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INCOME AND THE HAMILTONIAN[1]

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Among the many interpretations of real national income are (i) the return to national wealth and (ii) the Hamiltonian of an appropriately-chosen dynamic model of the economy. These interpretations are sometimes alleged to be equivalent and to constitute the self-evidently ideal definition to which statistics of real national income should conform as closely as possible., The allegation is correct on some very restrictive assumptions about technology and taste. Otherwise, these interpretations are inconsistent, inexpedient as definitions of real national income and significantly at variance with the usage in the national accounts. The return to wealth is unmeasurable with the currently-available data. The Hamiltonian is typically in the wrong units. It is an accurate reflection of neither productive capacity nor welfare in an intertemporal context. It is not well-defined in a tax-distorted economy. It is rarely an indicator of the return to wealth.

> A person's income is "the maximum value he can consume during a week and still be as well off at the end of the week as he was at the beginning"
>
> J. R. Hicks[2]

> "... the rigorous search for a meaningful income concept leads to a rejection of all current income concepts and ends up with something closer to a "wealth-like magnitude," such as the present discounted value of future consumption. ... a standard welfare interpretation of NNP is that it is the largest permanently maintainable value of consumption. ... What we have been calling net national product is just the Hamiltonian for a general optimization problem."
>
> Martin L. Weitzman[3]

Economists frequently employ a model of an economy without technical change and with only one good which does double duty as output and as capital. Within that model, national income is unambiguous. It is the total annual output of the economy, part of which is consumed in the current year and the rest added to the existing capital stock to augment output in the future. This paper, like the two quotations above, is about how to extend the definition of real income from the simple model to more complex and realistic models, and how to employ the extended definition as a guide for manipulating data available in the current year into aggregates that provide the user of the national accounts with the appropriate information about the economy. In particular, the paper is about how to interpret investment in real terms and how, in various circumstances, to fuse measures of

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[1]This paper is to a great extent the outcome of discussions with my colleague John Hartwick who virtually forced me to appreciate the significance of the Hamiltonian in national accounting. We disagree on certain matters, but his influence upon this paper is significant nonetheless. See Hartwick, (1990) and (1991).

[2]Hicks, (1946), 172.

[3]Weitzman, (1976), 159.

consumption and investment in their own units into a precise measure of real income.

Together, the opening quotations offer a definitive, simple and clear prescription: Real net national income is *both* the return to wealth and the Hamiltonian of an appropriately-chosen dynamic model of the economy. Understood properly, these interpretations of income are revealed to be entirely consistent. This paper is a critique—and ultimately a rejection—of these assertions.

The starting point and major premise of this paper is that the meaning of national income, or net national product, must be grasped by induction rather than by deduction. A meaning cannot be imposed *ex cathedra*. Mathematical elegance may signify usefulness in practice, but that must be proved so, and not just assumed. Ultimately, the meaning of national income is inherent in the purposes of the national accounts. It must be inferred from the way the term is used in charting business cycles, comparing prosperity among nations, observing industrial structure, measuring factor shares and so on. The "right" meaning is that which rationalizes the organization of data to answer the question or questions at hand. In particular, the right measure of saving or investment is that which conveys to the user of statistics of, for instance, economic growth what he hopes to learn about the progress of the economy over time. As there is a multiplicity of uses of the national accounts, real income may be interpreted as a family of concepts, each member of which is best for some particular purpose.[4]

I. Digression on Real Consumption as Welfare and as Productivity

Investment is necessarily forward-looking, but, to get to the bottom of the intrinsically dynamic problem of choosing a measure of saving or investment in real terms, it is convenient to digress briefly to the simpler atemporal problem of comparing apples and oranges in a measure of real consumption. As Hicks (1940) showed long ago, this is really two distinct problems conflated in a common usage of words. Real consumption may be designed as a measure of productivity or as a measure of welfare. When measuring productivity, statistics of real consumption track changes over time in the location of the production possibility curve. When measuring welfare, statistics of real consumption track movements of the bundles of goods consumed through the space of assumedly invariant indifference curves. As long as there is no investment, real consumption and real income are one and the same.

Suppose we want to measure the growth of real income between year 1 and year 2 for an economy with two consumption goods, apples (z) and oranges (x), and with no investment at all. Alternative measures of real income are compared in Figure 1 with the good z on the vertical axis and the good x on the vertical

[4]In speaking of income as a family of concepts, I am not referring to the usual components of the national accounts; national income at market prices, national income at factor cost, personal income and so on. I am referring to different ways of understanding the same statistic—real national income or net national product in real terms. Income as a measure of welfare, income as a measure of production capacity, and income denominated in one good are members of a family in this sense. So too are the concepts Y_W, Y_M, Y_U, Y_P and Y_H as defined in the text.

axis. The technology in the two years is represented by the production possibility curves, T^1 and T^2, and there is a common time-invariant set of indifference curves, among which U^1 is tangent to T^1 at α^1 and U^2 is tangent to T^2 at α^2. Equilibrium relative prices of oranges in terms of apples, p^1 and p^2, are the common tangents of U^1 and T^1 and of U^2 and T^2 respectively. If the money price of apples is held constant at 1 in both periods, then money income and income in units of apples are necessarily one and the same. Money incomes in this sense are represented in the figure as the projections of the common tangents unto the vertical axis. Specifically, money incomes in years 1 and 2 are

$$Y_M^1 \equiv z^1 + p^1 x^1 \tag{1}$$

and

$$Y_M^2 \equiv z^2 + p^2 x^2 \tag{2}$$

where z^1, z^2, x^1 and x^2 are apples and oranges produced and consumed per head in the two years. As this paper is not about inflation, the term money income will be used throughout in a context where some price—later on it will be the price of consumption goods—is held constant at 1, and relative prices of other goods are allowed to vary. Note particularly, that money income tracks neither welfare nor productive capacity; constancy of over time of money income is indicative neither of the absence of productivity change nor that the representative consumer remains on the same indifference curve.

To measure real income as productive capacity or as welfare, one must first choose a "reference" price defined as a set of *relative* prices of all goods in terms of the numeraire. It is customary and convenient to let the reference price be the actual price in some chosen "base year," but the reference price could be chosen arbitrarily. Then with p^1 as the chosen reference price (that is, with year 1 as the chosen base year), real incomes in years 1 and 2 as measures of productive capacity are defined as

$$Y_P^1 = \max_{\{z,\, x\}} z + p^1 x \quad \text{s.t.} \quad T^1(z, x) = 0 \tag{3}$$

and

$$Y_P^2 = \max_{\{z,\, x\}} z + p^1 x \quad \text{s.t.} \quad T^2(z, x) = 0 \tag{4}$$

so that $Y_P^1 = Y_M^1$ but $Y_P^2 \neq Y_M^2$.

Similarly real incomes as measures of welfare become

$$Y_U^1 = \min_{\{z,\, x\}} z + p^1 x \quad \text{s.t.} \quad U^2(z, x) \geq U^1 \tag{5}$$

and

$$Y_U^2 = \min_{\{z,\, x\}} z + p^1 x \quad \text{s.t.} \quad U(z, x) \geq U^2 \tag{6}$$

so that $Y_U^1 = Y_M^1$ but $Y_U^2 \neq Y_M^2$.

In words, Y_p in any year is the largest value of goods that could be obtained with the technology in that year and at the given reference price, while Y_U is the cheapest bundle of goods at the given reference prices that would leave the

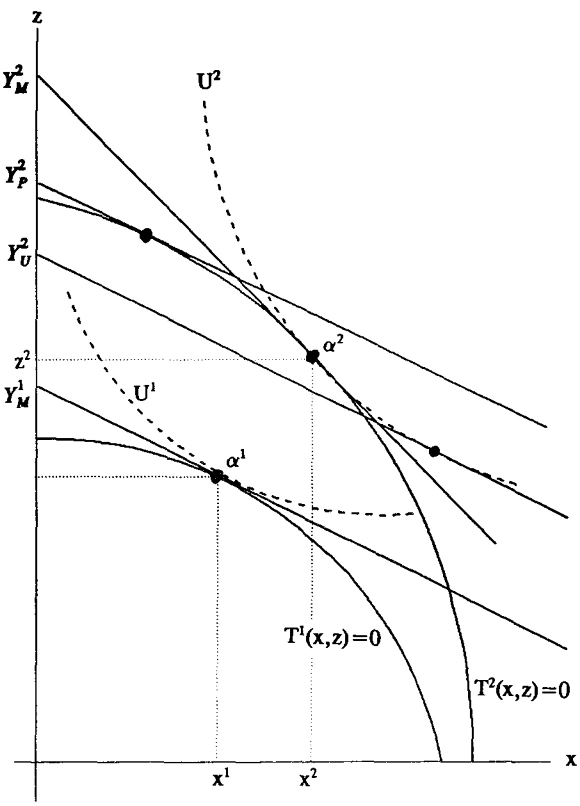

Figure 1

representative consumer no worse off than he actually was in that year. The values of Y_P^1, Y_P^2, Y_U^1 and Y_U^2 are shown in Figure 1 as distances or the vertical axis. Rates of economic growth as productivity and as welfare, are

$$(Y_P^2 - Y_P^1)/Y_P^1 \quad \text{and} \quad (Y_U^2 - Y_U^1)/Y_U^1$$

The latter is a well-defined cardinalization of utility; it increases whenever the representative consumer advances over time from a lower to a higher indifference curve.

The question at hand is what happens to these measures of real national income, the productivity measure and the welfare measure, when we allow for investment, when the two primary goods become not apples and oranges but consumption today and provision for the future. In this substitution, z becomes consumption, c; x becomes the change, $\dot{k}$, in the capital stock, k; the reference price becomes a rate of transformation between c and $\dot{k}$ in some chosen base year; c and $\dot{k}$ are looked upon as amounts of two homogenous, well-defined commodities. Productive capacity is easy, for equations (3) and (4) are *mutatis mutandis* unchanged. Indeed, for some purposes, real income is best interpreted as a measure of the location of the production possibility curve for c and $\dot{k}$ in the current year.

Income as a measure of welfare is not so easily generalized. One cannot write $U(x, z)$ when $z \equiv c$ and $x \equiv \dot{k}$, as one could when z was interpreted as apples and x was interpreted as oranges, because $\dot{k}$ does not contribute directly to utility. It contributes indirectly, and the same $\dot{k}$ in two different years may give rise to different flows of additional consumption goods. To generalize income as a measure of welfare, one must introduce an intertemporal measure of utility, a wealth-like measure encapsulating the benefit of a stream of consumption from now until the end of the world, and one must derive a shadow price of $\dot{k}$ as a rate of trade-off in use between this wealth-like measure and $\dot{k}$.

II. The Hamiltonian Measure of Income and the Return to Wealth

To generalize from the atemporal context we have been considering to an intertemporal context, we require replacements for utility, U, and for productive capacity, T. There are any number of plausible intertemporal generalizations of utility, but there is one very convenient interpretation for which (with appropriate assumptions about productivity) both of the quotations at the outset of this paper turn out to be correct. Let intertemporal utility be the present value of the entire time-stream of consumption discounted at an invariant real rate of interest. Specifically, if the current year is τ, this measure of utility becomes

$$W(\tau) \equiv \int_{\tau}^{\infty} c(t)\, e^{-r(t-\tau)}\, dt \qquad (7)$$

where r is the assumedly-invariant rate of interest, t is any time in the future and $c(t)$ is consumption in the year t. Note that $W(\tau)$ is at once a measure of intertemporal utility and a measure of wealth.

For continuous time and with two goods, c and $\dot{k}$, the production possibility frontier becomes a differential equation, $T(c(t), \dot{k}(t); k(t)) = 0$. A convenient simplification of this equation is

$$\dot{k}(t) = f(k(t)) - h(c(t)) \qquad (8)$$

where $f' > 0, f'' < 0, h' > 0$ and $h'' < 0$. The function f, which can be thought of as an ordinary production function, depends only on the stock of capital at any given time because population and labour force are assumed to be invariant. The function h, which is like an upward-sloping supply curve of new capital goods, is introduced to provide curvature to the production possibility frontier of c and $\dot{k}$ at any moment of time.

The representative consumer chooses a time stream of consumption from time τ until the end of the world to maximize intertemporal utility subject to every year's production constraint. Specifically, with an initial capital stock $k(\tau)$, he chooses a function $c(t)$ to maximize $W(\tau)$ in equation (7), subject to the intertemporal constraint in the differential equation (8). It is characteristic of such dynamic problems that the solution is discovered by a critical intermediate step. Instead of maximizing $W(\tau)$ all at once, the representative consumer may be looked upon as choosing $c(t)$ at each moment of time t (from τ to ∞) to maximize an expression called a Hamiltonian, written here as $Y_H(t)$ in conformity with other measures of income as defined above. Specifically, the Hamiltonian is

$$Y_H(t) \equiv c(t) + \pi(t)\dot{k}(t) \tag{9}$$

where $\pi(t)$ is the shadow price of $\dot{k}$ in terms of c at time t. In words, "the Hamiltonian measure of income is the sum of current consumption and discounted future consumption generated by the activity in the economy during the current year."

Two salient characteristics of the Hamiltonian justify treating the Hamiltonian as a measure of income and (within the simple model in this section) account for the claims in the quotations at the outset of this paper. When intertemporal utility and wealth are one and the same in equation (7) and when the representative consumer is seen as maximizing this measure of intertemporal utility subject to the technology in equation (8), it follows that;

(i) The shadow price, $\pi(t)$, in equation (9) is $\partial W/\partial \dot{k}$ as long as the entire time stream $c(t)$ is chosen to maximize $W(\tau)$. It is the present value of all changes over time in c, from τ to the end of the world, brought about by an increase in $\dot{k}$ at time t. Thus $Y_H(t)$ in equation (9) can be seen as the natural generalization of Y^1_M and Y^2_M in equations (1) and (2) to an intertemporal context.

(ii) The Hamiltonian is the return to wealth i.e.

$$Y_H(t) = rW(t) \tag{10}$$

where r is the discount factor in equation (7). One can think of the Hamiltonian measure of income as, to use the old fashioned terminology, the national dividend on the stock of human and physical capital. This is proved by Weitzman (1976), and is demonstrated somewhat more directly in Appendix 1 below. In this context, Hicks is entirely vindicated in describing a person's income as "the maximum he can consume during the week and still be as well off at the end of the week as he was at the beginning."

From here on, these desirable properties of the Hamiltonian disintegrate as the model is adjusted for various aspects of the economy that have so been assumed away: (a) the Hamiltonian is no longer the return to wealth, (b) the discrepancy is a point in the Hamiltonian's favour, for a thoroughgoing estimate of the return to wealth would be either fortune-telling or worthless altogether and (c) the Hamiltonian is a measure of money income in consumption units (like equations (1) and (2), not of real income as welfare or productive capacity. For convenience in exposition, define Y_w as the return to wealth, that is

$$Y_w \equiv rW \tag{11}$$

Thus, equation (10) can be interpreted as the statement that $Y_H = Y_w$. That statement is not true of other dynamic models of the economy.

III. The Capitalization of Future Technical Change

The Hamiltonian measure of income parts company from the return to wealth whenever economic growth cannot be directly attributed to capital formation. To see this, consider for the moment an "extreme case" of an economy with no capital formation but where exogenous technical change causes consumption to increase steadily at a rate of g percent per year, though the resources of the economy, land and labour, remain constant forever. By assumption, output and consumption are one and the same. Once again, let r be the rate of discount as a property of the intertemporal welfare function of the representative consumer. The value of wealth becomes

$$W(\tau) \equiv \int_{\tau}^{\infty} c(t)\, e^{-r(t-\tau)}\, dt = \int_{\tau}^{\infty} c(\tau)\, e^{-(r-g)(t-\tau)}\, dt = c(\tau)/(r-g) \qquad (12)$$

It is immediately evident that equation (10) is not valid in this case. As there are no state variables, the Hamiltonian is just equal to consumption.

$$Y_H(t) \equiv c(t) = (r-g)\, W(t) = r\, W(t) - g\, W(t) \qquad (13)$$

which is the return to wealth *less* that part of the return attributable to future technical change.

What is going on here? Why is the Hamiltonian equal to the return to wealth in the preceding model but not now? The answer would seem to depend on the reason why consumption increases over time. In the preceding model leading up to equation (10), consumption next year could be thought of as created by the stock of capital in existence today, and the net national product this year could be thought of as the sum of consumption this year and the value at today's prices of the newly-acquired capital goods. In the present model, consumption next year is autonomous; nothing done today affects tomorrow's output at all; the growth of consumption over time is an exogenous gift of nature, or of world-wide scientific progress, unconnected to the activity in the domestic economy today.

Consider carefully the definition of the Hamiltonian as "the sum of current consumption and discounted future consumption generated by the activity in the economy during the current year." The Hamiltonian is the return to wealth in so far as future consumption is procured with the stock of capital (and other resources) in existence today, it falls short of the return to wealth in so far as wealth, in equation (7), is the capitalization of future technical change. Weitzman's proof that $Y_H = rW$ was based on a model without technical change and for which the equality is, so far as I can tell, valid.

Nor is it necessary to abstract from current capital formation altogether. Consider an economy where the differential equation (8) representing the technology of the economy each year is replaced (suppressing t as an argument in c, k and $\dot{k}$) by

$$\dot{k} = A\, e^{gt} + f(k) - h(c) \qquad (14)$$

where the expression $A\, e^{gt}$ represents spontaneous future technical change and the rest of the equation remains as before. It is shown in Appendix 2 that the current value Hamiltonian becomes

$$Y_H(\tau) = r\, W(\tau) - \int_{\tau}^{\infty} [g/h']A\, e^{gt}\, e^{-r(t-\tau)}\, dt \qquad (15)$$

where the second expression on the right hand side of equation (15) can be interpreted as the present value of future technical change. Once again the Hamiltonian measure of income is less than the return to wealth though, in this case, it remains possible to augment consumption next year by investing today. Equation (15) implies that $Y_H(\tau) < Y_W(\tau)$ in the presence of exogenous future technical change.

IV. Should Income be Defined as The Return to Wealth?

There is one very strong reason why it should not. To know the return to wealth, one must know the magnitude of wealth itself, and, to know that, one must be able to predict the future from now until the end of the world. This may not be immediately evident to the reader because, in ordinary usage, the word wealth signifies the value of assets in the market today. But that is not the interpretation of wealth according to which income might conceivably be the return to wealth. For that proposition, wealth would have to be interpreted as the present value of future consumption regardless of why future consumption happens to be what it is. Consumption may be high ten years hence because workers will have become more productive or because of inventions that are unanticipated today. In either case, the present value of consumption ten years hence is a part of wealth today, and Y_W in equation (11) is the return to that comprehensive measure of wealth. It would, of course, be nice to have accurate statistics of the return to wealth, just as it would be nice to have accurate statistics of wealth as defined in equation (7). Such statistics would be even more useful than the statistics of income that are collected now, for they would foretell the future. The statistics we could actually construct on this principle might be worse than useless. By contrast, the Hamiltonian measure of income is a function of c, $\dot{k}$ and π all of which can be observed today.

There are circumstances, however, where wealth *is* more or less knowable today and where a measure of income as the return to wealth may be useful. Consider "Kuwait"—not the actual country as it is, but an idealized version with no resources other than a fixed stock of oil that is costless to discover, costless to produce and sold as required to finance the consumption each year. The idealized Kuwait is located in an idealized world where the rate of interest is fixed forever and the world price of oil, $\pi_s(t)$, appreciates each year at the rate of interest, as Hotelling said it must, because, otherwise, the oil might be sold off all at once to finance the purchase of interest-bearing assets.

$$\pi_s(t) = \pi_s(\tau)\, e^{r(t-\tau)} \tag{16}$$

Let $S(t)$ be Kuwait's stock of oil in the year t and let $\dot{S}(t)$ be, at once, the amount of oil sold (expressed negatively) and the change in the stock during the year. Clearly, consumption in Kuwait in the year t is just equal to $-\pi_s(t)\dot{S}(t)$, and the wealth of Kuwait is at once the value of the stock of oil in the ground and the present value of all future consumption.

$$W(\tau) = \pi_s(\tau)S(\tau) = \int_\tau^\infty -\pi_s(t)\, e^{-r(t-\tau)}\dot{S}(t)\, dt = \int_\tau^\infty c(t)\, e^{-r(t-\tau)}\, dt \tag{17}$$

The growth of wealth over time is

$$\dot{W}(\tau) = \pi_s(\tau)\dot{S}(\tau) + \dot{\pi}_s(\tau)S(\tau) = -c(\tau) + \dot{\pi}_s(\tau)S(\tau) \tag{18}$$

where $\dot{\pi}_s(\tau) = r\dot{\pi}_s(\tau)$ because the price of oil must increase at the going rate of interest. Thus, when income, $Y_W(\tau)$, is defined as the return to wealth, it must be the case that

$$Y_W(\tau) = rW(\tau) = r\pi_s(\tau)S(\tau) = \dot{\pi}_s(\tau)S(\tau) = c(\tau) + \dot{W}(\tau) \tag{19}$$

which is the sum of consumption plus the change in the course of the year of the *value* of the stock of oil. On this interpretation of income, future increases in the price of oil are capitalized as part of income today. Alternatively, if income is interpreted in the Hamiltonian sense as $Y_H = c(t) + \pi_s(t)\dot{S}(t)$, then the national income of Kuwait is automatically equal to 0 because $c(t) = -\pi_s(t)\dot{S}(t)$ signifying that consumption is just equal to the value of the oil that is extracted and sold. The national income of our hypothetical Kuwaitis either $rW(t)$ or 0 depending on whether national income is interpreted as the return to wealth or as the value of the stream of consumption attributable to activity in the current year.

Which of these interpretations is best? As there can be no God-given definition of anything, the answer to this question must turn on the usefulness of alternative definitions in organizing data to tell us what we hope to learn about the economy. The measure of national income as the return to wealth is presumably the more informative to the people of Kuwait, for there would not be much point in telling them that their national income is zero when they are, in fact, quite prosperous and have every expectation of remaining so for some time. Note however that the entire return to their wealth is in the form of capital gains; $Y_w(t) = \dot{\pi}_s(t)S(t)$. The mechanism of international trade enables Kuwait to convert a fixed stock of asset into a stream of consumption goods forever. The rise over time in the price of the assets creates an annual capital gain that can be consumed without diminishing the value of the stock in units of consumption goods.

The capital gain accrues to Kuwait rather than to the world as a whole, for it is matched by an equal and opposite capital loss to the producers of the consumption goods that Kuwait imports. The price of oil, π_s, is after all a relative price enumerated in consumption goods. By definition, that price cannot go up unless the relative prices of consumption goods and of the factors of production that make them go down. The capital gain in Kuwait has to be matched by an equal and opposite capital loss in other countries. For the world as a whole, the capital gain, $\dot{\pi}_s(t)S(t)$, is really no gain at all, and Kuwait's contribution to world national income, defined with full allowance for the using up of the world's natural resources, is equal to zero.

In principle, the Hamiltonian measure of income of an oil-producing country is easily generalized to take account of ordinary capital goods, undiscovered oil, the cost of discovery, and the cost of extracting oil from the ground. Consider a slightly more realistic Kuwait. This Kuwait is still without indigenous labour but it does own ordinary capital goods, k, and may acquire more in the course of the year. It begins the year with a "discovered" stock of oil, S, and an "undiscovered" stock of oil, U. In the course of the year, Kuwait discovers an amount D and extracts an amount E. By definition $\dot{U}=-D$ and $S=D-E$. On the Hamiltonian interpretation, the national income of Kuwait becomes

$$Y_H=c+\pi_k\dot{k}+\pi_U\dot{U}+\pi_s\dot{S}=c+\pi_k\dot{k}+(\pi_s-\pi_U)D-\pi_sE \qquad (20)$$

where π_k is the price of ordinary capital goods, π_s is the price of oil and π_U is the shadow price of undiscovered oil, the amount of money that one would pay God to hide an extra barrel of oil under the ground.[5] The term $(\pi_s-\pi_U)D$ is the contribution to the national income of the discovery of oil, where the cost of discovery must equal the difference between the shadow prices of discovered and undiscovered oil. The final term is the cost of the using up of part of the national stock of oil.

In the absence of borrowing, all consumption, investment and exploration cost must be financed by the sale of oil or the return to capital. Thus

$$c+\pi_k\dot{k}+(\pi_s-\pi_U)D=r\pi_kk+\pi_sE \qquad (21)$$

so that $Y_H=r\pi_kk$, signifying that the national income is just the return to capital. If as much oil is discovered as is exported, then $D=E=-\dot{U}$, $\dot{S}=0$, and the national income becomes

$$Y_H=c+\pi_k\dot{k}+\pi_U\dot{U} \qquad (22)$$

A country that discovers as much oil as it exports has not preserved its resource base in tact. There is a real social cost to a reduction in the stock of undiscovered oil. The general principle would seem to be this: Where, as in an oil-producing country, one can identify the stock of wealth from which future consumption is presumed to flow, it might be best to measure national income as the return to national wealth. Otherwise, future technical change and the beneficial capital gains from the repricing of resources are excluded from national income and the Hamiltonian measure is to be preferred.

V. Is the Hamiltonian a Measure of Real Income or of Money Income?

As mentioned above, one can identify two distinct interpretations of money income. Money income may be measured each year at current prices regardless of whether the price level is changing over time, or money income may be measured as current income in dollars deflated by the current price of one numeraire good, which, in this paper, is the supposedly all-purpose consumption good. We are ignoring the first of these interpretations, for none of the measures of income we are comparing is money income in that sense. However, the Hamiltonian measure of income in equation (9) is obviously a measure of money income in the latter

[5]As it is difficult bordering on impossible to determine the shadow price of undiscovered oil, estimates of Y_H in accordance with equation (20) are usually based on one of two extreme simplifications. The stock of undiscovered oil may be ignored, in which case, no distinction is drawn between discovery of oil and production of an ordinary good; Y_H would then be estimated as $c+\pi_k\dot{k}+\pi_s(E-D)$. Alternatively, the discovery of oil may be ignored, in which case Y_H would be estimated as $c+\pi_k\dot{k}+\pi_sE$. Repetto (1989) adopted the former procedure in his calculation of the depreciation of resource depletion in Indonesia. I adopted the latter procedure in Usher (1980), 300. Repetto discusses the matter in some detail.

sense because Y_H is invariant when c and $(\pi\dot{k})$ are invariant, regardless of what happens to $\dot{k}$ alone. With appropriate changes over time in π, Y_H could decrease over time though *both* c and $\dot{k}$ increase, or Y_H could increase though both c and $\dot{k}$ decrease. Thus, as a generalization of equation (1) rather than equation (3) or equation (5), the Hamiltonian measure of income is an ideal indicator of neither welfare or productive capacity and may be defective as a basis for the measurement of a country's economic growth.

One might even differentiate between "nominal" and "real" Hamiltonians, the latter being

$$Y_{HR}(t) \equiv c(t) + \pi(\tau)\dot{k}(t) \tag{23}$$

where t is the current year and τ is some chosen base year. A time series of $Y_{HR}(t)$ could be constructed for any given base year τ and reference price $\pi(\tau)$.

This difficulty is compounded if we abandon the identification of intertemporal welfare with wealth. Even in a world with only one consumption good, there is some question as to whether wealth, as represented by equation (7), is the appropriate intertemporal objective function. The implied indifference curves over consumption at any two periods of time would be downward sloping straight lines, implying that there would be no consumption whatsoever in any year for which the rate of return to investment exceeded the constant, taste-generated rate of interest, r. It was to avoid that unacceptable possibility that I introduced the concave function $h(c(t))$ in equation (8); the function h can be specified to keep the rate of discount well above r whenever c approaches zero.

This difficulty could be overcome to some extent, and the Hamiltonian made to look more like a measure of welfare, if $c(t)$ in equation (7) were replaced by an atemporal utility function $u(c(t))$, where $u' > 0$ and $u'' < 0$, in a new dynamic model with diminishing marginal rates of substitution between amounts of consumption at different moments of time. The new model would be constructed by changing the intertemporal objective function from wealth, $W(\tau)$ in equation (7), to $\hat{W}(\tau)$ defined as

$$\hat{W}(\tau) = \int_{\tau}^{\infty} u(c(t))\, e^{-a(t-\tau)}\, dt \tag{24}$$

where a is the rate of discount on utils.[6] The representative consumer chooses a time stream of $c(t)$ to maximize $\hat{W}$ subject to a technology that can be represented at each moment of time by the differential equation (8). The Hamiltonian of the new dynamic problem becomes

$$u(c(t)) - \hat{\pi}(t)\dot{k}(t) \tag{25}$$

[6]Even the objective function in equation (24) is arbitrarily imposed upon the consumer. In the conceptual experiment by which an intertemporal welfare function is "observed," a representative consumer is assumed to rank all possible time-streams of consumption, $c(t)$, just as alternative bundles of goods are ranked in the construction of ordinary indifference curves to summarize the representative consumer' answers to a long series of questions of the form, "Do you prefer this bundle of goods to that bundle of goods?" From the answers to such questions, the inquirer discovers whether any bundle $c_\alpha(t)$ is preferred, dispreferred or indifferent to any other bundle $c_\beta(t)$. Nothing in these answers enables the inquirer to determine by how much the one bundle is preferred or dispreferred to another. Specifically, the numbering of the intertemporal indifference curves requires more information than can be gleaned from our imaginary questionnaire. The economic significance of the intertemporal

where $\hat{\pi}$ is the shadow price of $\dot{k}$ with respect to $\hat{W}$, rather than W, as the numeraire. However, the new Hamiltonian would no longer be an indicator of income, as the term is commonly understood, because it would be in the wrong units. Deflation each year by $\delta u/\delta c$ (the current value of the derivative of utility with respect to consumption), would convert the Hamiltonian of equation (25) to consumption units, but that does not provide a consistent time series of net national product in consumption units because, as c changes, the correction factor u/u_c is not a fixed multiple of c.

VI. A Distorted Economy

The Hamiltonian measure of national income becomes especially problematic in a distorted economy where the shadow price of $\dot{k}$ in terms of c is different in production and in use. By construction, the Hamiltonian is a property of an economy on an efficient path undistorted by taxes, tariffs or other impediments to the free flow of goods and services; a representative consumer maximizes intertemporal welfare subject only to a sequence of annual production constraints. Actual economies are not like that. Outcomes emerge from the interactions of many agents. Governments must distort markets by taxation and may distort markets deliberately for the benefit of one group at the expense of another. The one well-defined shadow price $\pi(t)$ in equation (9) gives way to a pair of shadow prices; $\pi_W(t)$ reflecting rates of substitution in use and $\pi_P(t)$ reflecting rates of transformation in production.

This is shown in figure 2, a demand and supply diagram for investment, $\dot{k}$, with consumption, c, as the numeraire and the relative price of $\dot{k}$ in terms of c on the vertical axis. The demand curve shows how the marginal valuation of investment, $\pi_W(t)$, diminishes as the amount of investment in the current year increases, and the amount of consumption decreases accordingly, where the marginal valuation is the present value of all extra future consumption that could be obtained from a given increase in the capital stock in the current year. The supply curve shows how the marginal cost of investment, $\pi_p(t)$ increases with the amount of investment, where the marginal cost is the amount of current consumption that must be forgone to acquire the given increase in the capital stock. The slope of the supply curve is a reflection of the assumed concavity of the function $h(c)$ in equation (8).

Suppose a corporation income tax places a wedge between the rate of return to investment in production and in use. Without this distortion, investment would

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measure of welfare—as expressed in units of consumption goods in equation (7) or in utils in equation (24)—is that wealth is *assumed* to serve as a cardinalization of these intertemporal indifference curves. The time-stream of consumption $c_\alpha(t)$ lies on a higher indifference curve than the time-stream of consumption $c_\beta(t)$ if and only if the corresponding measure of wealth is larger. The key assumption in the wealth-like measures of intertemporal welfare in equations (7) and (24) is that the intertemporal utility function W is the weighted sum of values of a postulated temporal utility function $u(c)$. Actual tastes need not conform to this assumption. Intertemporal choice need not correspond to the maximization of any wealth-like measure in which the welfare derived from a stream of consumption over many years is representable as the weighted sum of the values each year of an assumedly-invariant temporal utility function. However, this assumption is crucial for the analogy between income and the Hamiltonian.

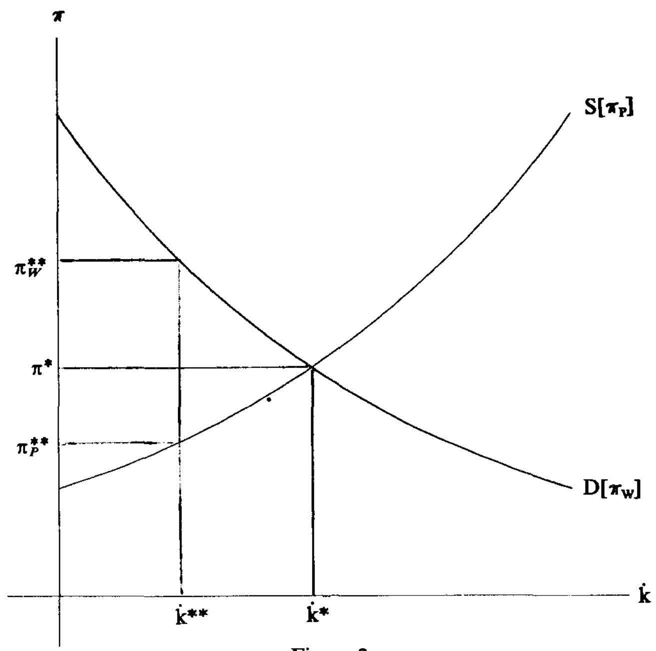

Figure 2

be $\dot{k}^*$ and the common price of investment in production and in use would be π^*. The distortion causes a reduction in investment to $\dot{k}^{**}$ and opens a gap between the demand price π_W^{**} and the supply price π_P^{**}. Which of these prices is appropriate for the Hamiltonian measure of income? Not the undistorted price, π^*, because actual c and $\dot{k}$ are not what they would be in the absence of distortions. Possibly the demand price, π_W^{**}, but this might lead to a very large value of investment and to a grossly misleading picture of economic growth if the supply curve is steep and the distortion considerable. Possibly the supply price, π_P^{**}, which is at least available in the current year, but it is arguable that $c+\pi_P^{**}\dot{k}$ is a measure of instantaneous productive capacity and not really a Hamiltonian at all.

The Hamiltonian measure of income tends to crumble when the optimizing assumptions that gave rise to the Hamiltonian are violated. Perhaps the Hamiltonian can be restored. I have seen no attempt at such a restoration.

VII. Depreciation, Obsolescence and Capital Gains

Imagine a burst of scientific discovery with no immediate effect upon the productive capacity of the economy, but expected to make most people better off tomorrow than they would otherwise be. There is, by definition, a change in wealth as the present value of future consumption, including the consumption of people who are as yet unborn. A measure of income as the return to wealth would reflect this change automatically. A Hamiltonian measure may or may not be affected and the direction of the effect, if any, is ambiguous. The shadow price π

may remain unchanged, in which case Y_H remains unchanged as well. Otherwise Y_H may rise or fall depending on whether the anticipated technical change increases or decreases the present value of the stream of incremental future consumption resulting from a given increase in new capital goods produced today. Anticipated future technical change may diminish Y_H today by lowering the current value of newly-produced capital goods which must in time compete with better types of capital goods that the new technology brings forth.

In the usual procedure for measuring net national product, gross investment is reduced by the value of depreciation, where depreciation is a measure of the loss of value of the existing capital stock due to aging or to scrapping of obsolete equipment. In measuring depreciation, statisticians accept market valuation and are not inclined to ask why capital is scrapped or why it declines in value over time at the rate it does. Though there may be no practical alternative to this procedure, it should be recognized that the procedure can be perverse in causing the national income as it is measured to *fall* as a consequence of beneficial technical change, by recording some negative impacts on wealth while other positive impacts are ignored. The crux of the problem is that there are two definitions of income in play. Sensible statisticians who would have no truck with models or theorizing may pick and choose among alternative ways of measuring the components of national income on the basis of *ad hoc* arguments associated, now with one concept of income, now with another, to create a hybrid statistic corresponding to no clear principle and serving no purpose well. Unable to measure Y_W and unwilling to incorporate future technical change in current income, we settle for Y_H but fail to draw the full implications of that choice, specifically, that the term $\dot{\pi}k$ plays no role in the Hamiltonian. I think the appropriate procedure for converting gross to net national product within a Hamiltonian framework is to limit the measure of depreciation to the actual deterioration of the capital stock in the course of the year, ignoring obsolescence and accepting the discrepancy between this interpretation of depreciation and the interpretation that is appropriate for the balance sheet of the firm.

It is tempting to imagine that the difference between Y_W and Y_H is just capital gains, in which case the choice between these interpretations of income would boil down to deciding whether capital gains should or should not be included as part of real income. Of course, one could define capital gains as $Y_W - Y_H$, but that would not conform to the ordinary usage of the term, for not all future benefits are reflected in the value of capital today, and not all capital gains are indicative of future benefits.

As usually understood, a capital gain or loss is the increase or decrease over the year in the value of the capital goods that were available at the beginning of the year. A capital gain or loss may reflect some or all of the following: (i) the mere passage of time, as when an asset is expected to yield an especially high rate of return in some future year which, by definition, is closer at the end of the current year than it was at the beginning; (ii) the discovery during the year that the future yield of the capital good is greater than had been supposed; (iii) the accumulation of earnings not disbursed in the current year; (iv) changes in taste for the types of consumption goods made with the capital good in question; (v) changes in long-term interest rates; and (vi) the discovery during the year of new types of capital goods that are similar in their role in production to the capital good in question, but cheaper or more productive.

The first type of capital gain is excluded from the Hamiltonian measure of income and from the measures of income in the official national accounts because of the essential "this-yearness" of national income. National accountants do not want to say that income this year is higher than income last year because a bonanza due five years hence has become one year closer in the course of the current year. Let the value of the bonanza be B. Its contribution to wealth last year was $B/(1+r)^6$. Its contribution to wealth this year is $B/(1+r)^5$. Thus the wealth associated with the bonanza grows by $rB/(1+r)^6$, and income measured as the return to wealth grows accordingly. Practice in national accounting does not include this as part of income. Kuwait, as discussed above, might be an exception. For Kuwait, it may be reasonable to measure income as the return to wealth because the relevant future *is* assumed to be known today. Otherwise, the construction of statistics of Y_W would be an exercise in prophecy, not economics.

The second type of capital gains would be excluded for similar reasons. National accountants hold to the distinction between what is produced today and what is merely learned about the future. The Hamiltanian, $c+\pi\dot{k}$, explicitly excludes the term $\dot{\pi}k$ reflecting the change in the value, as distinct from the amount, of capital goods. The third type of capital gains is excluded because the return to capital is already accounted for elsewhere in the national accounts. The fourth is excluded because changes in national income are supposed to reflect changes in quantities produced, not changes in prices. The fifth would be excluded from the Hamiltonian measure of income because the measure takes no account of *why* the demand price of capital goods does or does not change over time. The sixth type—strictly speaking, a capital loss rather than a capital gain—ought also to be ignored in the Hamiltonian measure of income, but might be included in depreciation by accident, for obsolescence may be mistaken for the physical deterioration of capital goods.[7]

VIII. The Stock of Knowledge and the Stock of People

So far, both of these stocks have been ignored. Both could, in principle, be incorporated—analogously to physical capital or stocks or natural resources—into a Hamiltonian-type measure of income. Formally, the stock of knowledge might be treated as a second kind of capital good. The measure of income would become

$$Y_H = c + \pi_k\dot{k} + \pi_Q\dot{Q} \tag{26}$$

where Q is the quantity of knowledge and π_Q is its marginal valuation. In practice, the closest statistical representation of $\dot{Q}$ would be the expenditure on research and development which inevitably misses a great deal of knowledge-generating activity during the current year. The addition of "knowledge" as a state variable

[7]For the opposite view, see Eisner, (1988), 1624. I find it difficult to take issue with Eisner on this matter because I cannot tell what ideal of income is reflected in the statistics he constructs. For a critique of Eisner's views, see Hung (1991).

or factor of production would go some distance toward closing the gap between Y_W and Y_H, but the gap would not be closed altogether because knowledge that will in time be acquired from other countries or by plain dumb luck would still generate future income that cannot be attributed to a country's activities today.

Population growth gives rise to a different set of statistical problems. Population may grow because people live longer or because there is a surplus of births and immigrants over deaths and emigrants. Longevity might appropriately be incorporated as an argument in the utility function, in which case real income as an indicator of welfare might increase as a consequence of an increase in average life-expectancy. However, population growth *per se* might be looked upon as a shrinkage of the resource base per head, and as disinvestment leading to a decline in the standard of living. Much depends on the postulated utility function. I have written the atemporal utility function as $u(c)$ where c could equally well be total consumption or consumption per head because population was implicitly assumed to be constant. When population is not constant, the specification of the Hamiltonian depends critically on the interpretation of the utility function. If c and k are interpreted as consumption and capital stock per head and if the production function is assumed to exhibit constant returns to scale, then population growth is equivalent to a fall in the capital stock, and it must lead to a fall in Y_W per head. Population growth becomes problematic if c is interpreted as total consumption regardless of the number of people among whom the total consumption must be shared. A negative imputation for population growth is presented in Usher, (1980).

IX. Conclusion

One may identify three stages of definition for a term such as real national income: the general, half-specified meaning of the term in common use, its exact meaning as a component of a well-articulated model of the economy, and the procedure by which primary data are assembled and processed in the construction of statistics. Ideally, there are tight bonds between stages. In practice, the bonds are looser than we would like. This paper has been about the middle stage and its connections up and down. As a meaning within a model, the Hamiltonian interpretation of income turns out to be less compelling than its proponents have claimed, but of considerable interest nonetheless.

Long ago, Irving Fisher argued that the only useful concepts for national accounting are consumption (which he called "income") and wealth, and that no intermediate, income-like concept made any sense at all.[8] What might be called the Hamiltonian lesson is that there is a distinct and useful third concept: income as the consumption equivalent of the sum of present and future utilities generated

[8] Irving Fisher has argued that it is wrong to "regard 'savings' as income in the year the savings are accumulated.... The nature of the fallacy is seen as soon as we translate from money to other instruments. If a man saves up money and purchases an automobile, it is clearly double counting to call the automobile obtained 'real income', and then include its subsequent uses in the real income in ensuing years... ; it is always double counting to include the instrument and its uses. The saving may be invested in land or in confectionery. The only true income is the use of the land or the use of the confectionery." Fisher, (1906), 108–109.

within the current year. Unlike wealth in equations (7) and (24), and unlike income as the return to wealth, the Hamiltonian measure of national income can be constructed with the information at the disposal of the statistician during the current year. However, the Hamiltonian measure of income is not useful because it reflects the return to wealth. It is useful because it does not do so.

APPENDIX I

A proof that the Hamiltonian is equal to the return to wealth in a dynamic model where the intertemporal objective function is

$$W(\tau)=\int_{\tau}^{\infty} c(t)\, e^{-r(t-\tau)}\, dt$$

and the constraint is

$$\dot{k}(t)=f(k(t))-h(c(t))$$

Proof. The Hamiltonian becomes

$$H(t)=c(t)\, e^{-r(t-\tau)}+\phi(t)[f(k(t))-h(c(t))]$$

The optimality conditions are:

$$H_c(t)=e^{-r(t-\tau)}-\phi(t)h'(c(t))=0$$

$$\dot{k}(t)=-H_{\phi}=f(k(t)-h(c(t))$$

$$\dot{\phi}(t)=-H_k=-\phi(t)f'(k(t))$$

and

$$H(\infty)=0$$

which is a variant of the transversality condition.
It follows (suppressing the representation of functions as dependent on t) that

$$\begin{aligned} H(\tau) &\equiv H(\infty)-\int_{\tau}^{\infty}\frac{d}{dt}H\, dt \\ &=-\int_{\tau}^{\infty}[-cr\, e^{-r(t-\tau)}+\dot{c}\, e^{-r(t-\tau)}+\dot{\phi}(f-h)+\phi(f'\dot{k}-h'\dot{c})]\, dt \\ &=r\int_{\tau}^{\infty} c\, e^{-r(t-\tau)}\, dt-\int_{\tau}^{\infty}[\dot{c}(e^{-r(t-\tau)}-\phi h')]\, dt \\ &\qquad -\int_{\tau}^{\infty}[-\phi f'f+\phi f'h+\phi f'f-\phi f'h]\, dt \\ &=r\int_{\tau}^{\infty} c\, e^{-r(t-\tau)}\, dt \end{aligned}$$

Consequently,

$$H(\tau)=c(\tau)+\phi(\tau)\dot{k}(\tau)=r\int_{\tau}^{\infty} c(t)\, e^{-r(t-\tau)}\, dt=rW(\tau) \qquad \text{Q.E.D.}$$

This proof follows a similar demonstration in Davidson and Harris, (1981).

Appendix 2

A proof that

$$H(\tau)=rW(\tau)-\int_{\tau}^{\infty} [g/h']A\, e^{gt}\, e^{-r(t-\tau)}\, dt$$

when the constraint is changed to

$$\dot{k}=A\, e^{gt}+f(k)+h(c)$$

Proof. The Hamiltonian becomes

$$H(t)=c(t)\, e^{-r(t-\tau)}+\phi(t)[A\, e^{gt}+f(k(t))-h(c(t))]$$

The optimality conditions are the same as in Appendix 1, except that

$$\dot{k}=H_{\phi}=f-h+A\, e^{gt}$$

It then follows that

$$H(\tau)=-\int_{\tau}^{\infty} [-rc\, e^{-r(t-\tau)}+\dot{c}\, e^{-r(t-\tau)}+\dot{\phi}(f-h+A\, e^{gt})+\phi(f'\dot{k}-h'\dot{c}+gA\, e^{gt}]\, dt$$

$$=r\int_{\tau}^{\infty} c\, e^{-r(t-\tau)}\, dt-\int_{\tau}^{\infty}[\dot{c}\, e^{-r(t-\tau)}-\dot{c}\phi h']\, dt$$

$$-\int_{\tau}^{\infty}[-\phi f'(f-h+A\, e^{gt})+\phi(f'(A\, e^{gt}+f-h)+gA\, e^{gt})]\, dt$$

$$=r\int_{\tau}^{\infty} c\, e^{-rt}\, dt-\int_{\tau}^{\infty}[g/h']\, e^{-r(t-\tau)}A\, e^{gt}dt$$

$$=rW(\tau)-\int_{\tau}^{\infty}[g/h']A\, e^{gt}\, e^{-r(t-\tau)}\, dt \qquad \text{Q.E.D.}$$

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