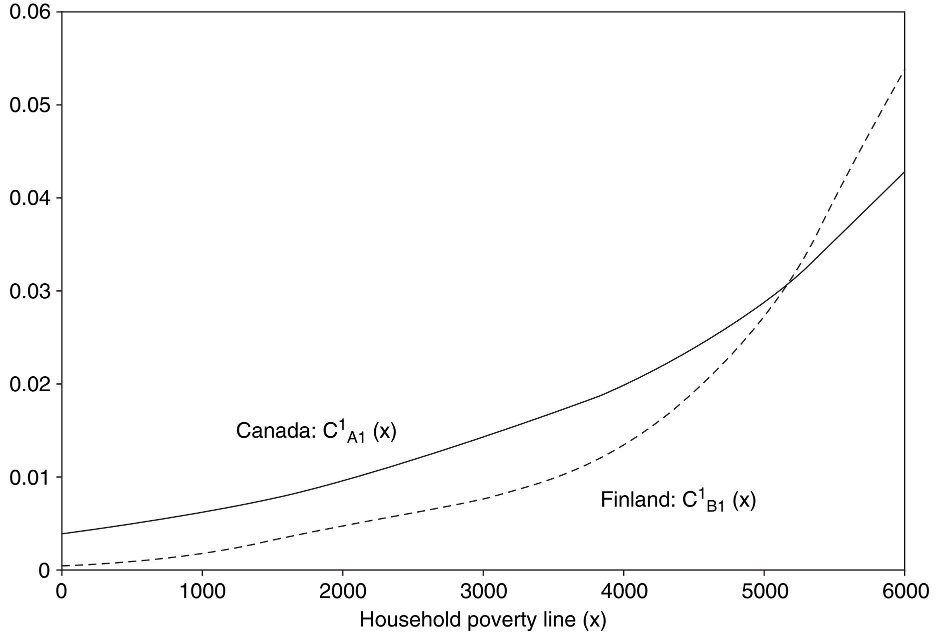

Figure 2. Poverty Headcounts in Canada (A) and Finland (B)

use sequential stochastic dominance (of and order[12]) to rank poverty robustly between these two countries. Italy cannot be ranked with the U.S. for the same reason. Finally, we can affirm that poverty is lower in Finland than in the U.S. for every poverty line lower than \$7242 for households of one individual, for all equivalence scales such that $m(k) \in [1, k]$ and $m(k) \leq m(k+1)$, for $k = 1, \ldots, n-1$, and for every of the poverty indices belonging to class Ξ_H^1.

We now compare poverty in Finland with poverty in Canada to illustrate our methodology in greater details. For low z_k^+, the data first indicate that Finland (country B) dominates Canada (country A) for all relevant cumulative household types. Figures 2 to 4 illustrate this through the curves $C_{Al}^1(x)$ and $C_{Bl}^1(x)$ for $l = 1$ to 3. As noted above, these curves show the contribution of cumulative groups to total poverty (as measured here by the headcount index). The difference between $C_{A1}^1(x)$ (Figure 2) and $C_{A2}^1(x)$ (Figure 3) then gives the contribution of households of 1 person to total poverty in A. Condition D1 says that as long as curve $C_{Bl}^1(x)$ is located under curve $C_{Al}^1(x)$, Finland (B) dominates Canada (A). The threshold values for the poverty lines $\hat{z}_l^1$ are determined by the intersection of the two curves $C_{Al}^1(x)$ and $C_{Bl}^1(x)$.

Table 5 shows estimated thresholds $\hat{z}_k^s$ for sequential stochastic dominance tests of order 1, 2, 3 and 4. To be able to say that poverty in Finland is lower than

[12]Lemma 1 of Davidson and Duclos (2000) shows that when a distribution dominates another distribution over a range of incomes [0, d] for an order of dominance s, it will eventually dominate the other distribution for arbitrarily large d as s increases to infinity. Hence, as s is increased, Canada will keep dominating the U.S. for households of more than one to five people, but the U.S. will keep dominating Canada for households of six people or more.

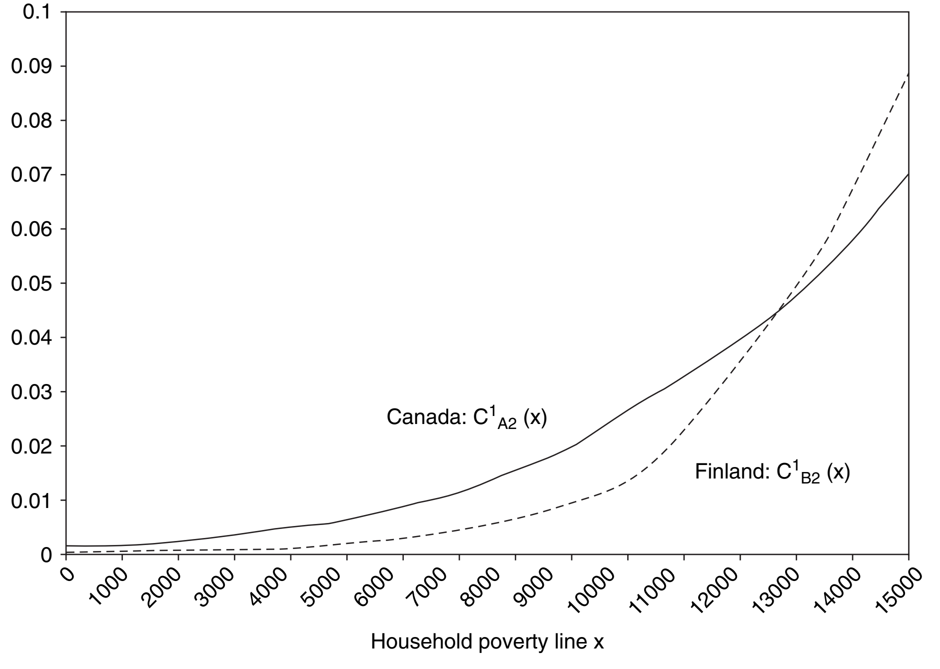

Figure 3. The Contribution of Households of 2 Persons and More to the Poverty Headcount (Canada = A, Finland = B)

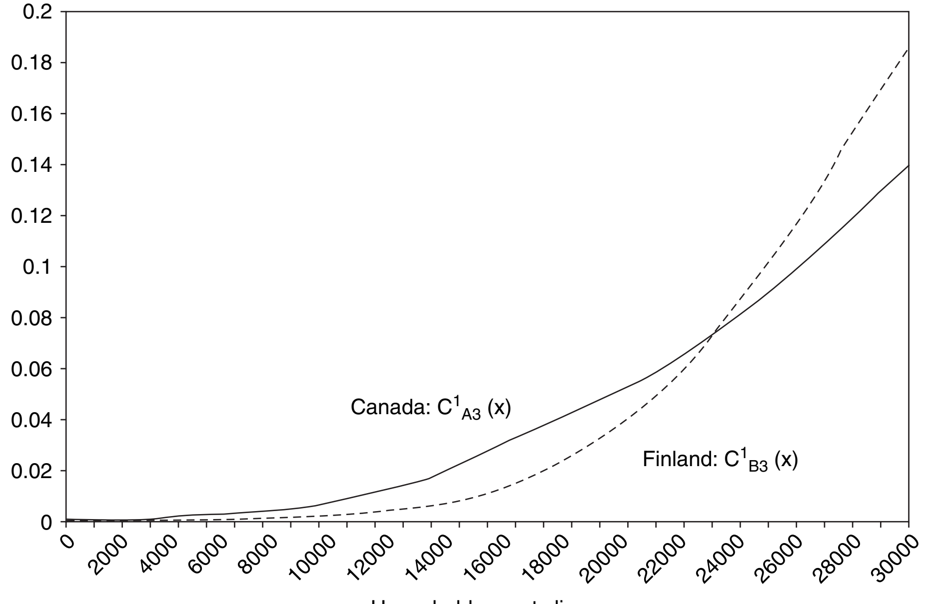

Figure 4. The Contribution of Households of 3 Persons and More to the Poverty Headcount (Canada = A, Finland = B)

TABLE 5

CRITICAL THRESHOLDS $\hat{z}_k^s$ UNDER WHICH FINLAND DOMINATES CANADA WHEN AGGREGATING HOUSEHOLDS (1991 US$)

	$s = 1$	$s = 2$	$s = 3$	$s = 4$
$\hat{z}_1^s$	5,337	7,058	9,043	11,198
$\hat{z}_2^s$	12,841	17,244	22,309	27,442
$\hat{z}_3^s$	22,829	30,509	39,739	50,501
$\hat{z}_4^s$	26,145	34,493	48,311	–
$\hat{z}_5^s$	8,920	–	–	–
$\hat{z}_6^s$	–	–	–	–

TABLE 6

CRITICAL RANGES OF EQUIVALENCE SCALES $[1, z_k^s/z_1^\dagger]$ FOR WHICH FINLAND DOMINATES CANADA WHEN AGGREGATION IS OVER HOUSEHOLDS

	$z_1^\dagger = 3000$	$z_1^\dagger = 4000$	$z_1^\dagger = 5000$
m (2)	[1, 2.97]	[1, 2.23]	[1, 1.78]
m (3)	[1, 2.97]	[1, 2.23]	[1, 1.78]
m (4)	[1, 2.97]	[1, 2.23]	[1, 1.78]
m (5)	[1, 2.97]	[1, 2.23]	[1, 1.78]
m (6)	[1, ∞]	[1, ∞]	[1, ∞]

in Canada for any of the poverty indices belonging to class Ξ_H^1 and for all equivalence scales consistent with (10), we should fix $z_1^\dagger = \hat{z}_5^1/5 = \$1784$, which is probably too low a threshold to have confidence in the ordering. For classes Ξ_H^2, Ξ_H^3 and Ξ_H^4, we may fix $z_1^\dagger = \hat{z}_1^2 = \$7058$, $z_1^\dagger = \hat{z}_1^3 = \$9043$ and $z_1^\dagger = \hat{z}_1^4 = \$11198$ respectively, which are clearly more robust upper bounds.

To increase the robust upper poverty line bound for single persons derived from Table 5, we can also choose to restrict the class of equivalence scales over which the Canada–Finland poverty ordering is robust for Ξ_H^1 indices. For classes Ξ_H^2, Ξ_H^3 and Ξ_H^4, restricting the intervals of equivalence scales yields no gain in robustness since for orders of dominance 2, 3 and 4, $\hat{z}_k^s \geq k\hat{z}_1^s$ for all household types. Table 6 shows the intervals of equivalence scales $[1, z_k^s/z_1^\dagger]$ for which poverty comparisons are robust over class Ξ_H^1 and up to maximum poverty lines for a single person of \$3000, \$4000 and \$5000. Figure 5 illustrates the relationship between the upper bounds $z_k^s/z_1^\dagger$ of the intervals to which the equivalence scales $m(2)$, $m(3)$, and $m(4)$ must belong and the maximum poverty line for households of one individual when we restrict ourselves to indices in class Ξ_H^2. Table 6 and Figure 5 both show that the intervals of acceptable equivalence scales are increasingly restricted as the poverty line $z_1^\dagger$ for singles increases. Figure 6 also illustrates how the upper limit of the interval for $m(2)$ changes when the maximum poverty line for singles and the order of dominance vary. Clearly, the higher the order of dominance or the lower the upper bound $z_1^\dagger$, the larger the robust sets of equivalence scales.

Let us now consider the aggregation of poverty over individuals. Table 7 shows estimates of the thresholds $\hat{z}_k^s$ for dominance tests of order 1, 2, 3 and 4. To be able to say that poverty in Finland is lower than in Canada for any of the poverty indices belonging to class Ξ_I^1 and for any of the equivalence scales

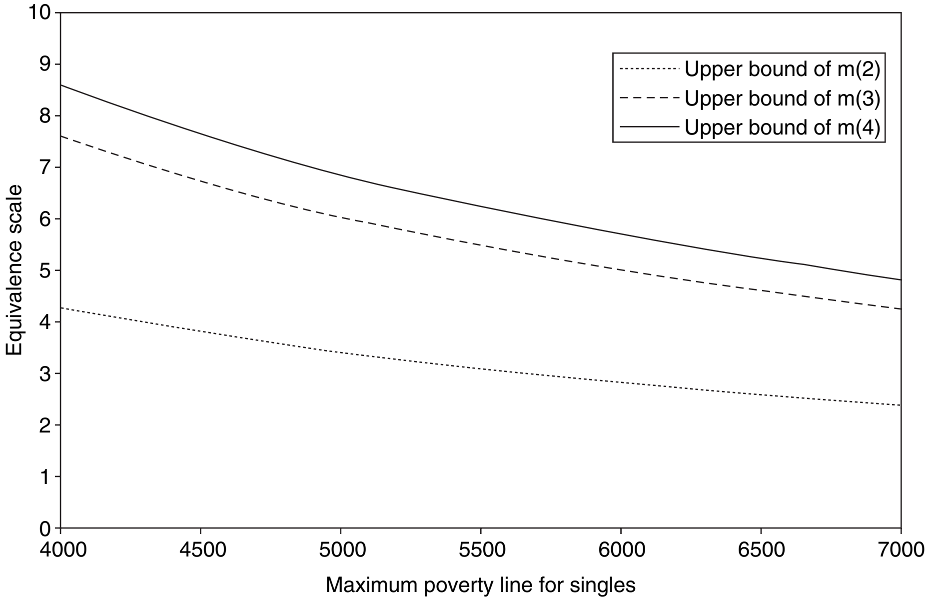

Figure 5. Equivalence Scales and Maximum Poverty Line for Singles for which Finland Dominates Canada (second-order dominance, household aggregation)

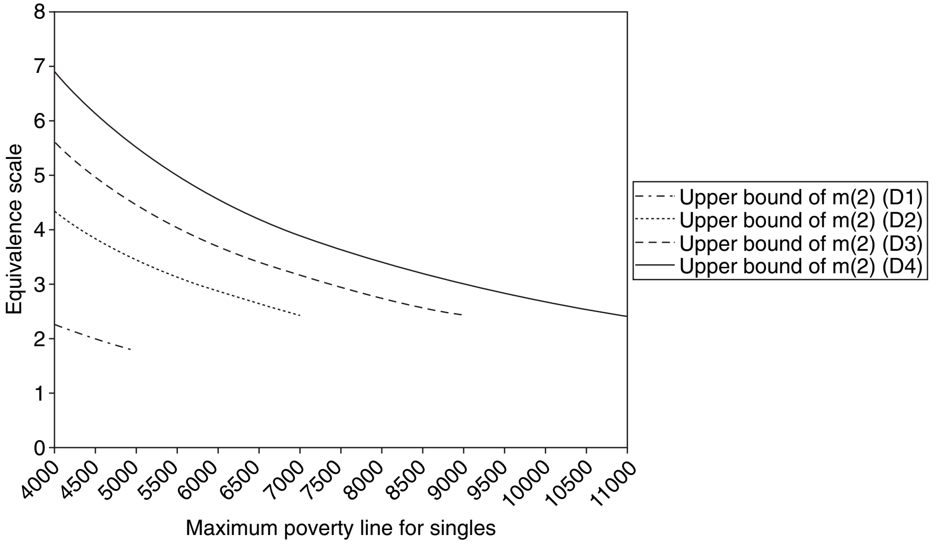

Figure 6. Equivalence Scales, Order of Dominance and Maximum Poverty Lines for Singles for which Finland Dominates Canada (household aggregation)

TABLE 7

CRITICAL THRESHOLDS $\hat{z}_k^s$ UNDER WHICH FINLAND DOMINATES CANADA WHEN AGGREGATING IS OVER INDIVIDUALS (1991 US$)

	$s = 1$	$s = 2$	$s = 3$	$s = 4$
$\hat{z}_1^s$	4,092	5,891	7,422	9,030
$\hat{z}_2^s$	5,197	6,389	7,885	9,488
$\hat{z}_3^s$	5,558	7,033	8,702	10,482
$\hat{z}_4^s$	5,548	7,117	8,820	10,662
$\hat{z}_5^s$	5,481	7,065	8,819	10,710
$\hat{z}_6^s$	5,513	7,190	9,060	11,088

satisfying (10), we should fix $z_1^+ = \hat{z}_1^1 = \$4092$. For classes Ξ_I^2, Ξ_I^3 and Ξ_I^4, it is necessary to fix $z_1^+ = \hat{z}_1^2 = \$5891$, $z_1^+ = \hat{z}_1^3 = \$7422$ and $z_1^+ = \hat{z}_1^4 = \$9030$ respectively. Since the poverty line upper bounds for households of one person are always lower than the poverty line upper bounds for the other types of households, restricting the intervals of equivalence scales is not useful here for increasing the maximum acceptable poverty line for single persons.

Comparing Tables 5 and 7, note also that the critical upper bound for the poverty lines of one-person households appears lower (whatever the order of dominance) when we use individuals rather than households as the aggregating units. Recall, however, that in Table 5 poverty lines are set for total household income, whereas they are set in terms of *per capita* income in Table 7. Moreover, we noted in the discussion of Table 5 above that for robustness over the class of equivalence scales defined by assumptions (10), we needed to set $z_1^+ = \$1784$. For the I approach to aggregation, z_1^+ can be set as high as $4092 for the same degree of equivalence scale robustness. Hence, it would appear that poverty is more robustly higher in Canada than in Finland when it is individuals rather than households that are the aggregating units. This also conforms with the results of Table 6 which indicated that it is in part the higher Canadian proportion of larger households which makes poverty higher than in Finland. Besides being ethically preferable, weighting households by their size in the aggregation exercise then reinforces the effect of this demographic difference.

5. CONCLUSION

This paper develops methods for testing whether ordinal poverty orderings are robust over large sets of equivalence scales, poverty lines and poverty indices. The methods rely on well-known criteria of first-order multidimensional stochastic dominance and extend them to any arbitrary order of dominance, to the alternative choice of individual or household based aggregation, and to the estimation of "critical sets" of measurement assumptions. The latter exercise provides estimates of critical bounds for the sets of equivalence scales and poverty lines over which poverty comparisons may be considered robust at a given order of dominance. These estimates are useful to show the trade-offs involved in delimiting the critical sets of poverty indices, poverty lines and equivalence scales over which robustness may be inferred. Generally speaking, the higher the order of dominance or the lower the upper bound for the poverty lines of single individuals, the larger the robust set of equivalence scales over which poverty orderings may be considered robust.

The theoretical results are illustrated using data for four countries drawn from the Luxembourg Income Study databases. The sequential dominance conditions fail to order Canada and the U.S., or Italy and the U.S., and this, whatever the selected order of dominance. We can confidently infer, however, that poverty is lower in Finland than in the U.S. for a wide range of poverty indices, poverty lines and equivalence scales. The comparison of Finland and Canada also serves to illustrate how the size of the robust intervals of equivalence scales is affected by the size of the intervals of admissible poverty lines for single persons and by the size of the class of admissible poverty indices. Poverty is also found to be more robustly higher in Canada than in Finland when it is individuals rather than households that are the aggregating units, highlighting the effect on poverty comparisons of demographic differences in household composition and of alternative approaches to assessing differential household needs.

Appendix

A. *Proof of Proposition 1*

To prove sufficiency, we first need to integrate by parts the integral for subgroup k in assumption (A1):

$$\int_0^a p_k(x)dF_{Ak}(x) = p_k(x)F_{Ak}(x)\Big|_0^a - \int_0^a p_k^{(1)}(x)F_{Ak}(x)dx. \tag{30}$$

We know that $F_{AK}(0) = 0$ and that $p_k(0)$ is finite. Also, $F_{AK}(a) = 1$ and, from (A2) and the definition of a, we know that $p_k(a) = 0$. The first term on the right-hand side of (30) is thus nil. Consequently, equation (30) may be rewritten as

$$\int_0^a p_k(x)dF_{Ak}(x) = -\int_0^a p_k^{(1)}(x)D_{Ak}^1(x)dx. \tag{31}$$

Now, assume that $s > 1$, and that for some $i \in \{2, 3, \ldots, s-1\}$ we have:

$$\int_0^a p_k(x)dF_{Ak}(x) = (-1)^{(i-1)}\int_0^a p_k^{(i-1)}(x)D_{Ak}^{(i-1)}(x)dx. \tag{32}$$

Integrating by parts equation (32), we get

$$\int_0^a p_k(x)dF_{Ak}(x) = (-1)^{(i-1)}\left\{p_k^{(i-1)}(x)D_{Ak}^i(x)\Big|_0^a - \int_0^a p_k^{(i)}(x)D_{Ak}^i(x)dx\right\}. \tag{33}$$

$p_k^{(i-1)}(0)$ is finite and $D_{AK}^i(0) = 0$. We have $p_k(x) = 0\ \forall x \geq z_k$ and we know, from assumption (A2), that $p_k(x) \in \hat{C}^i$. This means that $p^{(i-1)}(\mathrm{a}) = 0$. Finally, $D_{Ak}^i(a)$ is finite. We can rewrite this equation as

$$\int_0^a p_k(x)dF_{Ak}(x) = (-1)^i\int_0^a p_k^{(i)}(x)D_{Ak}^i(x)dx. \tag{34}$$

When $i = 2$, equation (32) is simply equation (31) and we have shown that if (32) is true, then (34) is also true. When $s > 1$, this implies that equation (34) is true for all integer $i \in \{2,3, \ldots, s-1\}$. But when $i = 0$, (34) is valid by definition, and when $i = 1$, (34) is just (31). Thus, (34) is true for all integers $i \in \{0, 1, 2, 3, \ldots, s-1\}$ and for any $s = 1, 2, \ldots$ Integrating by parts equation (34), for $i = s-1$, yields:

$$(35) \qquad \int_0^a p_k(x)dF_{Ak}(x) = (-1)^{(s-1)}\left\{p_k^{(s-1)}(x)D_{Ak}^s(x)\Big|_0^a - \int_0^a p_k^{(s)}(x)D_{Ak}^s(x)dx\right\},$$

We know that, by definition of the domain, $D_{Ak}^s(0) = 0$. Furthermore, we have $p_k(x) = 0\ \forall x \geq z_k$ and we know, from assumption (A2), that $p_k(x) \in C^s$. This means that $p_k^{(s-1)}(a) = 0$. The first term within the braces is thus nil. We then have:

$$(36) \qquad \int_0^a p_k(x)dF_{Ak}(x) = (-1)^{(s)}\int_0^a p_k^{(s)}(x)D_{Ak}^s(x)dx.$$

From equation (36) and equation (5), we get

$$(37) \qquad \Delta P_{AB} = (-1)^s \sum_{k=1}^{n}\left[\theta_{Bk}\int_0^a p_k^{(s)}(x)D_{Bk}^s(x)dx - \theta_{Ak}\int_0^a p_k^{(s)}(x)D_{Ak}^s(x)dx\right].$$

We can rewrite equation (37) as

$$(38) \qquad \Delta P_{AB} = (-1)^s \int_0^a \sum_{k=1}^{n} p_k^{(s)}(x)[\theta_{Bk}D_{Bk}^s(x) - \theta_{Ak}D_{Ak}^s(x)]dx.$$

Using (7), (A3) and Abel's Lemma,[13] it is sufficient for $\Delta P_{AB} \leq 0$ under the conditions of the proposition that $\Sigma_{k=l}^{n}[\theta_{Ak}D_{Ak}^s(x) - \theta_{Bk}D_{Bk}^s(x)] \geq 0\ \forall x \leq z_l^+,\ \forall l$.

To establish necessity, we use a set of functions $p_k(x)$, the $(s-1)$-th derivative of which is:

$$(39) \qquad p_k^{(s-1)}(x) = \begin{cases} (-1)^{s-1}\varepsilon & \text{if } x \leq \bar{x} \\ (-1)^{s-1}(\bar{x}+\varepsilon-x) & \text{if } \bar{x} < x \leq \bar{x}+\varepsilon, \\ 0 & x > \bar{x}+\varepsilon \end{cases} \quad \text{for all } k = 1, 2, \ldots, n.$$

Poverty indices whose functions $p_k(x)$ have the above form for $p_k^{(s-1)}(x)$ belong to the class Ξ_H^s. This yields:

$$(40) \qquad p_k^{(s)}(x) = \begin{cases} 0 & \text{if } x < \bar{x} \\ (-1)^s & \text{if } \bar{x} < x < \bar{x}+\varepsilon, \\ 0 & x > \bar{x}+\varepsilon \end{cases} \quad \text{for all } k = 1, 2, \ldots, n.$$

Imagine now that $\Sigma_{k=l}^{n}[\theta_{Ak}D_{Ak}^s(x) - \theta_{Bk}D_{Bk}^s(x)] < 0$ on an interval $[\bar{x}, \bar{x} + \in]$ for some l, for $\bar{x} < z_l^+$, and for ε that can be arbitrarily close to 0. For $p_k(x)$ defined as in (40), expression (38) is then positive and poverty increases with a movement from distribution A to distribution B. Hence it cannot be that $\Sigma_{k=l}^{n}[\theta_{Ak}D_{Ak}^s(x) - \theta_{Bk}D_{Bk}^s(x)] < 0$ for some l, $x \in [\bar{x}, \bar{x}+\varepsilon]$ when $\bar{x} < z_l^+$. This proves the necessity of the condition. ■

[13]Abel's Lemma is proved in Jenkins and Lambert (1993):

Abel's Lemma: *If* $x_n \geq x_{n-1} \geq \ldots \geq x_2 \geq x_1 \geq 0$, *a sufficient condition for* $\Sigma_{i=1}^{n} x_i y_i \geq 0$ *is* $\Sigma_{i=j}^{n} y_i \geq 0$ *for each j. If* $x_n \leq x_{n-1} \leq \ldots \leq x_2 \leq x_1 \leq 0$, *the same condition is sufficient for* $\Sigma_{i=1}^{n} x_i y_i \leq 0$.

B. *Proof of Proposition 2*

To prove sufficiency we first need to integrate by parts the integral for subgroup k in Assumption (A4):

$$\int_0^a \pi_k(y)d\Phi_{Ak}(y) = \pi_k(y)\Phi_{Ak}(y)\Big|_0^a - \int_0^a \pi_k^{(1)}(y)\Phi_{Ak}(y)dy. \tag{41}$$

We know that $\Phi_{Ak}(0) = 0$ and $\pi_k(0) > 0$ is finite. Also, $\Phi_{Ak}(0) = 1$ and, from assumption (A5) and the definition of a, we know that $\pi_k(a) = 0$. The first term on the right side of (41) is thus nil. Consequently, equation (41) may be rewritten as

$$\int_0^a \pi_k(y)d\Phi_{Ak}(y) = -\int_0^a \pi_k^{(1)}(y)G_{Ak}^1(y)dy. \tag{42}$$

Now assume that $s > 1$, and that for some $i \in \{2, 3, \ldots, s-1\}$, we have:

$$\int_0^a \pi_k(y)d\Phi_{Ak}(y) = (-1)^{(i-1)}\int_0^a \pi_k^{(i-1)}(y)G_{Ak}^{(i-1)}(y)dy. \tag{43}$$

Integrating by parts equation (43), we get

$$\int_0^a \pi_k(y)d\Phi_{Ak}(y) = (-1)^{(i-1)}\left\{\pi_k^{(i-1)}(y)G_{Ak}^i(y)\Big|_0^a - \int_0^a \pi_k^{(i)}(y)G_{Ak}^i(y)dy\right\}. \tag{44}$$

$\pi_k^{(i-1)}(0)$ is finite and $G_{Ak}^i(0) = 0$. We have $\pi_k(y) = 0\ \forall y \geq z_k$ and we know, from assumption (A5), that $\pi_k(y) \in \hat{C}^i$. This means that, $\pi_k^{(i-1)}$ (a) = 0. Finally, G_{Ak}^i (a) is finite. We can rewrite this equation as

$$\int_0^a \pi_k(y)d\Phi_{Ak}(y) = (-1)^s\int_0^a \pi_k^{(i)}(y)G_{Ak}^i(y)dy. \tag{45}$$

When $i = 2$, equation (43) is simply equation (42) and we have shown that if (43) is true, then (45) is also true. When $s > 1$, this implies that equation (45) is true for all integer $i \in \{2, 3, \ldots, s-1\}$. But, when $i = 0$, (45) is valid by definition, and when $i = 1$, (45) is just (42). Thus, (45) is true for all integers $i \in \{0, 1, 2, 3, \ldots, s-1\}$ and for any $s = 1, 2, \ldots$ Integrating by parts equation (34), for $i = s - 1$, yields:

$$\int_0^a \pi_k(y)d\Phi_{Ak}(y) = (-1)^{(s-1)}\left\{\pi_k^{(s-1)}(y)G_{Ak}^s(y)\Big|_0^a - \int_0^a \pi_k^{(s)}(y)G_{Ak}^s(y)dy\right\}, \tag{46}$$

We know that, by definition of the domain, $G_{Ak}^s(0) = 0$. Furthermore, we have $\pi_k(y) = 0\ \forall y \geq z_k$ and we know, from assumption (A5), that $\pi_k(y) \in \hat{C}^s$. This means that $\pi_k^{(s-1)}(a) = 0$. The first term in the braces is thus nil. We then have:

$$\int_0^a \pi_k(y)d\Phi_{Ak}(y) = (-1)^{(s)}\int_0^a \pi_k^{(s)}(y)G_{Ak}^s(y)dy. \tag{47}$$

From equation (47) and equation (6), we get

$$\Delta P_{AB} = (-1)^s \sum_{k=1}^{n}\left[\gamma_{Bk}\int_0^a \pi_k^{(s)}(y)G_{Bk}^s(y)dy - \gamma_{Ak}\int_0^a \pi_k^{(s)}(y)G_{Ak}^s(y)dy\right]. \tag{48}$$

We can rewrite equation (48) as

$$\Delta P_{AB} = (-1)^s \int_0^a \sum_{k=1}^{n} \pi_k^{(s)}(y)[\gamma_{Bk} G_{Bk}^s(y) - \gamma_{Ak} G_{Ak}^s(y)]dy. \tag{49}$$

Using (13), (A6) and Abel's Lemma, it is sufficient for $\Delta P_{AB} \leq 0$ that $\Sigma_{k=1}^{l} [\gamma_{Ak} G_{Ak}^s(y) - \gamma_{Bk} G_{Bk}^s(y)] \geq 0 \ \forall y \leq z_l^+, \forall l$.

For necessity, use a set of functions $\pi_k(y)$, the $(s-1)$-th derivative of which is:

$$\pi_k^{(s-1)}(y) = \begin{cases} (-1)^{s-1}\varepsilon & \text{if } y \leq \bar{y} \\ (-1)^{s-1}(\bar{y} + \varepsilon - y) & \text{if } \bar{y} < y \leq \bar{y} + \varepsilon, \\ 0 & y > \bar{y} + \varepsilon \end{cases} \quad \text{for all } k = 1, 2, \ldots, n. \tag{50}$$

Poverty indices whose function $\pi_k(y)$ have the above form for $\pi_k^{(s-1)}(y)$ belong to the class Ξ_l^s. This yields:

$$\pi_k^{(s)}(y) = \begin{cases} 0 & \text{if } y < \bar{y} \\ (-1)^s & \text{if } \bar{y} < y < \bar{y} + \varepsilon, \\ 0 & y > \bar{y} + \varepsilon \end{cases} \quad \text{for all } k = 1, 2, \ldots, n. \tag{51}$$

Imagine now that $\Sigma_{k=1}^{l} [\gamma_{Ak} G_{Ak}^s(y) - \gamma_{Bk} G_{Bk}^s(y)] < 0$ on an interval $[\bar{y}, \bar{y} + \varepsilon]$ for some l, for $\bar{y} < z_l^+$, and for ε that can be arbitrarily close to 0. For $\pi_k(y)$ defined as in (51), expression (49) is then positive and poverty increases with a movement from distribution A to distribution B. Hence it cannot be that $\Sigma_{k=1}^{l} [\gamma_{Ak} G_{Ak}^s(y) - \gamma_{Bk} G_{Bk}^s(y)] < 0$ for some l, $y \in [\bar{y}, \bar{y} + \varepsilon]$ when $\bar{y} < z_l^+$. This concludes the proof of Proposition 2. ■

References

Aaberge, R. and I. Melby, "The Sensitivity of Income Inequality to Choice of Equivalence Scales," *Review of Income and Wealth*, 44, 565–70, 1998.

Ahmad, E. and N. H. Stern, "The Theory of Tax Reform and Indian Indirect Taxes," *Journal of Public Economics*, 25, 259–98, 1984.

———, *The Theory and Practice of Tax Reform in Developing Countries*, Cambridge University Press, Cambridge, 1991.

Atkinson, A. B., "On the Measurement of Inequality," *Journal of Economic Theory*, 21, 244–63, 1970.

———, "On the Measurement of Poverty," *Econometrica*, 55, 759–64, 1987.

———, "Measuring Poverty and Differences in Family Composition," *Economica*, 59, 1–16, 1992.

Atkinson, A. B. and F. Bourguignon, "Income Distribution and Differences in Needs," in G. R. Feiwel (ed.), *Arrow and the Foundations of the Theory of Economic Policy*, Macmillan, New York, 1987.

Banks, J. and P. Johnson, "Equivalence Scale Relativities Revisited," *The Economic Journal*, 104, 883–90, 1994.

Bourguignon, F., "Family Size and Social Utility: Income Distribution Dominance Criteria," *Journal of Econometrics*, 42, 67–80, 1989.

Bradbury, B., "Measuring Poverty Changes with Bounded Equivalence Scales: Australia in the 1980's," *Economica*, 64, 245–64, 1997.

Buhmann, B., L. Rainwater, G. Schmaus, and T. M. Smeeding, "Equivalence Scales, Well-Being, Inequality, and Poverty: Sensitivity Estimates Across Ten Countries Using the Luxembourg Income Study (LIS) Database," *Review of Income and Wealth*, 34, 115–42, 1987.

Burkhauser, R. V., T. M. Smeeding, and J. Merz, "Relative Inequality and Poverty in Germany and the United States Using Alternative Equivalence Scales," *Review of Income and Wealth*, 42, 381–400, 1996.

Chambaz, C. and E. Maurin, "Atkinson and Bourguignon Dominance Criteria: Extended and Applied to the Measurement of Poverty in France," *Review of Income and Wealth*, 44, 77–124, 1998.

Clark, S., R. Hemming, and D. Ulph, “On Indices for the Measurement of Poverty,” *The Economic Journal*, 91, 515–26, 1981.
Coulter, F. A. E., F. A. Cowell, and S. P. Jenkins, “Differences in Needs and Assessment of Income Distributions,” *Bulletin of Economic Research*, 44, 77–124, 1992a.
———, “Equivalence Scale Relativities and the Extent of Inequality and Poverty,” *The Economic Journal*, 102, 1067–82, 1992b.
Dasgupta, P., A. Sen, and D. Starret, “Notes on the Measurement of Inequality,” *Journal of Economic Theory*, 6, 180–7, 1973.
Davidson, R. and J. Y. Duclos, “Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality,” *Econometrica*, 68, 1435–65, 2000.
De Vos, K. and M. A. Zaidi, “Equivalence Scale Sensitivity of Poverty Statistics for the Member States of the European Community,” *Review of Income and Wealth*, 43, 319–33, 1997.
Duclos, J.-Y. and M. Mercader-Prats, “Household Needs and Poverty: With Application to Spain and the UK,” *Review of Income and Wealth*, 45, 77–98, 1999.
Ebert, U. “Sequential Generalized Lorenz Dominance and Transfer Principles,” Working Paper, University of Oldenburg, 1997.
Fleurbaey, M., C. Hagneré, and A. Trannoy, “Welfare Comparisons with Bounded Equivalence Scales,” Working Paper 9823, THEMA, Université de Cergy-Pontoise, 1998.
Fishburn, P. C. and R. D. Willig, “Transfer Principles in Income Redistribution,” *Journal of Public Economics*, 25, 323–8, 1984.
Foster, J. E. and A. F. Shorrocks, “Poverty Orderings and Welfare Dominance,” *Social Choice and Welfare*, 5, 179–98, 1988a.
———, “Poverty Orderings,” *Econometrica*, 56, 173–7, 1988b.
Foster, J. E., J. Greer, and E. Thorbecke, “A Class of Decomposable Poverty Measures,” *Econometrica*, 52, 761–76, 1984.
Jenkins, S. P. and P. J. Lambert, “Ranking Income Distributions When Needs Differ,” *Review of Income and Wealth*, 39, 337–56, 1993.
Jenkins, S. P. and F. A. Cowell, “Parametric Equivalence Scales and Scale Relativities,” *Economic Journal*, 104, 891–900, 1994.
Jenkins, S. P. and P. J. Lambert, “Three ’I’s of Poverty Curves, with an Analysis of UK Poverty Trends,” Oxford Economic Papers, 49, 317–27, 1997.
Kolm, S. C., “The Optimal Production of Justice,” in H. Guitton and J. Margolis (eds), *Public Economics*, Macmillan, St Martin’s Press, London, 1969.
Lambert, P. J. and X. Ramos, “Welfare Comparisons: Sequential Procedures for Heterogeneous Populations,” mimeo, 2000.
Lanjouw, P. and M. Ravallion, “Poverty and Household Size,” *The Economic Journal*, 105, 1415–34, 1995.
Ok, E. and P. Lambert, “On Evaluating Social Welfare by Sequential Generalized Lorenz Dominance,” *Economics Letters*, 63, 45–53, 1999.
Phipps, S. A., “Measuring Poverty Among Canadian Households: Sensitivity to Choice of Measure and Scale,” *The Journal of Human Resources*, 28, 162–84, 1991.
Sen, A. K., *On Economic Inequality*, expanded edition, Clarendon Press, Oxford, 1997.
———, *Poverty and Famines: An Essay on Entitlement and Deprivation*, Clarendon Press, Oxford, 1981.
Shorrocks, A. F., “Ranking Income Distributions,” *Economica*, 50, 3–17, 1983.
Summers, R. and A. Heston, “The Penn World Table (Mark 5): An Expanded Set of International Comparisons, 1950–1988,” *The Quarterly Journal of Economics*, 106, 327–68, 1991.
Watts, H. W. “An Economic Definition of Poverty,” in D. P. Moynihan (ed.), *On Understanding Poverty*, Basic Books, New York, 1968.
Zheng, B. “On the Power of Poverty Orderings,” *Social Choice and Welfare*, 3, 349–71, 1999.
———, “Poverty Orderings,” *Journal of Economic Surveys*, 14, 427–66, 2000a.
———, “Minimum Distribution-Sensitivity, Poverty Aversion, and Poverty Orderings,” *Journal of Economic Theory*, 95, 116–37, 2000b.