A TAXONOMY OF MULTILATERAL METHODS FOR MAKING INTERNATIONAL COMPARISONS OF PRICES AND QUANTITIES

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A taxonomy of multilateral methods is developed. All the main methods are classified within the taxonomy. It is then shown that methods in the same class share many of the same properties. Therefore the taxonomy should help practitioners to appreciate the wide range of alternative methods for making international comparisons, the relationships between methods, and the relative strengths and weaknesses of each method.

1. INTRODUCTION

Many different methods have been proposed for making multilateral international comparisons of prices and quantities. The various methods were generally advocated by different authors from different perspectives using different notations. Hence the relationships between methods in the literature are frequently unnecessarily obscured. This paper develops a taxonomy that builds on the pioneering work of Van Ijzeren (1987), and rationalizes the literature, by classifying all the main methods within a general framework. The taxonomy groups methods together if and only if they can be shown to be special cases of a more general method thereby revealing the underlying generic similarities between apparently different methods. The taxonomy reveals five main classes of methods, which are called here Multilateral Generalized Törnqvist, Average Basket, Average Price, Fisher Star and Mean Asymmetric Star. Methods in each class are shown to share many of the same strengths and weaknesses. Therefore, the taxonomy serves both to rationalize the literature, and to help users find the multilateral method most suited to their purpose.

2. NOTATION AND DEFINITIONS

2.1. *Multilateral PPPs and Quantity Indices*

Consider the problem of calculating multilateral (transitive) Purchasing Power Parities (PPPs) denoted by P_k and quantity indices denoted by Q_k, over a set of K countries. If the quantity indices are normalized to sum to one, then they are called output shares. It is assumed that each country indexed by $k = 1, \ldots, K$, supplies price and quantity data (p_{ki}, q_{ki}), defined over the same set of goods and services, indexed by $i = 1, \ldots, N$.

The performance of each class of multilateral method identified in the taxonomy is evaluated with reference to five index number tests and the Gerschenkron effect. These tests and the Gerschenkron effect are described below.[1]

(i) Weak Factor Reversal Test

A multilateral method satisfies the Weak Factor Reversal Test if its price and quantity indices are related as shown in (1).

$$(1) \qquad \frac{P_k}{P_b} \frac{Q_k}{Q_b} = \frac{\sum_{i=1}^{N} p_{ki} q_{ki}}{\sum_{i=1}^{N} p_{bi} q_{bi}} \quad \forall b, k = 1, \ldots, K$$

(ii) Strong Factor Reversal Test

A multilateral method satisfies the Strong Factor Reversal Test if first, it satisfies the Weak Factor Reversal Test and second, the PPP formula can be obtained from the quantity index formula by simply interchanging the price and quantity vectors. The Strong Factor Reversal Test is important because if satisfied, it implies that a method calculates quantity indices and PPPs in a symmetric or balanced manner.

(iii) Additivity Test

A multilateral method is additive if its quantity indices literally add up over different levels of aggregation when measured in value terms. Additivity requires a method to compare all countries using a single vector of prices. In other words, the quantity index formula for each country must have the form: $Q_k = \alpha \sum_{i=1}^{N} p_i q_{ki}$, where α denotes an arbitrary constant. For example, suppose GDP is separated into three components: consumption (C), investment (I) and government expenditure (G). Let $i = 1, \ldots, L$ index the categories of consumption, $i = L+1, \ldots, M$ the categories of investment, and $i = M+1, \ldots, N$ the categories of government expenditure. Now if the quantity index formula of country k is of the form $Q_k = \alpha \sum_{i=1}^{N} p_i q_{ki}$, then as (2) demonstrates, the results obtained at the level of GDP are consistent with the results obtained at lower levels of aggregation.[2]

$$(2) \qquad Q_k^{GDP} = \sum_{i=1}^{N} p_i q_{ki} = \sum_{i=1}^{L} p_i q_{ki} + \sum_{i=L+1}^{M} p_i q_{ki} + \sum_{i=M+1}^{N} p_i q_{ki}$$
$$= Q_k^C + Q_k^I + Q_k^G.$$

Additivity is very useful if international comparisons are made at varying levels of aggregation as for example in national accounts comparisons.

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[1]For a more extensive list of index number tests of relevance to multilateral international comparisons, see Diewert (1986) and Balk (1996). In particular, it is worth noting that all the methods discussed in this paper satisfy Diewert's Commensurability or Invariance to Changes in the Units of Measurement Test.

[2]For convenience in (2) the parameter α is set equal to one.

(iv) *Average Test (Quantity Indices)*

This Average Test seems like a reasonable minimum requirement to impose on a multilateral method. Therefore, it is perhaps surprising to discover that many multilateral methods sometimes violate it.

$$(3) \quad \mathrm{Min}\left(\frac{q_{k1}}{q_{b1}},\dots,\frac{q_{kN}}{q_{bN}}\right)\le\frac{Q_k}{Q_b}\le\mathrm{Max}\left(\frac{q_{k1}}{q_{b1}},\dots,\frac{q_{kN}}{q_{bN}}\right)\quad \forall b,k=1,\dots,K.$$

(v) *Average Test (PPPs)*

This Average Test also seems like a reasonable minimum requirement to impose on a multilateral method. However, again many multilateral methods violate it. Moreover, methods that always satisfies the Average Test for PPPs will sometimes violate the Average Test for quantity indices and vice versa.[3]

$$(4) \quad \mathrm{Min}\left(\frac{p_{k1}}{p_{b1}},\dots,\frac{p_{kN}}{p_{bN}}\right)\le\frac{P_k}{P_b}\le\mathrm{Max}\left(\frac{p_{k1}}{p_{b1}},\dots,\frac{p_{kN}}{p_{bN}}\right)\quad \forall b,k=1,\dots,K.$$

(vi) *The Gerschenkron Effect*

The Gerschenkron effect is only applicable to methods that compare all countries using either a reference price or quantity vector. For methods of the former type, the output share of a country tends to fall as the reference price vector becomes relatively more characteristic of its own price vector. Conversely, for methods of the latter type, the output share of a country tends to rise as the reference quantity vector becomes relatively more characteristic of its own quantity vector. This systematic relationship between the reference price or quantity vector and the resulting output shares is known as the Gerschenkron effect.

2.2. *Bilateral PPPs and Quantity Indices*

The following bilateral formulae defined over two countries b and k are referred to repeatedly. It should be noted that unlike their multilateral counterparts, bilateral indices are intransitive.

$$(5) \quad \text{Laspeyres:}\ Q_{bk}^{L}=\frac{\sum_{i=1}^{N}p_{bi}q_{ki}}{\sum_{i=1}^{N}p_{bi}q_{bi}}, \qquad P_{bk}^{L}=\frac{\sum_{i=1}^{N}p_{ki}q_{bi}}{\sum_{i=1}^{N}p_{bi}q_{bi}}$$

$$(6) \quad \text{Paasche:}\ Q_{bk}^{P}=\frac{\sum_{i=1}^{N}p_{ki}q_{ki}}{\sum_{i=1}^{N}p_{ki}q_{bi}}, \qquad P_{bk}^{P}=\frac{\sum_{i=1}^{N}p_{ki}q_{ki}}{\sum_{i=1}^{N}p_{bi}q_{ki}}$$

$$(7) \quad \text{Fisher:}\ Q_{bk}^{F}=(Q_{bk}^{P}Q_{bk}^{L})^{1/2}, \qquad P_{bk}^{F}=(P_{bk}^{P}P_{bk}^{L})^{1/2}$$

$$(8) \quad \text{Törnqvist:}\ Q_{bk}^{T}=\prod_{i=1}^{N}\left(\frac{q_{ki}}{q_{bi}}\right)^{(v_{bi}+v_{ki})/2}, \qquad P_{bk}^{T}=\prod_{i=1}^{N}\left(\frac{p_{ki}}{p_{bi}}\right)^{(v_{bi}+v_{ki})/2},$$

$$\text{where } v_{ki}=\frac{p_{ki}q_{ki}}{\sum_{i=1}^{N}p_{ki}q_{ki}}.$$

[3]See Drechsler and Krzeczkowska (1982).

Also the Country (Time) Reversal Test is referred to in Section 7. This test is defined below in (9). Fisher and Törnqvist satisfy (9) while Paasche and Laspeyres do not.

$$(9) \qquad Q_{bk} = \frac{1}{Q_{kb}}, \qquad P_{bk} = \frac{1}{P_{kb}}, \qquad \forall b, k = 1, \ldots, K$$

2.3. Symmetric and Weighted Means

It is assumed here that a mean $M(w_1, \ldots, w_K, x_1, \ldots, x_K) = M_{k=1}^{K}(w_k, x_k)$ is a function of K variables, $x_1, \ldots, x_K$ and K parameters or weights $w_1, \ldots, w_K$, such that $w_k \geq 0$, $\forall k = 1, \ldots, K$ and $\sum_{k=1}^{K} w_k = 1$. A mean is symmetric if: $w_1 = w_2 = \cdots = w_K = 1/K$. Otherwise a mean is weighted. A mean has at least the following three properties.[4]

    (i) Min $(x_1, \ldots, x_K) \leq M(w_1, \ldots, w_K, \quad x_1, \ldots, x_K) \leq$ Max $(x_1, \ldots, x_K)$ (Min-Max Property);

    (ii) $M(\mathscr{P}_1 w, \mathscr{P}_1 x) = M(\mathscr{P}_2 w, \mathscr{P}_2 x)$, where $\mathscr{P}_1$ and $\mathscr{P}_2$ denote different permutations of the components of the vectors w and x (Symmetry);

    (iii) If $x_k > y_k$, $\forall k = 1, \ldots, K$, then:

$$M(w_1, \ldots, w_K, x_1, \ldots, x_K) > M(w_1, \ldots, w_K, y_1, \ldots, y_K) \text{ (Monotonicity)}.$$

The three best known examples of means are the arithmetic, geometric and harmonic means. These means are used extensively in the calculation of multilateral PPPs and quantity indices.

Arithmetic Mean: $M_A(w_1, \ldots, w_K, x_1, \ldots, x_K) = \sum_{k=1}^{K} w_k x_k$

Geometric Mean: $M_G(w_1, \ldots, w_K, x_1, \ldots, x_K) = \prod_{k=1}^{K} x_k^{w_k}$

Harmonic Mean: $M_H(w_1, \ldots, w_K, x_1, \ldots, x_K) = \left(\sum_{k=1}^{K} w_k (1/x_k)^{-1} \right)^{-1}$

These three means are related as follows: $M_H(x) \leq M_G(x) \leq M_A(x)$.

3. MULTILATERAL GENERALIZED TÖRNQVIST (MGT) METHODS

Consider first the following bilateral generalization of the Törnqvist index defined in (10). $M(\cdot, \cdot)$ denotes a symmetric mean as defined in Section 2.3, while v_{ki} denotes the share of total expenditure in country k spent on commodity i as defined in (8). Törnqvist is a special case of the Generalized Törnqvist (GT) obtained by setting $M(\cdot, \cdot)$ equal to the arithmetic mean.

$$(10) \qquad Q_{bk}^{GT} = \prod_{i=1}^{N} \left(\frac{q_{ki}}{q_{bi}} \right)^{M(v_{bi}, v_{ki})/(\Sigma_{i=1}^{N} M(v_{bi}, v_{ki}))} \qquad P_{bk}^{GT} = \prod_{i=1}^{N} \left(\frac{p_{ki}}{p_{bi}} \right)^{M(v_{bi}, v_{ki})/(\Sigma_{i=1}^{N} M(v_{bi}, v_{ki}))}$$

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[4]For a more detailed discussion on the properties of means, see Diewert (1993a).

The bilateral Generalized Törnqvist is turned into a multilateral index by simply redefining $M(\cdot,\cdot)$ to be a function of all K countries rather than just two countries. The Multilateral Generalized Törnqvist (MGT) quantity index and PPP are defined below in (11).

$$(11) \qquad \frac{Q_k}{Q_b} = \prod_{i=1}^{N} \left(\frac{q_{ki}}{q_{bi}}\right)^{M_{j=1}^{K}(v_{ji})/(\Sigma_{i=1}^{N} M_{j=1}^{K}(v_{ji}))} \qquad \frac{P_k}{P_b} = \prod_{i=1}^{N} \left(\frac{p_{ki}}{p_{bi}}\right)^{M_{j=1}^{K}(v_{ji})/(\Sigma_{i=1}^{N} M_{j=1}^{K}(v_{ji}))}$$

MGT methods differ only in the choice of symmetric mean formula $M_{j=1}^{K}(v_{ji})$. The two best known methods of this type are the Walsh method defined in (12) which uses the geometric mean formula, and the Walsh method with arithmetic weights defined in (13) which uses the arithmetic mean formula.[5]

$$(12) \qquad M_{j=1}^{K}(v_{ji}) = \prod_{j=1}^{K} (v_{ji})^{1/K} \to \frac{Q_k}{Q_b} = \prod_{i=1}^{N} \left(\frac{q_{ki}}{q_{bi}}\right)^{\pi_{j=1}^{K}(v_{ji})^{1/K}/(\Sigma_{i=1}^{N}\{\Pi_{j=1}^{K}(v_{ji})^{1/K}\})}$$

$$(13) \qquad M_{j=1}^{K}(v_{ji}) = \frac{1}{K}\sum_{j=1}^{K} v_{ki} \to \frac{Q_k}{Q_b} = \prod_{i=1}^{N} \left(\frac{q_{ki}}{q_{bi}}\right)^{(1/K)\Sigma_{j=1}^{K} v_{ji}}$$

MGT methods violate the Weak and Strong Factor Reversal Tests, the Additivity Test and both Average Tests. However, at least they are not subject to the Gerschenkron effect.

4. STAR METHODS

Star methods compare countries by linking bilateral indices using the "star" spanning tree depicted in Figure 1.[6] One important implication of using the star spanning tree is that *a priori* one country must be selected and placed at the center of the star. The country in the center is labelled country X. Comparisons between each pair of countries in the set are made indirectly via country X. The quantity index and PPP between countries b and k are calculated as follows:

$$(14) \qquad \frac{Q_k}{Q_b} = \frac{Q_{Xk}}{Q_{Xb}}, \qquad \frac{P_k}{P_b} = \frac{P_{Xk}}{P_{Xb}}.$$

Q_{Xk} and P_{Xk} denote arbitrary bilateral indices. Country X may be one of the countries in the comparison or an artificially created "average" country. In the taxonomy a distinction is drawn between Asymmetric and Symmetric Star methods. Asymmetric Star methods place one of the countries in the comparison at the center of the star, while Symmetric Star methods place an artificial average country at the center of the star.

5. SYMMETRIC STAR METHODS

Symmetric Star methods may differ in two respects. First they may differ with respect to the bilateral index number formula used in (14). Second they may

[5]For a more detailed discussion on these two methods, see Ruggles (1967).
[6]A spanning tree is a set of K vertices (countries) that are connected by exactly $K-1$ edges.
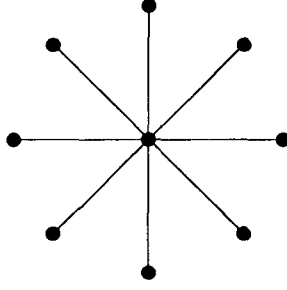
Figure 1. The Star Spanning Tree

differ in the formula used to calculate the average price and/or quantity vectors (p_X, q_X) that define country X. Methods that use the bilateral Paasche quantity index formula are considered first.

5.1. *Average Basket Methods*

All Average Basket methods calculate the quantity index and PPP between countries b and k as follows:

$$(15) \qquad \frac{Q_k}{Q_b} = \frac{Q^P_{Xk}}{Q^P_{Xb}}, \qquad \frac{P_k}{P_b} = \frac{P^L_{Xk}}{P^L_{Xb}},$$

where Q^P_{Xk} is the Paasche quantity index defined in (6), while P^L_{Xk} is the Laspeyres PPP defined in (5). Alternatively, (15) may be written thus:

$$\frac{Q_k}{Q_b} = \frac{\sum_{i=1}^N p_{ki} q_{ki}}{\sum_{i=1}^N p_{ki} q_{Xi}} \frac{\sum_{i=1}^N p_{bi} q_{Xi}}{\sum_{i=1}^N p_{bi} q_{bi}},$$

$$\frac{P_k}{P_b} = \frac{\sum_{i=1}^N p_{ki} q_{Xi}}{\sum_{i=1}^N p_{Xi} q_{Xi}} \frac{\sum_{i=1}^N p_{Xi} q_{Xi}}{\sum_{i=1}^N p_{bi} q_{Xi}} = \frac{\sum_{i=1}^N p_{ki} q_{Xi}}{\sum_{i=1}^N p_{bi} q_{Xi}}.$$

Neither Q_k/Q_b nor P_k/P_b as defined in (15) depends on the price vector p_X of country X. Therefore Average Basket methods need only define an average quantity vector q_X. Van Ijzeren (1987) calls methods of this type, "q-combining" methods. Three natural formulae for q_X are respectively, the arithmetic, geometric and harmonic means of the quantity vectors of all the countries in the set. These formulae are given below.[7]

$$(16) \qquad q_{Xi} = \alpha \sum_{k=1}^K q_{ki} \quad \forall i = 1, \ldots, N$$

$$(17) \qquad q_{Xi} = \alpha \prod_{k=1}^K q_{ki}^{1/K} \quad \forall i = 1, \ldots, N$$

$$(18) \qquad q_{Xi} = \alpha \left(\sum_{k=1}^K \frac{1}{q_{ki}} \right)^{-1} \quad \forall i = 1, \ldots, N.$$

[7]Rescaling the average basket q_X has no effect on the results. Hence the presence of an arbitrary positive constant α in each average basket formula.

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The Average Basket method that calculates q_X using (16) is usually called the ECLAC method. ECLAC stands for United Nations Economic Commission for Latin America and the Caribbean. This method was advocated by Walsh (1901), who called it Scrope's method, and by Van Ijzeren (1956) who called it the weighted heterogenous group method. It is also discussed in Ruggles (1967) and Diewert (1993b). The Average Basket method based on (17) was advocated by Walsh (1901) who called it Scrope's method with geometric weights.[8] However, it is referred to here as the Geometric Average Basket method. The Average Basket method based on (18) does not seem to have been advocated in the literature. A logical name for it would be the Harmonic Average Basket method.

Although these three methods differ only in the choice of symmetric mean formula, there is one very important difference between them. The ECLAC method (16) gives countries with larger baskets greater weight in the q_X formula. In contrast, the Harmonic Average Basket method (18) gives countries with smaller baskets larger weight in the q_X formula. Meanwhile, the Geometric Average Basket method (17) gives equal weight to the baskets of all countries. Therefore, which of these formulae is to be preferred depends on the desired weighting of countries.

Equal weighting across countries can be imposed on (16) and (18) by dividing each quantity vector q_k by its corresponding quantity index.[9] These formulae are given below. The average baskets q_{Xi} and Paasche quantity indices Q_{Xk}^P are obtained by solving the system of $N + K$ simultaneous equations.

$$(19) \qquad q_{Xi} = \alpha \sum_{k=1}^{K} \left(\frac{q_{ki}}{Q_{Xk}^P} \right) \quad \forall i = 1, \ldots, N$$

$$(20) \qquad q_{Xi} = \alpha \prod_{k=1}^{K} \left(\frac{q_{ki}}{Q_{Xk}^P} \right)^{1/K} \quad \forall i = 1, \ldots, N$$

$$(21) \qquad q_{Xi} = \alpha \left(\sum_{k=1}^{K} \left(\frac{q_{ki}}{Q_{Xk}^P} \right)^{-1} \right)^{-1} \quad \forall i = 1, \ldots, N.$$

The de-weighting procedure has no effect on the Geometric Average Basket method since it already gives all countries equal weight. In other words, (17) and (20) are equivalent. The Average Basket method that calculates q_X using (19) is known as the unweighted Van Ijzeren (1956) heterogenous group method. Van Ijzeren (1956) is a highly original paper that proposes three different methods. The first of these is the heterogenous group method. The homogenous group and balanced methods are discussed later in the paper. The heterogenous group method finds the set of PPPs that equate the ratio of the cost of buying the basket of country j in all countries in the set, to the cost of buying the basket of country j in country j, across all countries $j = 1, \ldots, K$ in the set. The equations defining the heterogenous group PPPs, $P_{X1}, \ldots, P_{XK}$ may be written as follows:

$$(22) \qquad \sum_{k=1}^{K} (w_k P_{kj}^L P_{Xk}) = \lambda P_{Xj} \quad \forall j = 1, \ldots, K,$$

[8] Again see Diewert (1993b).
[9] This point is made by Van Ijzeren (1987). He calls this procedure "de-weighting."

where w_k denote country weights and λ is the Perron–Frobenius root of the system.[10] Similarly the equations defining the heterogenous group quantity indices, $Q_{X1}, \ldots, Q_{XK}$ may be written as follows:

$$(23) \qquad \sum_{k=1}^{K} \left(w_k Q_{jk}^L \frac{1}{Q_{Xk}} \right) = \frac{\lambda}{Q_{Xj}} \quad \forall j = 1, \ldots, K.$$

Although Van Ijzeren (1956) approaches the multilateral international comparison problem from a completely different perspective, the solution to (22) and (23) is nevertheless an Average Basket method. The heterogenous group average basket formula is given in (24). This result is proved in the Appendix, Proof 1.

$$(24) \qquad q_{Xi} = \alpha \sum_{k=1}^{K} \left(w_k \frac{q_{ki}}{Q_{Xk}^P} \right) \quad \forall i = 1, \ldots, N.$$

If $w_k = 1$, $\forall k = 1, \ldots, K$, then the average basket formula reduces to (19). If on the other hand the weights w_k equal the output shares $Q_{Xk}^P / (\sum_{j=1}^{K} Q_{Xj}^P)$, then the average basket formula reduces to (16).

Average Basket methods satisfy the Weak Factor Reversal Test and the Average Test for PPPs. However, Average Basket methods fail the Additivity Test, the Average Test for quantity indices, and the Strong Factor Reversal Test. Furthermore, Average Basket methods are subject to the Gerschenkron effect.[11] According to the Gerschenkron effect, the output share of a country tends to rise as the average basket used in the comparison becomes relatively more characteristic of its own basket as compared with the baskets of the other countries. This implies that measured *per capita* income differentials across countries will tend to be larger if the average basket is similar to the basket of the richest country than if it is similar to the basket of the poorest country in the comparison. The Gerschenkron effect arises because expenditure patterns change in response to changes in relative prices. Consumers tend to switch their consumption towards relatively cheaper goods. Figure 2(a) provides an intuitive example of the Gerschenkron effect as it applies to Average Basket methods. The basket is defined over two goods X and Y. For convenience, countries A and B are placed on the same indifference curve $v(p_X, p_Y)$. The baskets of countries A and B are denoted respectively by q_A and q_B. Similarly, the price vectors are denoted by p_A and p_B. If A and B are compared using A's basket q_A, then Figure 2(a) implies that: $p_A \cdot q_A < p_B \cdot q_A \Rightarrow P_{AB}^I > 1$. Now applying the Weak Factor Reversal Test, the following inequality is obtained: $Q_{AB}^P < (p_B \cdot q_B)/(p_A \cdot q_A)$. In contrast, if A and B are compared using B's basket q_B, then: $p_B \cdot q_B < p_A \cdot q_B \Rightarrow P_{BA}^L > 1$. The Weak Factor Reversal Test now implies that: $1/Q_{BA}^P > (p_B \cdot q_B)/(p_A \cdot q_A)$. Therefore, $1/Q_{BA}^P > Q_{AB}^P$. In other words, B appears richer relative to A if B's basket q_B is used to make the comparison than if A's basket q_A is used.

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[10]The Perron–Frobenius root measures the strength of the substitution effect across countries. If there is no substitution effect then λ equals K. In general, λ rises as the substitution effect gets stronger. For a more detailed discussion on the properties of Perron–Frobenius roots, see Debreu and Herstein (1953).

[11]The Gerschenkron effect is discussed in more detail in Nuxoll (1994) and Hill (1995).
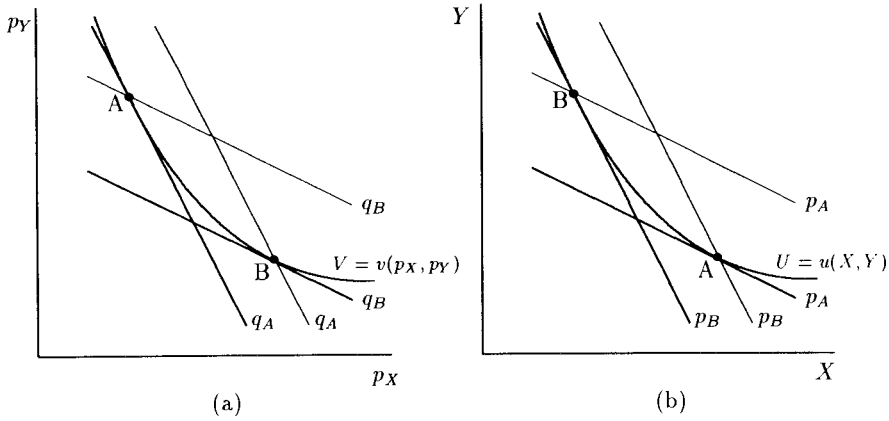
Figure 2. The Gerschenkron Effect

5.2. *Average Price Methods*

All Average Price methods calculate the quantity index and PPP between countries b and k as follows:

(25)
$$\frac{Q_k}{Q_b} = \frac{Q^L_{Xk}}{Q^L_{Xb}}, \qquad \frac{P_k}{P_b} = \frac{P^P_{Xk}}{P^P_{Xb}},$$

where Q^L_{Xk} is the Laspeyres quantity index defined in (5), while P^P_{Xk} is the Paasche PPP defined in (6).[12] Alternatively, (25) may be written thus:

$$\frac{Q_k}{Q_b} = \frac{\sum_{i=1}^{N} p_{Xi} q_{ki}}{\sum_{i=1}^{N} p_{Xi} q_{Xi}} \frac{\sum_{i=1}^{N} p_{Xi} q_{Xi}}{\sum_{i=1}^{N} p_{Xi} q_{bi}} = \frac{\sum_{i=1}^{N} p_{Xi} q_{ki}}{\sum_{i=1}^{N} p_{Xi} q_{bi}}$$

$$\frac{P_k}{P_b} = \frac{\sum_{i=1}^{N} p_{ki} q_{ki}}{\sum_{i=1}^{N} p_{Xi} q_{ki}} \frac{\sum_{i=1}^{N} p_{Xi} q_{bi}}{\sum_{i=1}^{N} p_{bi} q_{bi}}.$$

Neither Q_k/Q_b nor P_k/P_b as defined in (25) depends on the quantity vector q_X of country X. Therefore Average Price methods need only define an average price vector p_X. Van Ijzeren (1987) calls methods of this type, "*p*-combining" methods.

Unweighted Average Prices

The unweighted average price counterparts to the unweighted average basket formulae in (19), (20) and (21) are given below.[13] The average prices p_{Xi} and Paasche PPPs P^P_{Xk} are obtained by solving the system of $N + K$ simultaneous

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[12]Comparing (15) with (25) it can be seen that a symmetrical relationship exists between Average Basket and Average Price methods. Average Basket methods use a Paasche quantity index and a Laspeyres PPP, while Average Price methods use a Laspeyres quantity index and a Paasche PPP.

[13]Again rescaling the average price vector has no effect on the results. Hence the presence of an arbitrary positive constant α in each average price formula.

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equations.

$$(26) \qquad p_{Xi} = \alpha \sum_{k=1}^{K} \left(\frac{p_{ki}}{P_{Xk}^P} \right) \quad \forall i = 1, \ldots, N$$

$$(27) \qquad p_{Xi} = \alpha \prod_{k=1}^{K} \left(\frac{p_{ki}}{P_{Xk}^P} \right)^{1/K} \quad \forall i = 1, \ldots, N$$

$$(28) \qquad p_{Xi} = \alpha \left(\sum_{k=1}^{K} \left(\frac{p_{ki}}{P_{Xk}^P} \right)^{-1} \right)^{-1} \quad \forall i = 1, \ldots, N.$$

The Average Price methods based on (26) and (27) are better known respectively as the unweighted Van Ijzeren (1956) homogenous group method and the Gerardi method.[14] An intriguing feature of the Gerardi method is that deflating each national price by its corresponding Paasche PPP, P_{Xk}^P before calculating average prices has no effect on the Gerardi average prices. In other words, the two formulae in (29) are equivalent.

$$(29) \qquad p_{Xi} = \alpha \prod_{k=1}^{K} (p_{ki})^{1/K}, \qquad p_{Xi} = \alpha \prod_{k=1}^{K} \left(\frac{p_{ki}}{P_{Xk}^P} \right)^{1/K} \quad \forall i = 1, \ldots, N.$$

The equivalence of these two geometric Average Price methods is analogous to the equivalence of the two Average Basket methods given in (17) and (20). The method based on (28) has not been advocated in the literature.

Weighted Average Prices

Van Ijzeren (1987) outlines a weighted version of the Gerardi method (27). The formula is given below:

$$(30) \qquad p_{Xi} = \alpha \prod_{k=1}^{K} (p_{ki}^{Q_{Xk}^L})^{1/(\Sigma_{j=1}^{K} Q_{Xj}^L)} \quad \forall i = 1, \ldots, N.$$

The most widely used Average Price method is Geary (1958)–Khamis (1972). In particular, Geary–Khamis is the preferred multilateral PPP method of the International Comparison Program (ICP), World Bank, IMF and until recently also the OECD.[15] The Geary–Khamis method is a weighted version of (26). Geary–Khamis average prices are calculated using the following formula.[16]

$$(31) \qquad p_{Xi} = \alpha \sum_{k=1}^{K} \left(\frac{q_{ki}}{\sum_{j=1}^{K} q_{ji}} \frac{p_{ki}}{P_{Xk}^P} \right) \quad \forall i = 1, \ldots, N.$$

A second weighted variant on (26) calculates average prices as follows:

$$(32) \qquad p_{Xi} = \alpha \sum_{k=1}^{K} \left(\frac{Q_{Xk}^L}{\sum_{j=1}^{K} Q_{Xj}^L} \frac{p_{ki}}{P_{Xk}^P} \right) \quad \forall i = 1, \ldots, N.$$

[14]The Gerardi method was first proposed by Walsh (1901). It was later popularized by Gerardi (1982). For a more detailed discussion on the evolution of the early literature, see Diewert (1993b).

[15]The OECD now publishes both Geary–Khamis and Elteö, Köves and Szulc (EKS) results. The EKS method is defined in (50).

[16]Khamis (1972) proves existence and uniqueness for the Geary-Khamis system.

The method based on (32) is the weighted Van Ijzeren homogenous group method. The rationale behind the homogenous group method is as follows. It finds the set of PPPs that equate the ratio of the cost of buying the baskets of all the countries in the set in country j, to the cost of buying the basket of country j in country j, across all countries $j = 1, \ldots, K$ in the set. The equations defining the homogenous group PPPs, $P_{X1}, \ldots, P_{XK}$, may be written as follows:

$$(33) \qquad \sum_{k=1}^{K} \left(w_k P_{jk}^L \frac{1}{P_{Xk}} \right) = \frac{\lambda}{P_{Xj}} \quad \forall j = 1, \ldots, K,$$

where again, w_k denote country weights and λ is the Perron–Frobenius root of the system. Similarly the equations defining the homogenous group quantity indices, $Q_{X1}, \ldots, Q_{XK}$, may be written as follows:

$$(34) \qquad \sum_{k=1}^{K} \left(w_k Q_{kj}^L Q_{Xk} \right) = \lambda Q_{Xj} \quad \forall j = 1, \ldots, K.$$

The general formula for heterogenous group average prices is given below:[17]

$$(35) \qquad p_{Xi} = \alpha \sum_{k=1}^{K} \left(w_k \frac{p_{ki}}{P_{Xk}^P} \right) \quad \forall i = 1, \ldots, N.$$

If $w_k = 1$, $\forall k = 1, \ldots, K$, then the average price formula reduces to (26). If on the other hand the weights w_k equal the output shares $Q_{Xk}^L / (\sum_{j=1}^{K} Q_{Xj}^L)$, then the average price formula reduces to (32).

A further variant on (26) is the Iklé (1972) method. The Iklé method calculates p_{Xi} as follows:[18]

$$(36) \qquad p_{Xi} = \alpha \sum_{k=1}^{K} \left(\frac{q_{ki} / Q_{Xk}^L}{\sum_{j=1}^{K} q_{ji} / Q_{Xj}^L} \frac{p_{ki}}{P_{Xk}^P} \right) \quad \forall i = 1, \ldots, N.$$

Iklé is derived from Geary-Khamis by applying Van Ijzeren's de-weighting procedure. Therefore, Iklé is rather similar to Van Ijzeren's unweighted homogenous group method. Dikhanov (1994) demonstrates that the Iklé method can also be expressed as a variant on the harmonic mean formula in (28).[19]

$$(37) \qquad p_{Xi} = \alpha \left\{ \sum_{k=1}^{K} \left[\left(\frac{v_{ki}}{\sum_{j=1}^{K} v_{ji}} \frac{p_{ki}}{P_{Xk}^P} \right)^{-1} \right] \right\}^{-1} \quad \forall i = 1, \ldots, N.$$

The final Average Price method considered here approaches the problem of calculating average prices in a radically different way. The "Ideal Price" (IP) method was proposed by Gerardi.[20] Average prices are obtained as the solution

[17]Van Ijzeren (1956) proves existence and uniqueness for the heterogenous and homogenous group systems. (35) is derived in an analogous manner to (24).
[18]Balk (1996) proves existence and uniqueness for the Iklé system.
[19]v_{ki} in (37) again denotes the value share of commodity i in the basket of country k as defined in (8).
[20]See Eurostat (1983).

to the following minimization problem:

$$(38) \qquad \mathrm{Min}_{pX}\left[\sum_{k=1}^{K}\sum_{b=1}^{K}\left(\log\frac{\sum_{i=1}^{N}p_{Xi}q_{ki}}{\sum_{i=1}^{N}p_{Xi}q_{bi}}-\log Q_{bk}^{F}\right)^{2}\right],$$

where Q_{bk}^{F} is the Fisher quantity index defined in (7). The rationale behind the IP method is to maintain additivity while minimizing the Gerschenkron effect. This is achieved by finding the Average Price (additive) method whose quantity indices most closely approximate Fisher quantity indices. However, the IP method does not generate economically meaningful results, since in general even assuming the existence and uniqueness of a strictly positive solution, the IP average price vector is unlikely to bear any resemblance to the actual national price vectors of the countries in the comparison.[21] Therefore the IP quantity indices obtained at lower levels of aggregation will be essentially meaningless, even though they sum to something close to their corresponding Fisher quantity indices at the highest level of aggregation.

Average Price methods satisfy the Weak Factor Reversal Test, the Additivity Test and the Average Test for quantity indices, but violate the Strong Factor Reversal Test and the Average Test for PPPs. Average Price methods have received more attention in the literature than Average Basket methods because they are additive. However, Average Price methods also suffer from the Gerschenkron effect. According to the Gerschenkron effect, the output share of a country tends to fall as the average price vector used in the comparison becomes relatively more characteristic of its own price vector as compared with the price vectors of the other countries. This implies that measured per capita income differentials across countries will tend to be larger if the average price vector is similar to the price vector of the poorest country than if it is similar to the price vector of the richest country in the comparison. In fact, concern over the Gerschenkron effect has recently prompted the World Bank to consider switching from Geary–Khamis to Iklé on the grounds that Iklé may generate more reliable estimates of per capita income differentials across countries than Geary–Khamis.[22] The Gerschenkron effect as it applies to Average Price methods is illustrated in Figure 2(b). Which country has the larger quantity index, A or B? If A and B are compared using the price vector p_A of country A, then B has the larger quantity index, since: $p_A \cdot q_A < p_A \cdot q_B \Rightarrow Q_{AB}^{I} > 1$. In contrast, if they are compared using B's price vector p_B, then A has the larger quantity index, since: $p_B \cdot q_B < p_B \cdot q_A \Rightarrow 1/Q_{BA}^{I} < 1$.

5.3. Fisher Star Methods

Fisher Star methods differ only in the way they calculate the price and quantity vectors p_X and q_X of country X. All Fisher Star methods calculate the quantity index and PPP between countries b and k as follows:

$$(39) \qquad \frac{Q_k}{Q_b}=\frac{Q_{Xk}^{F}}{Q_{Xb}^{F}}, \qquad \frac{P_k}{P_b}=\frac{P_{Xk}^{F}}{P_{Xb}^{F}},$$

[21] In a 1993 Eurostat mimeo, Cuthbert claims that the IP method does not always have a unique solution, and that sometimes one or more of the average prices may equal zero.
[22] See Dikhanov (1994).

where Q_{Xk}^F and P_{Xk}^F denote the Fisher quantity index and PPP defined in (7). Alternatively, (39) may be written thus:

$$(40) \qquad \frac{Q_k}{Q_b} = \left[\left(\frac{Q_{Xk}^P}{Q_{Xb}^P} \right) \left(\frac{Q_{Xk}^L}{Q_{Xb}^L} \right) \right]^{1/2}, \qquad \frac{P_k}{P_b} = \left[\left(\frac{P_{Xk}^P}{P_{Xb}^P} \right) \left(\frac{P_{Xk}^L}{P_{Xb}^L} \right) \right]^{1/2}.$$

A comparison of (40) with (15) and (25) reveals that any Fisher Star quantity index (PPP) may be expressed as the geometric mean of a pair of Average Basket and Average Price quantity indices (PPPs).

A plethora of Fisher Star methods may be generated by combining pairs of Average Basket and Average Price methods. However, for a Fisher Star method to be meaningful, its respective Average Basket and Average Price methods must have the same weighting structure. As an example of a "bad" Fisher Star method consider combining the ECLAC method (16) with the Gerardi method (27). The Gerardi method gives all countries in the comparison equal weight. In contrast the ECLAC method gives larger countries larger weights. The Fisher Star method formed by combining these two methods is a confused mixture of weighting schemes. On the other hand, the Fisher Star method formed by combining the Geometric Average Basket (17) and Gerardi methods has a consistent weighting structure since both of its constituent methods give all countries equal weights. This method could be called the Geometric Fisher Star method. Similarly the Fisher Star method formed by combining the ECLAC and Geary-Khamis (31) methods also has a consistent weighting structure, since both methods give countries with larger baskets larger weights. This method could be called the Weighted Arithmetic Fisher Star method.

From (40) it follows that all Fisher Star methods can be decomposed into a pair of Average Basket and Average Price methods. However, this process of decomposition is not always straightforward. If the average basket formula of a Fisher Star method is a function of average prices, while simultaneously average prices are a function of the average basket, then it may be difficult to disentangle the Average Basket and Average Price methods from each other. An example of such a Fisher Star method is Van Ijzeren's (1956) balanced method.

Van Ijzeren's balanced method exploits the fact that the Perron-Frobenius roots in (22) and (33) are the same. This is because the matrix of Laspeyres indices in (33) is the transpose of the matrix of Laspeyres indices in (22) and hence has the same eigenvalues. Therefore (22) and (33) may be combined as follows:

$$(41) \qquad \sum_{k=1}^{K} \left(w_k \, P_{kj}^L \, \frac{P_{Xk}}{P_{Xj}} \right) = \sum_{k=1}^{K} \left(w_k \, P_{jk}^L \, \frac{P_{Xj}}{P_{Xk}} \right) \quad \forall j = 1, \ldots, K.$$

Similarly, (24) and (34) may be combined as follows:

$$(42) \qquad \sum_{k=1}^{K} \left(w_k \, Q_{jk}^L \, \frac{Q_{Xj}}{Q_{Xk}} \right) = \sum_{k=1}^{K} \left(w_k \, Q_{kj}^L \, \frac{Q_{Xk}}{Q_{Xj}} \right) \quad \forall j = 1, \ldots, K.$$

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The average basket and average price formulae are given below in (43).[23] q_X in (43) is a function of Q^F_{Xk} and hence p_X. Similarly, p_X is a function of P^F_{Xk} and hence q_X.

$$(43) \qquad q_{Xi} = \alpha \sum_{k=1}^{K} \left(w_k \frac{q_{ki}}{Q^F_{Xk}} \right), \qquad p_{Xi} = \alpha \sum_{k=1}^{K} \left(w_k \frac{p_{ki}}{P^F_{Xk}} \right) \qquad \forall i = 1, \ldots, N.$$

The two natural choices of weights are either $w_k = 1$ or $w_k = Q^F_{Xk}/(\sum_{j=1}^{K} Q^F_{Xj})$, $\forall k = 1, \ldots, K$.

Two variants on Van Ijzeren's balanced method are the YKS and YKS-Q methods proposed by Kurabayashi and Sakuma (1990).[24] The equations defining the YKS PPPs, $P_{X1}, \ldots, P_{XK}$, are given below:

$$(44) \qquad \sum_{k=1}^{K} w_k P^L_{kj} = \sum_{k=1}^{K} w_k P^{L}_{jk} \left(\frac{P_{Xj}}{P_{Xk}} \right)^2 \qquad \forall k = 1, \ldots, K.$$

A comparison of (44) with (41) reveals that the unweighted YKS method (i.e. $w_k = 1$) is equivalent to the Van Ijzeren method with the following weights: $w_k = 1/P^F_{Xk}$. Meanwhile, the equations defining the YKS-Q method are given below:

$$(45) \qquad \sum_{k=1}^{K} w_k Q^L_{kj} = \sum_{k=1}^{K} w_k Q^{L}_{jk} \left(\frac{Q_{Xj}}{Q_{Xk}} \right)^2 \qquad \forall k = 1, \ldots, K.$$

A comparison of (45) with (42) reveals that the unweighted YKS-Q method is equivalent to the Van Ijzeren method with the following weights: $w_k = 1/Q^F_{Xk}$. The economic rationale for the weighting structures of the YKS and YKS-Q methods is not clear.

Fisher Star methods dominate MGT methods because in addition to not suffering from the Gerschenkron effect, they also satisfy the Weak and Strong Factor Reversal Tests. However, Fisher Star methods do not satisfy the Additive Test and the two Average Tests. Also, some Fisher Star methods such as Van Ijzeren's balanced method are complicated and difficult to calculate. Nevertheless, both Diewert (1986) and Balk (1996) favour Van Ijzeren's weighted balanced method, on the grounds that it satisfies the most multilateral index number tests.

6. ASYMMETRIC STAR METHODS

Asymmetric Star methods a priori put one of the countries in the comparison at the center of the star spanning tree. Quantity indices and PPPs are then calculated using (14). The problem with Asymmetric Star methods is that the quantity indices and PPPs depend critically on which country is at the center of the star. Nevertheless, an Asymmetric Star method has been used for many years to make comparisons in Eastern Europe, with Austria at the center. It is still being used. The reasons have more to do with resources, costs and politics

[23]This result is proved in the Appendix, Proof 2.
[24]YKS stands for Van Yzeren–Kurabayashi Sakuma, (Van Izjeren is sometimes spelt Van Yzeren). See Kurabayashi and Sakuma (1990).

than with methodology. Austria can be seen as a link between Eastern and Western Europe and the Austrian Statistical Office has invested large resources in these comparisons.

7. MEAN ASYMMETRIC STAR METHODS

Mean Asymmetric Star methods proceed by placing sequentially each country at the center of the star. Hence K sets of quantity indices are obtained. Each set is associated with a different center country. Mean Asymmetric Star quantity indices are then obtained by taking a symmetric or weighted mean of these K sets of quantity indices. Four possible general functional forms for a Mean Asymmetric Star method are given below in equations (46) to (49).

$$(46) \qquad \frac{Q_k}{Q_b} = \frac{M_{j=1}^{K}(Q_{jk})}{M_{j=1}^{K}(Q_{jb})}$$

$$(47) \qquad \frac{Q_k}{Q_b} = \frac{M_{j=1}^{K}(Q_{bj})}{M_{j=1}^{K}(Q_{kj})}$$

$$(48) \qquad \frac{Q_k}{Q_b} = \frac{M_{j=1}^{K}[m(Q_{jk}, 1/Q_{kj})]}{M_{j=1}^{K}[m(Q_{jb}, 1/Q_{bj})]}$$

$$(49) \qquad \frac{Q_k}{Q_b} = \frac{M_{j=1}^{K}[m(1/Q_{jb}, Q_{bj})]}{M_{j=1}^{K}[m(1/Q_{jk}, Q_{kj})]}.$$

In the above equations, Q_{jk} denotes a generic bilateral quantity index, while $M_{j=1}^{K}(Q_{jk})$ denotes a symmetric or weighted mean of the bilateral quantity indices between country k and each of the countries in the set. $m(\cdot, \cdot)$ denotes a symmetric mean over its two arguments. If Q_{jk} satisfies the Country Reversal Test defined in (9), then (48) reduces to (46), while (49) reduces to (47).

7.1. *Geometric Mean Asymmetric Star Methods*

Suppose both M and m denote geometric means. Under this scenario, equations (48) and (49) are equivalent. In addition suppose that the bilateral quantity index satisfies the Country Reversal Test. Under this scenario, all four general forms are equivalent. EKS quantity indices are obtained then by using the Fisher quantity index. The EKS quantity index and PPP formulae may be written thus:

$$(50) \qquad \frac{Q_k}{Q_b} = \prod_{j=1}^{K}\left(\frac{Q_{jk}^F}{Q_{jb}^F}\right)^{1/K}, \qquad \frac{P_k}{P_b} = \prod_{j=1}^{K}\left(\frac{P_{jk}^F}{P_{jb}^F}\right)^{1/K}.$$

The EKS method was first proposed by Gini (1931), although credited to Elteö and Köves (1964) and Szulc (1964). It is the preferred method of Eurostat, and increasingly also of the OECD. A weighted version of the EKS method due to Van Ijzeren (1983) calculates quantity indices and PPPs as follows:

$$(51) \qquad \frac{Q_k}{Q_b} = \prod_{j=1}^{K}\left\{\left(\frac{Q_{jk}^F}{Q_{jb}^F}\right)^{Q_j/(\Sigma_{l=1}^{K} Q_l)}\right\}, \qquad \frac{P_k}{P_b} = \prod_{j=1}^{K}\left\{\left(\frac{P_{jk}^F}{P_{jb}^F}\right)^{Q_j/(\Sigma_{l=1}^{K} Q_l)}\right\}.$$

Another variant on the EKS method was proposed by Caves, Christensen and Diewert (1982). They proposed the Multilateral Translog method, which uses the Törnqvist as opposed to the Fisher index.[25] The Multilateral Translog quantity index and PPP are defined as follows:

$$(52) \qquad \frac{Q_k}{Q_b} = \prod_{j=1}^{K} \left(\frac{Q_{jk}^T}{Q_{jb}^T} \right)^{1/K} , \qquad \frac{P_k}{P_b} = \prod_{j=1}^{K} \left(\frac{P_{jk}^T}{P_{jb}^T} \right)^{1/K} .$$

Mean Asymmetric Star methods may be further classified according to the bilateral quantity index formula they use. According to such a classification, the EKS method is a Fisher Mean Asymmetric Star method, while the Multilateral Translog method is a Törnqvist Mean Asymmetric Star method. Törnqvist Mean Asymmetric Star methods are rather similar to Generalized Multilateral Törnqvist methods. In particular, both classes of methods are subject to the criticism that their quantity indices and PPPs do not satisfy the Weak Factor Reversal Test.

Mean Asymmetric Star methods may also be derived as the solution to (53).

$$(53) \qquad \text{Min}_{\log(Q_a/Q_b)} \left\{ \sum_{a=1}^{K} \sum_{b=1}^{K} \left[\left(\log \frac{Q_a}{Q_b} - \log Q_{ba} \right)^2 \right] \right\}.$$

The solution to (53) alters bilateral (intransitive) indices by the logarithmic least squares amount necessary to obtain transitivity. The solution is given below in (54). This result is proved in Prasada Rao and Banerjee (1986).

$$(54) \qquad \frac{Q_k}{Q_b} = \prod_{j=1}^{K} \left(\frac{Q_{bj} Q_{jk}}{Q_{kj} Q_{jb}} \right)^{1/(2K)} .$$

(54) is the geometric mean version of (48). If the bilateral quantity index Q_{ba} in (53) is set equal to any of the Paasche, Laspeyres or Fisher quantity index formulae Q_{ba}^P, Q_{ba}^L or Q_{ba}^F, then (54) reduces to the EKS quantity index formula in (50). This result is due originally to Van Ijzeren (1987). Therefore EKS quantity indices deviate from Fisher quantity indices by the logarithmic least squares amount necessary to obtain transitivity. If on the other hand the bilateral quantity index Q_{ba} in (53) is set equal to the Törnqvist quantity index formula Q_{ba}^T, then (54) reduces to the Multilateral Translog quantity index formula in (52). Analogous results apply for PPPs.

7.2. *Arithmetic and Harmonic Mean Asymmetric Star Methods*

Suppose now that both M and m denote harmonic means. In addition suppose that the bilateral index satisfies the Country Reversal Test. Under this scenario, in contrast to the geometric mean case, equations (46) and (47) are not equivalent. The Own Share method, which was proposed by Diewert (1986), uses the Fisher index in equation (46). Own Share quantity indices may be written as follows:

$$(55) \qquad \frac{Q_k}{Q_b} = \left(\frac{\sum_{j=1}^{K} (1/Q_{jk}^F)}{\sum_{j=1}^{K} (1/Q_{jb}^F)} \right)^{-1} = \frac{\sum_{j=1}^{K} Q_{bj}^F}{\sum_{j=1}^{K} Q_{kj}^F} .$$

[25]The Törnqvist index is exact for a translog cost function.

It is interesting to note from (55) that the Own Share method could equally well be defined using an arithmetic mean in equation (47). Similarly, using a harmonic mean in equation (47) is equivalent to using an arithmetic mean in (46). This variant on the Own Share method may be written as follows:

$$(56) \qquad \frac{Q_k}{Q_b} = \left(\frac{\sum_{j=1}^{K} (1/Q_{bj}^F)}{\sum_{j=1}^{K} (1/Q_{kj}^F)} \right)^{-1} = \frac{\sum_{j=1}^{K} Q_{jk}^F}{\sum_{j=1}^{K} Q_{jb}^F}.$$

Mean Asymmetric Star methods do not suffer from the Gerschenkron effect. In fact, Mean Asymmetric Star methods that use the Fisher index are rather similar to Fisher Star methods. They satisfy the Weak and Strong Factor Reversal Tests, but violate the Additivity Test and both Average Tests.

8. CONCLUSION

The multilateral international comparison literature has a well defined structure. All the main methods can be divided into five classes called here Multilateral Generalized Törnqvist, Average Basket, Average Price, Fisher Star and Mean Asymmetric Star. The taxonomic structure is depicted in Figure 3.

Methods in the same class share many of the same properties as illustrated in Table 1. Therefore, the taxonomy should help practitioners to appreciate the range of alternative methods for making multilateral international comparisons, and the relative merits of each method. Table 2 lists all the methods discussed in the paper with their corresponding equation numbers, according to their classification within the taxonomy.

As regards the issue of weighting, both Diewert (1986) and Balk (1996) favour weighted methods since they satisfy Diewert's *Irrelevance of tiny countries test*, which essentially states that small countries must not unduly influence the output shares of large countries. Furthermore, certain weighted methods
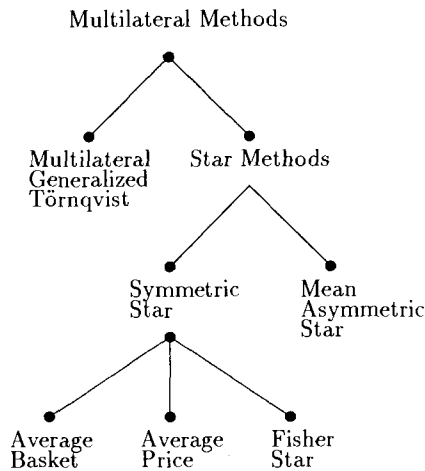

Figure 3. Taxonomic Structure

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TABLE 1

STRENGTHS AND WEAKNESSES OF EACH CLASS OF METHOD

	WFRT	SFRT	Add	AvQ	AvP	GEff
MGT	×	×	×	×	×	√
Av. Basket	√	×	×	×	√	×
Av. Price	√	×	√	√	×	×
Fisher Star	√	√	×	×	×	√
Fisher MAS	√	√	×	×	×	√

Note: A √ indicates that the class of methods in question satisfies this particular test, or in the case of GEff, that the class is not subject to the Gerschenkron effect. The abbreviations in Table 1 are as follows: WFRT—Weak Factor Reversal Test, SRFT—Strong Factor Reversal Test, Add—Additivity Test, AvQ—Average Test for Quantity Indices, AvP—Average Test for PPPs, GEff—Gerschenkron Effect, MGT—Multilateral Generalized Törnqvist, Av. Basket—Average Basket, Av. Price—Average Price, Fisher MAS—Fisher Mean Asymmetric Star.

TABLE 2

CLASSIFICATION OF METHODS

MGT	General formula—Section 3: (11) Walsh: (12), Walsh with arithmetic weights: (13)
Av. Basket	General formula—Section 5.1: (15) ECLAC—Scrope—Weighted Van Ijzeren heterogenous group: (16) Geometric Average Basket: (17) and (20) Harmonic Average Basket: (18) Unweighted Van Ijzeren heterogenous group: (19) Unweighted Harmonic Average Basket: (21)
Av. Price	General formula—Section 5.2: (25) Unweighted Van Ijzeren homogenous group: (26) Geometric Average Price—Gerardi: (29) Unweighted Harmonic Average Price: (28) Weighted Gerardi: (30), Geary-Khamis: (31) Weighted Van Ijzeren homogenous group: (32) Iklé: (36) and (37), Ideal Price (IP): (38)
Fisher Star	General formula—Section 5.3: (39) Geometric Fisher Star: Uses both (17) and (27) Weighted Arithmetic Fisher Star: Uses both (16) and (31) Van Ijzeren balanced: (43), YKS: (44), YKS-Q: (45)
MAS	General formula—Section 7: Either (46) or (47) EKS: (50), Weighted EKS: (51), Multilateral translog: (52) Own share: (55), Variant on own share: (56)

also satisfy Diewert's *Country partitioning test*.[26] In particular, Van Ijzeren's weighted balance method defined in (43), ECLAC defined in (16) and Geary-Khamis defined in (31) all satisfy this test, while the weighted EKS method defined in (51) does not. Ultimately both Diewert and Balk prefer Van Ijzeren's

[26]According to this test, if country k with quantity vector q_k is partitioned into two parts r and s with quantity vectors $q_r = \mu q_k$ and $q_s = (1 - \mu)q_k$, and both r and s still face the same price vector, i.e. $p_r = p_s = p_k$, then the quantity indices of k, r and s should be related as follows: $Q_r = \mu Q_k$, and $Q_s = (1 - \mu)Q_k$. The quantity indices of all the other countries in the comparison should be unaffected.

weighted balanced method since it satisfies the most multilateral index number tests.

Using simulations, Van Ijzeren (1987) demonstrates that Average Price and Average Basket methods can be quite sensitive to changes in the weighting scheme. In contrast, weighting has little impact on either his balanced method or EKS. Furthermore, Van Ijzeren also shows that his balanced method is closely approximated by EKS. Therefore in practice it will make little difference whether the theoretically preferred Van Ijzeren weighted balanced method or EKS (either weighted or unweighted) is used. This result provides a strong rationale for using the EKS method, since it is conceptually rather simpler and much easier to calculate than the theoretically preferred Van Ijzeren balanced method.

APPENDIX

Proof 1

(23) may be written as follows:

$$\sum_{k=1}^{K} \left(w_k \frac{\sum_{i=1}^{N} p_{ji} q_{ki}}{\sum_{i=1}^{N} p_{ji} q_{ji}} \frac{1}{Q_{Xk}} \right) = \frac{\lambda}{Q_{Xj}} \quad \forall j = 1, \ldots, K.$$

Now reverse the order of the summation signs in the numerator on the left hand side.

$$\Rightarrow \frac{\sum_{i=1}^{N} (p_{ji} \{\sum_{k=1}^{K} w_k q_{ki}/Q_{Xk}\})}{\sum_{i=1}^{N} p_{ji} q_{ji}} = \frac{\lambda}{Q_{Xj}} \quad \forall j = 1, \ldots, K$$

$$\Rightarrow \frac{Q_{Xj}}{Q_{Xb}} = \frac{\sum_{i=1}^{N} p_{ji} q_{ji}}{\sum_{i=1}^{N} p_{ji} q_{Xi}} \frac{\sum_{i=1}^{N} p_{bi} q_{Xi}}{\sum_{i=1}^{N} p_{bi} q_{bi}} = \frac{Q_{Xj}^{P}}{Q_{Xb}^{P}}, \quad \text{where } q_{Xi} = \sum_{k=1}^{K} w_k \frac{q_{ki}}{Q_{Xk}^{P}}.$$

Proof 2

(42) may be rewritten as follows:

$$Q_{Xj}^2 \sum_{k=1}^{K} \left(\frac{w_k}{Q_{Xk}} \frac{\sum_{i=1}^{N} p_{ji} q_{ki}}{\sum_{i=1}^{N} p_{ji} q_{ji}} \right) = \sum_{k=1}^{K} \left(w_k Q_{Xk} \frac{\sum_{i=1}^{N} p_{ki} q_{ji}}{\sum_{i=1}^{N} p_{ki} q_{ki}} \right) \quad \forall j = 1, \ldots, K.$$

Now reverse the order of the summation signs in the numerators on both sides of the equation, and replace Q_{Xk} with P_{Xk} on the right hand side using the Weak Factor Reversal Test (1).

$$\Rightarrow Q_{Xj}^2 \frac{\sum_{i=1}^{N} [p_{ji} \sum_{k=1}^{K} (w_k q_{ki}/Q_{Xk})]}{\sum_{i=1}^{N} p_{ji} q_{ji}}$$

$$= \frac{\sum_{i=1}^{N} [\sum_{k=1}^{K} (w_k(p_{ki}/P_{Xk})) q_{ji}]}{\sum_{i=1}^{N} p_{Xi} q_{Xi}} \quad \forall j = 1, \ldots, K.$$

$$\Rightarrow Q_{Xj} = \left(\frac{\sum_{i=1}^{N} p_{ji} q_{ji} \; \sum_{i=1}^{N} p_{Xi} q_{ji}}{\sum_{i=1}^{N} p_{ji} q_{Xi} \; \sum_{i=1}^{N} p_{Xi} q_{Xi}} \right)^{1/2}$$

$$= (Q_{Xj}^{P} \, Q_{Xj}^{L})^{1/2} = Q_{Xj}^{F} \quad \forall j = 1, \ldots, K,$$

where

$$q_{Xi} = \sum_{k=1}^{K} \left(w_k \frac{q_{ki}}{Q_{Xk}^{F}} \right), \qquad p_{Xi} = \sum_{k=1}^{K} \left(w_k \frac{p_{ki}}{P_{Xk}^{F}} \right), \quad \forall i = 1, \ldots, N.$$

Using an analogous method it can be shown that $P_{Xj} = P_{Xj}^{F}$, $\forall j = 1, \ldots, K$.

REFERENCES

Balk, B. M., A Comparison of Ten Methods for Multilateral International Price and Volume Comparison, *Journal of Official Statistics*, Vol. 12, No. 2, 199–222, 1996.

Caves, D. W., L. R. Christensen and W. E. Diewert, Multilateral Comparisons of Output, Input and Productivity using Superlative Index Numbers, *Economic Journal* 92, 73–87, March 1982.

Cuthbert, J. E., IP and EKS Aggregation Methods: Preservation of Price Structure under the IP Method, *Eurostat Mimeo*, Eurostat, Luxembourg, 1993.

Debreu, G. and I. N. Herstein, Nonnegative Square Matrices, *Econometrica* 21, 597–607, October, 1953.

Diewert, W. E., Microeconomic Approaches to the Theory of International Comparisons, *NBER Technical Working Paper No. 53*, 1986.

———, Symmetric Means and Choice Under Uncertainty, in *Essays in Index Number Theory*, Vol. 1, W. E. Diewert, and A. Nakamura (eds.), North Holland, Amsterdam, 1993a.

———, The Early History of Price Index Research, in *Essays in Index Number Theory*, Vol. 1, W. E. Diewert, and A. Nakamura (eds.), North Holland, Amsterdam, 1993b.

Dikhanov, Y., *Sensitivity of PPP-Based Income Estimates to Choice of Aggregation Procedures*, Paper presented at the Twenty-Third General Conference of the International Association for Research in Income and Wealth, held at St. Andrews, New Brunswick, Canada, August 1994.

Drechsler, L. and E. Krzeczkowska, Purchasing Power Parities in International Comparisons, *The Review of Income and Wealth* 28, 253–259, September, 1982.

Elteto, O. and P. Köves, On a Problem of Index Number Computation Relating to International Comparison, *Statisztikai Szemle* 42, 507–518, 1964.

Eurostat, *Comparison in Real Values of the Aggregates of the European System of Integrated Economic Accounts*, Eurostat, Luxembourg, 1983.

Geary, R. G., A Note on the Comparison of Exchange Rates and PPPs between Countries, *Journal of the Royal Statistical Society*, Series A, 121, 97–99, 1958.

Gerardi, D., Selected Problems of Inter-Country Comparisons on the Basis of the Experience of the EEC, *Review of Income and Wealth* 28, 381–405, December 1982.

Gini, C., On the Circular Test of Index Numbers, *International Review of Statistics*, Vol. 9, No. 2, 3–25, 1931.

Hill, R. J., International Comparisons, Additive PPPs and the Gerschenkron Effect, Discussion Paper 95/41, School of Economics, The University of New South Wales, 1995.

Iklé, D. M., A New Approach to the Index Number Problem, *Quarterly Journal of Economics*, Vol. 86, 188–211, 1972.

Khamis, S. H., A New System of Index Numbers for National and International Purposes, *Journal of the Royal Statistical Society*, Series A, 135, 96–121, 1972.

Kurabayashi, Y. and I. Sakuma, *Studies in International Comparisons of Real Product and Prices*, Kinokuniya Company Ltd., Tokyo, 1990.

Nuxoll, D. A., Differences in Relative Prices and International Differences in Growth Rates, *American Economic Review* 84, 1423–1436, 1994.

Prasada Rao, D. S. and K. S. Banerjee, A Multilateral Index Number System Based on the Factorial Approach, *Statistische Hefte/Statistical Papers* 27, 297–313, 1986.

Ruggles, R., Price Indexes and International Price Comparisons, in *Ten Economic Studies in the Tradition of Irving Fisher*, Fellner, W. (ed.), John Wiley, New York, 1967.

Szulc, B., Indices for Multiregional Comparisons, *Przeglad Statystyczny* 3, *Statistical Review* 3, 239–254, 1964.

Van Ijzeren, J., Three Methods of Comparing the Purchasing Power of Currencies, *Statistical Studies* No. 7, The Netherlands Central Bureau of Statistics, The Hague, 1956.

——, Index Numbers for Binary and Multilateral Comparison, *Statistical Studies* No. 34, The Netherlands Central Bureau of Statistics, The Hague, 1983.

——, *Bias in International Index Numbers: A Mathematical Elucidation*, Dissertation submitted to the Committee of Scientific Qualifications of the Hungarian Academy of Sciences for award of the degree of Candidate of Sciences (Ph.D) in Economics, 1987.

Walsh, C. M., *The Measurement of General Exchange Value*, Macmillan and Co., New York, 1901.