INDEX NUMBERS AND THE COMPUTATION OF FACTOR PRODUCTIVITY[1]

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For purposes of analyzing the nature and meaning of index number formulas to be used in the calculation of factor productivity, a distinction is made between intertemporal comparison of factor productivity for a single country and contemporaneous comparison of factor productivity in two different countries. In the former case, the country in question is supposed ideally to be realizing fully its production possibilities, and the concern is seen as appraisal of shifts in such possibilities over time due to the advance of technological knowledge. Following Moorsteen such an advance is taken to be represented by the change in capacity to produce a standard mix of outputs per unit of a standard mix of inputs. Any mix might be standard, but those actually realized at the times in question are of particular interest.

The index number formulas to be applied then depend on the assumed shape of the functions representing production possibilities. The conventional practice of aggregating output arithmetically and inputs geometrically, for example, is in order where production possibilities are given by an elaborated Cobb–Douglas function, but achieves only more or less approximate results otherwise. The analysis necessarily bears also on the prices at which inputs and outputs are to be valued.

For the case of contemporaneous comparison of different countries, technological knowledge is taken ideally to be the same in the countries considered. Hence the concern is to gauge differences in production efficiency, i.e., realization of production possibilities. With production capacity understood to reflect any shortfall from possibilities, and hence production inefficiency in that sense, the analysis proceeds much as before, but given the fact of inefficiency determination of suitable prices for valuation of inputs and outputs becomes relatively difficult. Alternative expedients, none entirely satisfactory, are explored.

I. INTRODUCTION

The calculation of factor productivity is by now the subject of a voluminous methodological literature, but only rarely is there any systematic treatment of a basic aspect. I refer to the problem posed by the fact that the summary data compiled for such a calculation on factor inputs and outputs take the form of index numbers. That, of course, is almost always so, at least for inputs. For output as well as inputs, the index number problem arises in an especially striking way when, as is often the case, the calculation is more or less comprehensive of the economy generally. What is in question is how properly to construct and interpret the needed index numbers. Further inquiry into this matter would seem in order.

The calculation of factor productivity, if at all inclusive in scope, represents but an extension of that of real national income. Anyone inquiring into principles of index number construction and interpretation in the case of factor productivity data, therefore, must become indebted to the classic formulation of such principles for index numbers of real national income that we owe to Hicks (1940) and Samuelson (1950). Where writers on factor productivity have considered the

[1]A revised and expanded version of Section 2, Theoretic Considerations, in Abram Bergson, "Comparative Productivity and Efficiency in the U.S.S.R. and the U.S.A.," in Alexander Eckstein, ed., *Comparison of Economic Systems*, Berkeley, California, 1971. I draw on that earlier essay with the permission of the University of California Press and of the Regents of the University of California. Work on the present version was facilitated by an award (Contract G-1525) from the National Science Foundation. I am indebted to Martin Weitzman for a critical review of an earlier version. As this essay was about to go to press, the sad news came of the untimely death of the writer whose analysis I take as a point of departure. This is not the place to try to memorialize Richard Moorsteen, but perhaps I can begin to suggest the sense of loss that all who knew him and his work must feel if I dedicate this essay to his memory.

problem of index numbers, however, use has been made of diverse approaches. Among these, I find especially illuminating one adopted in a rather neglected essay of Richard Moorsteen (1961). I myself may have been able previously (Bergson, 1961) to contribute to this sort of analysis. While drawing attention to the approach in question, this essay may carry the analysis further in some interesting ways.

Writings on factor productivity are not always explicit as to what it is that the data compiled are supposed to indicate, but the concern fairly clearly is to measure, so far as possible, the joint impact on the volume of production, relatively to the corresponding factor inputs, of variations in two related phenomena: technological knowledge, as that affects production possibilities, and production efficiency, understood as the degree of realization of such possibilities.

That is the concern whether reference is to different periods in the same community, or as is also occasionally so, to different communities at the same time. But, for obvious reasons, the two phenomena are apt to differ in relative importance depending on which sort of comparison is made. Thus, variations in technological knowledge might well be the more important variable for a single community at different times, but differences in production efficiency should often be paramount where different communities are compared at the same time. At least that should be so among communities at all modern, for technological knowledge seems to be disseminated rather rapidly among such communities.

It may not be entirely unrealistic, therefore, to focus in turn here on two ideal cases. In each, two economic situations are compared, but in the first these situations relate to two different periods for the same community. Technological knowledge varies between the two intervals, but there is no inefficiency in production in either. In the second case, reference is to two communities at one and the same time. Technological knowledge is the same in the two communities, but they differ in respect of production efficiency.

The purpose for which productivity is measured, then, also varies. Where the calculation is for one community at two different times, the concern is to gauge the increase in technological knowledge over the interval in question. Where productivity is calculated for two communities at one time, the concern is rather to appraise comparative production efficiency in the two communities.

However unrealistic, delineation of two such ideal cases is analytically convenient here. Any consequential inefficiency must pose special difficulties for the calculation of factor productivity. We need not ponder long to see that that is so, though the fact is rarely considered. It is well, therefore, to inquire first into the principles applying when there are no such difficulties, and then to consider how those principles are affected by such difficulties.

II. The Coefficient of Factor Productivity

Turning to the comparison of two different periods for a single community where production is always fully efficient, I shall also assume for the present that in both periods the community in question produces but a single consumers' good X, the amounts of such output being X_1 in period 1 and X_2 in period 2. Similarly, only one capital good, I, is produced, and this in amounts I_1 and I_2 in the two periods.

For present purposes, I, is supposedly infinitely durable. Production takes place by use of a stock of the capital good amounting to K_1 in period 1 and K_2 in period 2, and a single kind of labor, L. Employment of the latter is L_1 and L_2 in the two periods.

Production possibilities in the community are, therefore, given for period 1 by the formula

$$F^1(X, I, L, K) = 0, \tag{2.1}$$

and for period 2 by the formula

$$F^2(X, I, L, K) = 0. \tag{2.2}$$

Thus, for any given volume of employment of the two factors, and any given output of one of the two goods, (2.1) indicates the maximum amount of the other that can be produced in period 1 with the technological knowledge available at that time. Similarly for (2.2) and period 2. Since production possibilities supposedly are always fully realized, the mixes of outputs and inputs actually experienced in 1 and 2 must also conform to (2.1) and (2.2), i.e.,

$$F^1(X_1, I_1, L_1, K_1) = 0; \quad F^2(X_2, I_2, L_2, K_2) = 0. \tag{2.3a,b}$$

Because of the advance of technological knowledge, however, some mixes open in 2 may not be realizable in 1. It is evidently just such a change in capacity that one would wish to gauge from factor productivity data for the case considered, but it is advisable to try to delineate somewhat more precisely than is customary the nature of the change in question. Consider some mix of outputs and inputs, X^*, I^*, L^*, K^*. Let us call this the standard mix. For 1, for any $\beta_1 > 0$, there should be some $\alpha_1 > 0$, such that

$$F^1(\alpha_1 X^*, \alpha_1 I^*, \beta_1 L^*, \beta_1 K^*) = 0. \tag{2.4}$$

Similarly for 2, for any $\beta_2 > 0$, there should be some $\alpha_2 > 0$, such that

$$F^2(\alpha_2 X^*, \alpha_2 I^*, \beta_2 L^*, \beta_2 K^*) = 0. \tag{2.5}$$

In other words, for any multiple of the standard inputs, the community should be capable in each period of producing some multiple of the standard outputs.

Consider the ratios

$$\pi_1 = \alpha_1/\beta_1; \quad \pi_2 = \alpha_2/\beta_2. \tag{2.6a,b}$$

The former indicates for period 1 and the latter for period 2 the volume of output per unit of inputs relative to that implied by the standrad mix. Consider now the further ratio

$$\pi_{12} = \pi_2/\pi_1. \tag{2.7}$$

This indicates the comparative output per unit of inputs in the two periods, and hence is properly referred to as the coefficient of Factor Productivity (CFP).

The coefficient may also be written in another form of interest:

$$\pi_{12} = (\alpha_2/\alpha_1)/(\beta_2/\beta_1). \tag{2.8}$$

As (2.8) underlines, while the individual period ratios, π_1 and π_2, depend on the levels of inputs and outputs in the standard mix, the CFP involves a comparison only of postulated inputs and of resultant outputs in the two periods. Only the structure of the standard mix matters, therefore, at this point.

So far as π_{12} represents relative factor productivity, it evidently also indicates the difference in capacity to produce outputs of the standard structure with inputs of the standard structure due to the difference between the two periods in technological knowledge. Thus, π_{12} varies directly with the degree to which production capacity in 2, owing to the advance of technological knowledge, has come to surpass that of 1.

That assumes, however, an absence of economies or diseconomies of scale in production. This assures that, given the standard mix, the magnitudes of π_1, π_2 and π_{12} are independent of the volume of inputs that is postulated for each period. In other words, π_1 is independent of β_1, π_2 of β_2 and π_{12} of both β_1 and β_2. With scale effects, π_{12} may still indicate the change in production capacity due to the advance of technological knowledge, but strictly speaking that presupposes that the volume of inputs considered in the two periods is the same, i.e., $\beta_1 = \beta_2$. Failing that, π_{12} depends not only on comparative technological knowledge, but on the comparative scale of inputs considered and the resulting scale economies in the two periods.[2]

Even without scale effects, reference is to inputs and outputs in each period that conform structurally to the standard mix. Conceivably, technological change might be of a neutral sort where the variation in production capacity does not depend on the structure of inputs or outputs. If technological change should be neutral in this sense, π_{12} would also be invariant of the structure of inputs and outputs, but so far as circumstances are otherwise π_{12} is relative to that structure. As reflected in production capacity, in other words, the advance of technological knowledge depends on the mix.[3]

In sum, the CFP reflects scale effects as well as technological change, but clearly it is still a metric on which we should wish to have empirical observations. So far as the coefficient depends on the standard mix, the moral must be that we should seek observations on as many mixes as possible. But note that we may take as standard one or the other of the mixes observed in the two periods, or some combination of the two. Such mixes are obviously of particular interest. How might we obtain the desired observations?

[2]With no scale effects, (1) and (2) are both homogeneous to the zero degree. It follows that π_1 is unaffected by a change from β_1 to $\lambda\beta_1$, for there is a corresponding change from α_1 to $\lambda\alpha_1$. Similarly, for π_2, β_2 and α_2. Note that if we take $\lambda = 1/\beta_1$, for period 1, and $\lambda = 1/\beta_2$ for period 2, π_1 and π_2 may be introduced explicitly in the production functions for the two periods:

$$(2.9a,b) \qquad F^1(\pi_1 X^*, \pi_1 I^*, L^*, K^*) = 0; \quad F^2(\pi_2 X^*, \pi_2 I^*, L^*, K^*) = 0.$$

[3]While the concept of neutral technological change is familiar in growth theory, reference is usually made to a community where only a single commodity is produced. The usage adopted here, however, seems to represent a natural extension to the case where there is more than one such commodity. Indeed, the two are closely related. Thus, it is easily seen that sufficient conditions for π_{12} to be invariant of the standard mix are: (i) F^1 and F^2 are each expressible as the sum of two separate linear homogeneous functions, one of outputs and the other of inputs; (ii) The change in technology represented by the shift from the input function of 1 to that of 2 is Hicks neutral, while the change in technology represented by the shift from the output function of 1 to that of 2 is of a similar character.

We owe chiefly to Moorsteen, I think, the formulation of the problem of factor productivity measurement in this plausible way. His 1961 essay, referred to above, is also illuminating on the solution, but it may be possible to offer a more complete analysis than he could in that pioneer inquiry of the specific issues that inevitably arise regarding the valuation standard or standards and the formulas to be applied in compiling the needed index numbers of inputs and outputs; and also of the relation between the resulting measures of factor productivity and the CFP for one or another mix. I turn to these questions.

III. Index Numbers of Inputs and Outputs

The needed index numbers of inputs and outputs supposedly are to be compiled from these data: (i) actual outputs and inputs during the two periods considered: X_1, I_1, L_1, K_1, and X_2, I_2, L_2, K_2; and (ii) corresponding prices p_1, q_1, w_1, r_1q_1, and p_2, q_2, w_2, r_2q_2. While q_1 and q_2 are the prices of capital goods in 1 and 2, r_1q_1 and r_2q_2 are the corresponding rental rates. In other words, the rate of interest is r_1 in 1 and r_2 in 2.

The question posed as to the valuation standard or standards to be applied in effect concerns the nature of these prices. As will appear, I fall in here with a familiar, though still not always accepted, view on that matter: ideally, the relative prices of goods produced should correspond in each period to their marginal rate of transformation. Similarly, the ratio of the wage rate to the rental for services of capital goods should correspond to the marginal rate of substitution between the two factors. The rates of transformation and substitution in the two periods are given by (2.1) and (2.2). Thus, we have

$$\text{MRT}_{XI} \equiv (F_1/F_2) = p/q, \tag{3.1}$$

and

$$\text{MRS}_{KL} \equiv (F_4/F_3) = rq/w. \tag{3.2}$$

Subscripts to F are used in the usual way to denote partial derivatives. Depending on the period, appropriate superscripts to F and subscripts to prices are understood.

Note that in the case of capital goods, (3.1) and (3.2) mean that valuation conforms to both of the two standards usually referred to: that represented by marginal cost and that represented by marginal value productivity. Thus, according to (3.1) the current output of the capital good is valued at marginal cost, or at least at a price that is the same in relation to such costs as the price for our consumers' good is to the marginal cost of that good. Similarly, with (3.2), the services of the capital good are valued as an input relatively to labor in accord with the comparative marginal productivities of the two factors.

As for the index number formulas to be applied and the relation of the resulting measures to the CFP, in inquiring into these matters we may conveniently begin with a special case: the production functions in (2.1) and (2.2) are assumed provisionally to have a specific form; particularly, (2.1) is supposed to reduce to

$$X + AI = CL^{\gamma}K^{1-\gamma}, \tag{3.3}$$

and (2.2), to

$$(3.4) \qquad X + BI = DL^{\rho}K^{1-\rho}.$$

In each period, therefore, "separability" of output and inputs obtains, and the marginal rate of transformation between the two goods is independent of the volume and structure of both inputs and outputs. That is also true of the elasticity of substitution (σ) between factors, which is everywhere equal to unity. In other words, reference is simply to a variant of the usual Cobb–Douglas function, aggregation of products, such as is assumed in the latter implicitly, being here represented explicitly.[4]

The case is illustrated in Figure 1. In 1.2 are shown the mixes of inputs observed in periods 1 and 2, g_1 and g_2, and the corresponding production isoquants, f^1f^1 and f^2f^2. Given by (3.3), f^1f^1 represents for period 1 alternative mixes of inputs yielding the same aggregate output as does g_1. Similarly for f^2f^2, though reference here is to (3.4), period 2, and g_2. In Figure 1.1 are shown the mixes of outputs observed in the two periods, G_1 and G_2, and the corresponding transformation schedules, F^1F^1 and F^2F^2. The transformation schedule F^1F^1 indicates for period 1 alternative mixes of outputs producible with the mix of inputs observed in that period or an equivalent mix. Similarly for F^2F^2, and period 2. As given by (3.3) and (3.4) both schedules are linear.

Even in the special case considered, as noted technological change is apt not to be neutral. Hence, as apparent from the figure, π_{12} depends on the standard mix. Four such mixes are of particular interest:

(i) X_2, I_2, L_2, K_2, i.e., outputs and inputs observed in 2;
(ii) X_1, I_1, L_1, K_1, i.e., outputs and inputs observed in 1;
(iii) X_2, I_2, L_1, K_1, i.e., outputs observed in 2 and inputs observed in 1;
(iv) X_1, I_1, L_2, K_2, i.e., outputs observed in 1 and inputs observed in 2.

[4]As well known, formulas such as (3.3) and (3.4) would in fact obtain if production in all industries conformed to one and the same Cobb–Douglas function, apart from the dimensional constant. That is hardly realistic, but (3.3) and (3.4) still serve here as a convenient point of departure.

As Professor Paul A. Samuelson has pointed out to me, (3.3) and (3.4) strictly speaking could not in any case fully represent production possibilities for a single community at two historically different times. Thus, in period 2 technological knowledge presumably would have increased or at least not have decreased over that in period 1, but, as is not difficult to see, while (3.4) might dominate (3.3) generally, there must be some input and output mixes for which (3.3) would be more productive than (3.4). For such mixes, therefore, (3.3), rather than being superceded by (3.4), still prevails as a representation of production possibilities.

This, however, does not preclude use of (3.3) and (3.4) here to represent production possibilities in periods 1 and 2 respectively. Rather the formulas may be so used, but simply on the understanding that for any mixes of inputs and outputs for which the implied CFP turns out to be < 1, that coefficient nevertheless has a magnitude equal to unity. As indicated, of particular interest here are the alternative mixes of inputs and outputs actually experienced in the two periods. Use of (3.3) and (3.4) to represent production possibilities in respect to those mixes rests on the foregoing understanding.

Assuming that (3.3) and (3.4) imply a CFP ≥ 1 for some mixes, for what mixes might those formulas nevertheless imply a CFP < 1? That, of course, is an empirical matter, but note that the possibility of such anomolous results arises from the variation in the production function over time in respect of the exponential coefficients for factor inputs and the constant terms relating to outputs. Also, unless such variation is very great, as readily seen, a CFP that is appreciably greater than unity for some mixes is apt to be associated with a CFP ≥ 1 for a wide range of neighboring mixes as well.

Note, that even where the implied CFP ≥ 1, (3.3) and (3.4) satisfy the first but not the second condition given in note 3, above, for π_{12} to be independent of the standard mix.

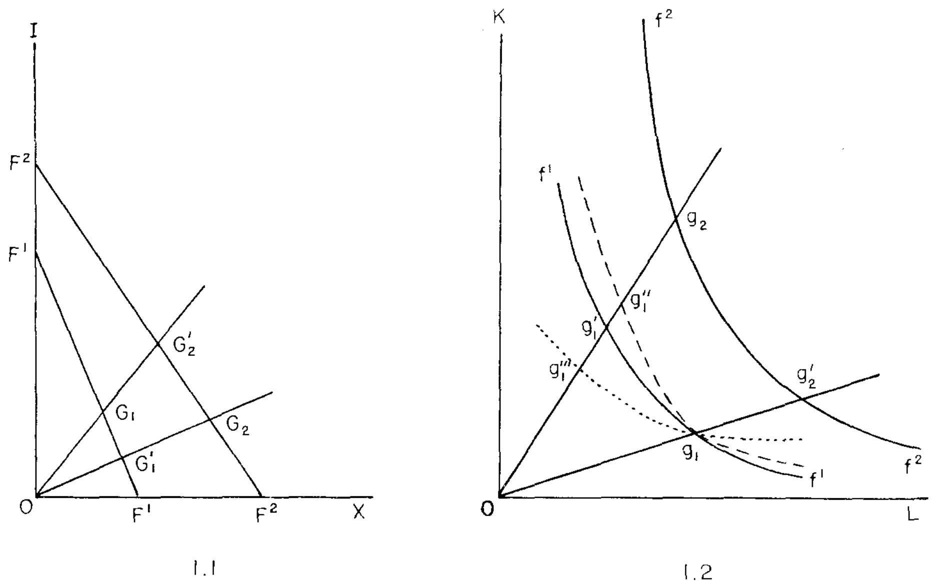

Figure 1. Output (1.1) and Input (1.2) Mixes in Periods 1 and 2

In Table 1, I show for each such standard mix the corresponding value of π_{12} as indicated by (2.8). Thus, as (2.8) requires, π_{12} is obtained for each standard mix by comparing for the two periods a relative of outputs with a relative of inputs, each of the standard structure. In the table, I also show index numbers calculated from observed prices and quantities that are supposed to correspond to the relatives of outputs and inputs conforming to (2.8). In other words, for each standard mix, by taking the ratio of the indicated index number of outputs to that of inputs, we obtain an observation on the corresponding π_{12}.

TABLE 1

CALCULATION OF π_{12} FOR ALTERNATIVE STANDARD MIXES

		Measurement in terms of observed prices and quantities	
Standard mix (1)	π_{12} (2)	Index of outputs (3)	Index of inputs[a] (4)
(i) X_2, I_2, L_2, K_2	$\frac{OG_2}{OG'_1} \div \frac{Og_2}{Og'_1}$	$\frac{p_1X_2+q_1I_2}{p_1X_1+q_1I_1}$	$\frac{L_2^\gamma K_2^{1-\gamma}}{L_1^\gamma K_1^{1-\gamma}}$
(ii) X_1, I_1, L_1, K_1	$\frac{OG'_2}{OG_1} \div \frac{Og'_2}{Og_1}$	$\frac{p_2X_2+q_2I_2}{p_2X_1+q_2I_1}$	$\frac{L_2^\rho K_2^{1-\rho}}{L_1^\rho K_1^{1-\rho}}$
(iii) X_2, I_2, L_1, K_1	$\frac{OG_2}{OG'_1} \div \frac{Og'_2}{Og_1}$	$\frac{p_1X_2+q_1I_2}{p_1X_1+q_1I_1}$	$\frac{L_2^\rho K_2^{1-\rho}}{L_1^\rho K_1^{1-\rho}}$
(iv) X_1, I_1, L_2, K_2	$\frac{OG'_2}{OG_1} \div \frac{Og_2}{Og'_1}$	$\frac{p_2X_2+q_2I_2}{p_2X_1+q_2I_1}$	$\frac{L_2^\gamma K_2^{1-\gamma}}{L_1^\gamma K_1^{1-\gamma}}$

[a] $\gamma = w_1L_1/(w_1L_1+r_1q_1K_1)$; $\rho = w_2L_2/(w_2L_2+r_2q_2K_2)$.

Index number formulas, of course, are in effect procedures for aggregation. In the calculation of factor productivity, so far as aggregation is considered explicitly, reference is usually to factor inputs, and as is proper the aggregation then conforms to the nature of the production function that is assumed. Although perhaps not always clearly grasped, the same principle evidently must apply to outputs as well. Moreover, with production functions given by (3.3) and (3.4) and prices as assumed, the appropriate index number formulas clearly must be of the sorts set forth in Table 1, but the corollary may deserve underlining that while aggregation of inputs is geometric that of outputs is arithmetic. It is sometimes assumed that one and the same aggregation procedure is indicated for both inputs and outputs. Note too that with outputs or inputs observed in one period as standard, the aggregation entails use of weights that relate to the other period—prices in the case of outputs and income shares in the case of inputs. This rather paradoxical feature of the calculation also follows from the nature of the production functions considered. Indeed, given these functions, the tabulated index number formulas turn out to be ideally appropriate in the sense that the resulting measures of π_{12} correspond precisely to those indicated by (2.8). The proof is readily found and is left to the reader.[5]

To try now to be more realistic, several departures from (3.3) and (3.4) are of interest:

(i) A transformation locus is ordinarily thought to be curvilinear rather than linear. Without any knowledge of its precise shape, we may still fall back on arithmetic aggregation in compiling index numbers of comparative outputs, but the correspondence to the output relatives indicated by (2.8) is now approximate, rather than exact. If the transformation locus is curvilinear, however, it is usually thought to be concave from below. Given that, the resulting error in the calculated π_{12} is predictable; that is, as can be seen at once, the calculated π_{12} is biased in favor of the community whose price weights are used in the aggregation of output. See Table 2, col. (2).

(ii) While I have assumed unity elasticity of substitution between factors, there are reasons to think that for an isoquant such as that considered σ might well be less than unity.[6] If that is so, and σ is known, the formulas for index numbers of inputs in Table 1 should be modified accordingly. Thus, the aggregation called for in that case is simply that given by the well-known CES production function. If σ is not known, and the formulas in Table 1 are still applied, the calculated π_{12} is subject to a further bias. Specifically, relative inputs tend to be overstated for the community whose inputs are taken as a standard, and the calculated π_{12} is biased accordingly. For example, with $\sigma < 1$, and g_2 as the standard for inputs, relative inputs according to (2.8) are indicated by (Og_2/Og_1'') in Figure 2.2. The corresponding index number formula in Table 1, however, still yields the larger ratio

[5]Hint: Given formulas (3.3) and (3.4), each transformation schedule and each isoquant in question is represented by a linear homogeneous function. Along any ray, the magnitude of such a function varies proportionately with its arguments. At the same time, with product and factor prices determined as assumed, alternative mixes of outputs which have the same total value arithmetically in terms of a community's own prices must be on one and the same transformation schedule for that community. Alternative mixes of inputs which have the same total value geometrically in terms of a community's own factor shares must be on one and the same isoquant for that community.

[6]See Arrow *et al.* (1961); David and van de Klundert (1965); Weitzman (1970).

(Og_2/Og_1').[7] It also follows that the calculated π_{12} is biased against period 2. More generally, the bias is as shown in Table 2, col. (3).

TABLE 2

BIASES IN MEASURES OF π_{12} OBTAINED FROM ALTERNATIVE INDEX NUMBER FORMULAS

Standard mix (1)	Bias due to concavity of transformation locus (2)	Bias due to less than unity elasticity of factor substitution (3)	Bias due to interdependence in production function (4)
(i) X_2, I_2, L_2, K_2	−	−	+
(ii) X_1, I_1, L_1, K_1	+	+	−
(iii) X_2, I_2, L_1, K_1	−	+	0
(iv) X_1, I_1, L_2, K_2	+	−	0

(iii) It was also assumed that for any given output mix the marginal rate of transformation does not depend on the input mix and that for any given input mix the marginal rate of substitution between factors does not depend on the output mix. Such rates and mixes are likely rather to be interdependent. To the extent that they are, the index number formulas in Table 1 may still be used to calculate π_{12} but that procedure is subject to a further error. Consider, for example, the computations where period 2 outputs and inputs, X_2, I_2, L_2, K_2, are standard. With interdependence, the output relative indicated by (2.8) is still approximated, as it was before, by the ratio (OG_2/OG_1') in Figure 1.1, and by the corresponding index number for outputs in Table 1. The related index number formula for inputs, however, yields a relative for inputs equal to (Og_2/Og_1') in Figure 1.2. That relative, as we saw, diverges from the measure called for by (2.8) so far as $\sigma < 1$. But even if $\sigma = 1$, the formula in question still errs, since strictly speaking the isoquant f^1f^1 relates to period 1 outputs, G_1. From this isoquant we determine g_1', the inputs of period 2 structure which in period 1 are equivalent to that period's actual inputs g_1 in the production of G_1. What we wish to determine rather is the inputs of period 2 structure which in period 1 are equivalent to g_1 in the production of G_1'. Generally, such inputs will be on an isoquant other than f^1f^1 through g_1.

How do the inputs we wish to know, say g_1''' compare with g_1'? A partial answer is provided by an argument due essentially to Moorsteen (1961) and Yasushi Toda (1964). Substitutions among factors which tend to make one period's factor endowment correspond structurally to that of another period should proceed more favorably to the increasing factor if the structure of output has already been modified to conform to that of the other period. While this rests on rather special

[7]With $\sigma < 1$, the marginal rate of factor substitutions is still supposed to correspond to relative factor prices at observed points. At other points, the isoquant for which $\sigma < 1$ necessarily is to the right of that for which $\sigma = 1$, as shown in Figure 1.2.

assumptions,[8] I show g_1''' accordingly in Figure 1.2, and have recorded in Table 2a corresponding bias in the calculated π_{12}. That bias is positive, for Og_1''' should fall short of Og_1'. Hence so far as our index number formula for inputs yields an observation on (Og_2/Og_1') it tends to overstate the volume of inputs of the standard structure that is needed in period 1 to produce a volume of outputs, OG_1', of a standard structure.

Thus far, period 2 outputs and inputs, X_2, I_2, L_2, K_2, have been taken as standard. By similar reasoning we find that, where X_1, I_1, L_1, K_1 are standard, the magnitude of π_{12} calculated from the index number formulas in Table 1 tends to be understated. Where the standard mix is a composite of outputs of one period and inputs of another, as readily seen, there is curiously no bias in the calculations due to interdependence.[9] The different biases due to interdependence are shown in Table 2, col. (4).

(iv) Formulas (3.3) and (3.4) exclude economies of scale. If there are such economies, all proceeds as before but the calculated value of π_{12} now reflects those economies as well as the advance of technological knowledge.

In considering departures from (3.3) and (3.4) I have assumed that, with outputs or inputs of one period as standard, the corresponding index number price or value weights should still relate to the other period. When production diverges from (3.3) and (3.4), we can no longer be certain that this is in order, but if we proceed as though it were, the direction of bias in the calculated π_{12} is predictable, and it may not be so otherwise. Moreover, the resultant observations, at least for

[8] In the present context, products that are relatively capital or labor intensive with one factor endowment are also that with the other; and the output structure reflects the factor endowment, in the sense that if, say, capital increases relatively to labor, the output of capital intensive products grows relatively to that of labor intensive products.

[9] Consider again the case where outputs and inputs in one period, say 2, are standard. In other words, the standard mix is X_2, I_2, L_2, K_2. By implication, in (2.8), α_2 and β_2 equal unity, while α_1 and β_1 are to be calculated. To refer first to α_1, this is implicitly given by the formula

$$F^1(\alpha_1 X_2, \alpha_1 I_2, L_1, K_1) = 0, \tag{3.5}$$

and is estimated from the relation

$$\alpha_1 \simeq OG_1'/OG_2 = (p_1 X_1 + q_1 I_1)/(p_1 X_2 + q_1 I_2). \tag{3.6}$$

Taking outputs in period 1 as given at the levels $\alpha_1 X_2$, $\alpha_1 I_2$, then, we wish also to determine β_1 so that

$$F^1(\alpha_1 X_2, \alpha_1 I_2, \beta_1 L_2, \beta_1 K_2) = 0. \tag{3.7}$$

This is estimated from the relation

$$\beta_1 \simeq Og_1'/Og_2 = (L_1^{\gamma} K^{1-\gamma})/(L_2^{\gamma} K_2^{1-\gamma}). \tag{3.8}$$

The equation on the right presupposes that $\sigma = 1$, but even if that were so, β_1 would only be approximated. Thus let us write

$$\beta_1^* = Og_1'/Og_2. \tag{3.9}$$

Then, β_1^* is such that

$$F^1(X_1, I_1, \beta_1^* L_2, \beta_1^* K_2), \tag{3.10}$$

and hence differs from β_1, which satisfies (3.7).

Suppose now we take as a standard outputs of one period and inputs of the other, e.g., X_2, I_2, L_1, K_1. In (2.8), then, α_2 and β_1 equal unity, and α_1 and β_2 are to be calculated. To begin with α_1, that is again given implicitly by (3.5) and estimated from (3.6). As for β_2, that is given implicitly by

$$F^2(X_2, I_2, \beta_2 L_1, \beta_2 K_1) = 0. \tag{3.11}$$

For $\sigma = 1$, β_2 then precisely corresponds to (Og_2'/Og_1) and is measured exactly by the corresponding index number in Table 1, so there is no further error at this point.

output, will probably turn out to be the more accurate ones. According to the analysis of Moorsteen (1961) and my own (Bergson, 1961) already referred to, the use of price weights for the period other than that whose mix is standard must yield the more reliable observations on the change in capacity to produce the standard mix whenever the transformation locus is concave from below, or at least not very convex. Also, the "Gerschenkron effect" is supposed to hold (i.e., in the comparison of output in the two periods, period 2 is favored by the use of period 1 prices as weights), but that is very often so. Figure 1.1 illustrates such a situation.[10]

I have referred to diverse index number formulas that might be applied in the calculation of factor productivity. The chief formulas in question are of a sort very often applied in practical work, but the analysis may have clarified their rationale and limitations, together with the valuation principles that they presuppose. It still remains, however, to explore some complexities so far excluded by the simple model on which I have focused.

IV. Capital Goods Valuation

We have been considering thus far a community in which but two products, one consumers' good and one capital good, are produced, and but two inputs, labor and the capital good, employed. What if there are many products and factors rather than just two? That more realistic case was not explored by Moorsteen, and I too have passed it by previously, but evidently all is as before where there are many products and inputs. Thus, the valuation standards considered still apply in that case, and so too does the analysis of index number formulas, though these must now be adapted to the many-product, many factor case in obvious ways.

In practical work, inputs of one or another sort are customarily grouped together in a sub-aggregate as a preliminary to the calculation of the aggregate of all inputs together. Moreover, the index number formula used in the sub-aggregation usually differs from that used in the aggregation of all inputs. For example, different kinds of capital goods may be aggregated arithmetically while such goods taken together may be aggregated with other inputs geometrically. Such a procedure, of course, is usually resorted to only as a practical expedient, but it may be of interest that, as seen here, it is just one of many possible ways of translating one observed mix of inputs into another, that is taken as standard; and so may be more or less valid depending on the degree to which the translation conforms to the shape of the production function, particularly the isoquant surface that the function defines.[11]

[10]With the concavity of the transformation locus and the Gerschenkron effect, however, the alternative index numbers in alternative price weights do not constitute, as we might wish them to, limits on the change in capacity to produce the output mix of either period. See Usher (1972) and Bergson (1972, pp. 216ff).

[11]In the light of Leontief (1947) and Solow (1955–56), it has often been supposed that the procedure in question assumes in any case that the marginal rate of substitution between any two inputs included in a sub-aggregate does not depend on the amounts of any inputs not included in the sub-aggregate. Leontief and Solow refer, however, to the conditions for collapsing variables in a production function. It should be observed that that is not quite the same thing as the problem that is germane here: how to translate one mix of inputs into another. Unless the Leontief–Solow condition is met, it is true that it becomes difficult to conceive of any sub-aggregate as an analytically distinct input, but depending on the nature of the production function, it is still imaginable that for purposes of translating one mix of inputs into another use of different index number formulas at different stages of aggregation might be appropriate.

For the two product, two factor case, as we saw, valuation properly is made in the familiar way where for outputs reference is to prices corresponding to their marginal rate of transformation and for inputs to prices corresponding to their marginal rate of substitution. While one of the products considered was capital, and hence also a source of inputs, the application of both standards simultaneously encounters no difficulty in principle in the simple case in question. As not always considered, however, where there is more than one capital good produced, the problem of valuation becomes somewhat more complex. Thus, the requirement for the valuation of outputs of two capital goods, I^a and I^b apparently is

$$\text{MRT}_{ab} = q^a/q^b, \tag{4.1}$$

while that for the valuation of the corresponding inputs of services would seem to be

$$\text{MRS}_{ab} = rq^a/rq^b = q^a/q^b \tag{4.2}$$

By implication, the relative prices of the two capital goods, q^a and q^b, must correspond at one and the same time to their marginal rate of transformation and the marginal rate of substitution between their corresponding services. Could a single set of prices conform to both conditions?

The answer is, of course, yes provided a well-known condition for dynamic efficiency is satisfied,[12] but such a requirement goes beyond the productive efficiency that has been assumed here, that is, realization of production possibilities, a purely static requirement. Hence, if we are to limit ourselves to that assumption we must consider it a possibility that $\text{MRT}_{ab} \neq \text{MRS}_{ab}$. What then? The rule is still the general one that has been applied, but it is understood that in the case of capital goods reference must be to two sets of prices: one in the valuation of outputs and the other in the valuation of inputs. To recur to an earlier formulation, the current output of capital goods is valued at marginal costs, while capital service inputs are valued proportionately to their marginal products. While reference may thus have to be made to two sets of prices for capital goods rather than one, only on that basis can the measures of factor productivity be construed as they were in the two product, two factor case.

Note that the need to refer to two sets of prices for capital goods may arise quite apart from whether any of the capital goods are new or not. In the literature on factor productivity, the issue posed for the valuation of capital goods by a divergence between their marginal cost and marginal value productivity is a familiar one, but it is usually considered in respect of situations where new capital goods are replacing old ones.

A divergence between marginal cost and marginal value productivity, however, necessarily arises also in such a situation, but in that case we are inevitably confronted too with a phenomenon that has an interest of its own, and which was also excluded from the simple model considered previously: "embodied" technological change. We in effect referred previously only to such change as was "disembodied," for the single capital good considered was available for use as an input in period 1 as well as in period 2. Over the interval considered, therefore, no new capital good embodying technological change was introduced.

[12]See Dorfman, Samuelson, Solow (1958, Ch. 12).

What if there are new capital goods embodying technological change? The moral here is essentially the same as that usually understood in such cases: in the case of new capital goods, we must value inputs not in accord with marginal rates of substitution, as was done previously, but in accord with marginal rates of transformation. As is not difficult to see, only in that way will the resultant measures of factor productivity indicate, as we should wish them to, a CFP reflecting embodied as well as disembodied technological progress. Note, however, that this principle applies only to new capital goods, or more precisely to the valuation of such goods relatively to old capital goods and to each other. In the case of old capital goods, valuation must still conform to marginal rates of substitution. In other words, the principle of valuation in accord with marginal value productivity still applies to inputs of the capital goods generally, but that principle is superceded by valuation at marginal cost in the case of inputs of new capital goods.[13]

New capital goods are but one example of new inputs and outputs generally, and all such inputs and outputs alike were excluded from the simple model with which we began. But what might be said here for new inputs and outputs other than capital goods should be evident, and need not be labored.

V. Contemporaneous Comparison

To come to comparative factor productivity for two communities at the same time, as explained I shall refer here again to an ideal case, though a different one from that considered previously. Thus, technological knowledge is now the same in the two communities. The communities may differ, though, in productive efficiency. In fact, it is that difference, rather than technological progress, that is now to be gauged from comparative data on factor productivity.

But may not the analysis even so proceed essentially as before? Thus, suppose 1 and 2 are seen as two different communities rather than two periods. May we not view production functions such as (2.1) and (2.2) essentially as before, but on the understanding that each formula reflects for the community concerned the alternative mixes of inputs and outputs that are open, after due allowance for inefficiency? And may we not also calculate and interpret factor productivity as before, but on the supposition that the π_{12}, on which the resultant data bear, relates to production functions as so construed?

Broadly speaking the answer in all cases, I believe, is in the affirmative, but as not often considered in writings on factor productivity appraisal of efficiency does sometimes pose novel problems. The problems, moreover, are not always very tractible, but it is well at least to be clear about them.

[13]In the case of new capital goods, then, inputs as well as outputs are to be valued at marginal cost. But the desideratum, to repeat, is that calculated factor productivity should reflect embodied as well as disembodied technological change. It should be observed, therefore, that that could also be achieved under an alternative procedure for new capital goods: valuation of both inputs and outputs in accord with marginal value productivity. The resultant representation of technological progress would differ, however, depending on which of the two approaches is employed. Thus, with valuation at marginal cost, such progress is manifest only when the new capital goods are used, while with valuation at marginal value productivity the progress is manifest when the new capital goods are produced.

On the valuation of new capital goods, while in essentials I subscribe here to a widely held view, one found for example in Denison (1957, pp. 218ff) and Kendrick (1961, p. 35), another standpoint still seems sometimes to be taken. See, for example, Nadiri (1972, p. 133).

According to familiar reasoning, a community may fail to realize its theoretic production capacities, and so suffer from inefficiency in production, in three ways:

(i) Owing to wasteful practices, a production unit may not obtain from the factor inputs at its disposal as large an output as available technological knowledge permits;

(ii) Because of misallocations of factors between production units within any industrial branch, marginal returns to any factor may differ in different production units;

(iii) Because of misallocations in the economy generally, the marginal rate of substitution between factors may vary as between different branches.

An initial question concerns the nature of the production functions to be considered where there is inefficiency of the foregoing sorts. Without such inefficiency, production functions such as (2.1) and (2.2) are determined solely by available technological knowledge. We are able, therefore, to delimit the contours of such functions simply by reference to meaningful alternative hypotheses as to the nature of such knowledge. With inefficiency, the mixes of inputs and outputs that are open depend as well on the working arrangements (institutions, policies and practices) governing resource use, for it is in those arrangements that the inefficiency originates. In the circumstances it is perhaps not entirely obvious that the relevant mixes are even sufficiently determinate to be properly represented by production functions such as (2.1) and (2.2), but assuming that they are, what may be said in a general way regarding the contours of those functions?

A usual supposition is that the functions with inefficiency must be similar in shape to what they are without it. On a theoretic plane perhaps that is the only assumption to make, but transformation loci might well be more or less concave and isoquants more or less convex with inefficiency than they are without it. In fact, it is not precluded that such schedules would be radically altered. Without inefficiency, for example, technological economies of scale might result in convexity of the transformation locus, but we cannot rule out that with inefficiency, such economies would give way to diseconomies resulting in concavity of that locus. Such diseconomies might result, for example, from bureaucratic ineptness in administering large enterprises.

Given production functions such as (2.1) and (2.2), the analysis formally may indeed proceed generally as before. Thus, π_{12} is defined just as it was previously in terms of those functions, and we also obtain measurements of that coefficient by applying index number formulas such as have been set forth. The principles to be observed in selecting and interpreting those formulas are entirely the same as those considered previously.

As before, too, however, the entire exercise presupposes valuation of inputs at prices corresponding to marginal rates of substitution and of outputs at prices corresponding to marginal rates of transformation. The rates in question are those given by production functions such as (2.1) and (2.2) and so relate to the economy generally. But without inefficiency, the marginal rate of substitution thus delineated obtains for substitutions within any production unit, while the corresponding marginal rate of transformation obtains for transformations of outputs of any two production units, one producing one of the two commodities in question and the other producing the other one. These well known relations are, of course, simply conditions for full efficiency, and it is their violation which results in the different sorts of waste itemized above. As it turns out, prices corresponding to rates of substitution and transformation that apply regardless of the production units affected are also identifiable with familiar behavioral norms, a feature much facilitating empirical inquiry.

With inefficiency, however, marginal rates of substitution and transformation evidently must vary depending on which production units are in question. Which of such rates are delineated by production functions such as (2.1) and (2.2) and so relate to the economy generally, therefore, must turn on the working arrangements, for it is those arrangements which determine which of all possible substitutions and transformations might actually occur in any particular instance. That is also to say that how to value inputs and outputs in factor productivity computation probably is a matter that must be dealt with in some degree in an *ad hoc* way, in the light of the working arrangements prevailing in the communities in question. Possibly the relevant valuations could be determined together with the production functions themselves, as some apparently assume, by econometric calculations. It may be illuminating, however, to consider in relation to the theoretic desiderata some specific valuation principles of an *a priori* sort that are formally similar to those applying where there is no inefficiency.

It suffices to refer at this point to but one community. Suppose that community is of the simple kind considered at the outset; that is, it produces two outputs, X and I, with two inputs, L and K, the different symbols having the same meaning as before. Alternative mixes open, with due allowance for inefficiency, are represented by:

$$F(X, Y, L, K) = 0. \tag{5.1}$$

Just what mixes of inputs and outputs might conform to (5.1) must depend, of course, on the manner in which the two inputs are allocated between the two outputs, and within each branch, on the allocation of the inputs among and their utilization by individual production units. We must now consider such activities explicitly, but it may suffice to refer summarily to the two branches. Production of X, then, supposedly conforms to

$$X = G(L_X, K_X), \tag{5.2}$$

and of I, to

$$I = H(L_I, K_I). \tag{5.3}$$

In each case, there is presumably waste of types (i) and (ii) above, and so far as there is, the waste is reflected in (5.2) and (5.3), but how that waste occurs and how great it might be are not here of special concern. By implication, we focus primarily on type (iii) waste. In any actual case, of course, that type of waste might well be overshadowed by the other types, but it is, I think, the most difficult to grapple with and hence conceptually the most interesting to consider for present purposes.

To come to valuation principles, ideally we should wish again to apply prices conforming to (3.1) and (3.2), which now correspond to the inefficient (5.1) rather than to (2.1) or (2.2). But owing to the inefficiency, such prices may not be directly observable, and two groups of valuation principles are to be considered provisionally as surrogates. We are supposedly able to determine prices corresponding to these principles, though that in practice might not be easy. The first group of principles constitute together what may be called the Own Factor Cost (OFC) standard of valuation. The relevant prices are designated p^0, q^0 for X and I, and w^0, for L. There are two rental charges for services of capital goods, resulting from the application of two interest rates, r_X^0 in the X industry and r_I^0 in the I industry. Among these OFC prices, w^0 is arbitrary, and serves in effect as a numéraire. For the rest, it is understood that:

(5.4a,b) $$r_X^0 q^0 / w^0 = G_2/G_1; \quad r_I^0 q^0 / w^0 = H_2/H_1.$$

Also,

(5.5a,b) $$p^0 = \frac{w^0 L_X + r_X^0 q^0 K_X}{X}; \quad q^0 = \frac{w^0 L_I + r_I^0 q^0 K_I}{I}.$$

While outputs are here priced at average cost, note that with inputs priced in accord with (5.4a,b), such prices also correspond to marginal costs provided that (5.2) and (5.3) are linear homogeneous. That, of course, follows at once from Euler's theorem.

How do OFC prices compare with those called for by (3.1) and (3.2)? From (5.2) and (5.3), we have

(5.6) $$MRT_{XI} = -\left(\frac{\Delta I}{\Delta X}\right)_{L,K} = -(H_1 \Delta L_I + H_2 \Delta K_I)/(G_1 \Delta L_X + G_2 \Delta K_X)$$

As indicated, reference is to small variations in L_X, K_X, L_I, K_I, where total employment (L) and the total stock of capital goods (K) are constant. The variations also conform to (5.1). Using (5.4a,b) and (5.5a,b), and assuming linear homogeneity,

(5.7) $$\mathrm{MRT}_{XI} = \alpha (p^0/q^0),$$

where

(5.8) $$\alpha = -(w^0 \Delta L_I + r_I^0 q^0 \Delta K_I)/(w_X^0 \Delta L_X + r_X^0 q^0 \Delta K_X).$$

Or, for relevant variations,

(5.9) $$\alpha = (w^0 \Delta L_X + r_I^0 q^0 \Delta K_X)/(w^0 \Delta L_X + r_X^0 q^0 \Delta K_X).$$

It follows that OFC product prices correspond fully to (3.1) and hence are theoretically ideal for factor productivity computation if there is waste of types (i) and (ii), but not of type (iii). In that case $r_X^0 = r_I^0$, $\alpha = 1$, and (p^0/q^0) precisely equals MRT_{XI}. Should there be type (iii) waste, however, $r_X^0 \neq r_I^0$, $\alpha \neq 1$, and (p^0/q^0) will generally diverge from MRT_{XI}. The extent of the divergence depends on the comparative magnitudes of r_X^0 and r_I^0 and of ΔL_X and of ΔK_X. The latter terms represent the transfers of labor and capital that are called for when I is transformed into X, and possibly could differ in sign, but that seems unlikely. Hence, for any given r_X^0 and r_I^0, α ordinarily should be between two extremes: that is, between $\alpha = 1$, which results when $\Delta K_X = 0$, and means that OFC prices are again ideal, and $\alpha = r_I^0/r_X^0$, which results when $\Delta L_X = 0$. It also follows that if rates of return on capital do not differ too much, p^0/q^0 should approximate 3.1 fairly closely. On the other hand, if rates of return do differ widely, p^0/q^0 could diverge appreciably from that norm, but whether and to what extent might perhaps be gauged by considering that the comparative magnitudes of ΔL_X and ΔK_X are determined in principle by (5.1) and so in effect by the working arrangements. The labor-capital ratios in the two branches in question, however, presumably might often be significant benchmarks.

To come to OFC factor prices and the marginal rate of substitution, we have from (5.2) and (5.3),

$$\text{MRS}_{KL} = -(\Delta L/\Delta K)_{X,I} = \frac{1}{\Delta K}\left(\frac{G_2}{G_1}\Delta K_X + \frac{H_2}{H_1}\Delta K_I\right). \tag{5.10}$$

Here, the variations of ΔK and ΔL and the division of ΔK between the two branches are such that X and I are constant. Also, formula (5.1) again holds. Using (5.4a,b),

$$\text{MRS}_{KL} = \frac{1}{w^0\Delta K}(r_X^0 q\Delta K_X + r_I^0 q^0\Delta K_I). \tag{5.11}$$

Let us designate by r^0 the average rate of interest in the economy generally, where

$$r^0q^0 = r_X^0 q^0(K_X/K) + r_I^0 q^0(K_I/K). \tag{5.12}$$

Should waste be only of types (i) and (ii) and not at all of type (iii), $r^0 = r_X^0 = r_I^0$, and

$$\text{MRS}_{KL} = r^0q^0/w^0, \tag{5.13}$$

so OFC prices correspond to (3.2) and so are at this point again ideal, it being understood that for the needed rental rate for capital goods reference is to the rate imputable to such goods in either industry or (what is the same thing) the average of such rates for the economy generally. Suppose now there is type (iii) waste. In that case, we again have $r_X^0 \neq r_I^0$, and r^0 will ordinarily differ from either. But (5.13) still holds, and OFC prices are still ideal with the rental rate for capital goods being r^0q^0 provided that the increment of capital that is supplanting labor in the economy generally is divided between the two branches proportionally to the stock already there. While such a division is hardly to be expected, if it is at all approximated, the divergence between r^0q^0/w^0 and the ratio called for by (3.2) might not be very great even when $r_X^0 \neq r_I^0$. The approximation is also the closer the smaller is the discrepancy between those rates of return.

The second group of valuation principles constitute what I have referred to elsewhere in a related context (Bergson, 1961, Ch. 3) as the Adjusted Factor Cost (AFC) standard of valuation. AFC closely resembles OFC, but has an interest of its own. Let us designate the relevant prices as p^*, q^*, w^*, $r_X^*q^*$, $r_I^*q^*$ and r^*q^*. Then, for AFC, we have

$$r_X^*q^*/w^* = G_2/G_1; \quad r_I^*q^*/w^* = H_2/H_1 \tag{5.14a,b}$$

Also,

$$r^*q^* = r_X^*q^*(K_X/K) + r_I^*q^*(K_I/K), \tag{5.15}$$

and

$$p^* = \frac{w^*L_X + r^*qK_X}{X}; \quad q^* = \frac{w^*L_I + r^*qK_I}{I}. \tag{5.16a,b}$$

Evidently, $r^*q^*/w^* = r^0q^0/w^0$, and so has the same claim as the latter, no better and no worse, to represent MRS_{KL}. The two product price ratios p^*/q^* and p^0/q^0 also are equal if there is no type (iii) waste and capital goods rental rates, under either price system, are the same in the two branches. Otherwise, however, $p^*/q^* \neq p^0/q^0$. Thus, while both sorts of prices cover average cost, in the case of p^* and q^* a uniform charge is made for capital at an average rental rate for the whole economy. In the case of p^0 and q^0 the charge for capital varies, and corresponds in each industry to the marginal productivity of capital there. But note that the discrepancy between p^*/q^* and p^0/q^0 might be such as to make the former ratio closer to the mark. Suppose, for example, that $r_I^0 > r_X^0$. Then from (5.7) and (5.9), $\alpha > 1$, and p^0/q^0 is too small. But, as readily seen, p^*/q^* must then also be greater than p^0/q^0. Similarly, if $r_I^0 < r_X^0$, p^0/q^0 is too large, but in that case p^*/q^* is less than p^0/q^0. Of course, in either case it is still possible that p^*/q^* deviates more than p^0/q^0 from the desired price ratio.

So far I have tacitly assumed that nothing is known about the way in which working arrangements cause inefficiency. So far as such information is available, clearly we might be able either to improve on or at least gauge more definitely the biases in our surrogate principles. For example, suppose as before that (5.2) and (5.3) are linear homogeneous. Suppose also that the relative divergence between r_x^0 and r_I^0 or r_X^* and r_I^* (the latter rates, of course, come to the same thing as the former) can be expected to be more or less stable as resources are reallocated. In that case the labor-capital ratios in the two branches not only provide, as it was suggested above that they might, benchmarks for the ratio of the increment of labor to the increment of capital transferred from one branch to another when, with given factor supplies, the output mix is changed. It can be shown that the labor-capital ratios in the two branches in fact delimit the latter ratio. Though (5.2) and (5.3) are hardly likely to be of the compatible Cobb–Douglas sort which, as noted, underlie (3.3) and (3.4), it is interesting to note that, if they should be, such an increment of capital replacing labor in the economy generally would be divided between the two branches in proportion to their existing capital stocks, which is a further relation that was considered above. I leave the proofs of these propositions to the reader.[14]

What if OFC prices, i.e., p^0 and q^0 for outputs, and w^0 and r^0q^0 for inputs, or AFC prices, i.e., p^* and q^* for outputs and w^* and r^*q^* for inputs, are applied even though they may not conform to MRT_{XI} and MRS_{KL}? There is simply still

[14] Of course, if we know the production functions in the two branches and also the precise manner in which relative rates of return vary in the two branches, we can in principle determine (5.1) and corresponding formulas for the theoretically ideal prices conforming to (3.1) and (3.2). Even in the case of compatible Cobb–Douglas functions, however, such formulas seem to turn out to be rather complex.

another source of bias in the computation, in addition to those already considered. The computation would have to be construed accordingly.

I have again passed by complexities. Concerning these, suffice it to say that if there are many products and factor inputs there is a possibility that was not very meaningful previously that divergence between OFC and AFC prices and (3.1) and (3.2) might not be highly correlated with comparative levels of outputs and inputs in the two communities considered, so that the bias in resulting measures of factor productivity computation. The mode of analysis derives from a 1961 essay different products and factor inputs. Regarding embodied technological change, it should be observed that we are concerned here with the contemporaneous comparison of two communities that have the same technological knowledge. Differences in the assortment of capital goods produced, nevertheless, are not precluded, and that means that variations in embodied technologies of the sort encountered in intertemporal comparison for a single community might also be found here, but if so presumably not so frequently. In any event, the analysis of embodied technological variation elaborated for the intertemporal case applies here as well, so no further consideration of that phenomenon is needed.

I have assumed throughout that performance in both communities considered falls short of production possibilities. What if such inefficiency should prevail for one community but not the other? In the real world no community is perfectly efficient, but perhaps the inefficiency sometimes is not so consequential for purposes of calculations such as are in question. If so, all calculations may proceed as in the case where an intertemporal comparison is made for a community always realizing fully its production possibilities. Or rather, that is so where the contemporaneous valuation is in terms of prices of the community that is more or less efficient. Where valuation is in terms of prices of the other community experiencing consequential inefficiency, the problem of valuation considered in this section still arises.

VI. Conclusions

I have sought in this essay to elaborate index number theory as it applies to factor productivity computation. The mode of analysis derives from a 1961 essay of Richard Moorsteen, but it may have been possible to deal more fully than has been done previously with the central question that arises regarding the nature and meaning of measures obtained by application of different index number formulas, together with the valuation principles that they presuppose. In the process, special attention has been given to a cardinal but relatively neglected aspect: the special problem posed where the ultimate concern is to appraise variations in productive efficiency as distinct from technological knowledge.

Index number theory tends to be abstract, and that elaborated here is no exception to that rule. That means among other things that I have followed a usual practice of assuming that all economic activities in a community take place at a single point in space. The analysis, thus, abstracts from the special problem posed by transportation cost. It may be hoped that before too long it will be possible to remove this important limitation.

The mode of index number analysis that I have employed is not the only one that might be adopted in respect of factor productivity computation. Lately, use has often been made of a rather different approach centering on the Divisia index. The methodological problem that is thus posed is properly the subject of a separate inquiry but in practice what is called for under the Divisia index approach is essentially the use of chained indices. It should be observed, therefore, that the analysis set forth in this essay does not preclude such calculations. The results, however, have to be interpreted in a complex way. In effect, observations are obtained on the cumulative variations in productive capacity in respect of a succession of changing standard mixes. Proponents of the Divisia index approach usually focus, moreover, on intertemporal changes in one country. That is understandable, for a chained index might be difficult to construct where reference is to contemporaneous variations between countries.[15]

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[15]On Divisia indices in relation to factor productivity computation see Griliches and Jorgenson (1967, pp. 250ff); Merrilees (1971); Griliches and Jorgenson (1971). For a related approach, see Domar (1961).